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An integral operator preserving functions with bounded argument rotation [1]

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Abstract

Let A be the class of functions f, which are analytic in the unit disc $\Delta = \{|z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$. Let B be the class of functions f, which are analytic in the unit disc Δ, with $f(0) = 1$. Let $R = \{f \in A : \Re f'(z) > 0, z \in \Delta\}$. Let $g \in A$ with $g(z)/z \neq 0$ in Δ, $h \in B$, and consider the integral operator $I : A \to A$, where $F = I(f)$ is defined by

$$F(z) = \int_0^z \frac{f(t)h(t)}{g(t)} dt$$

We obtain certain sufficient conditions on g and h so that $I(R) \subset R$. For $h(z) = 1, z \in \Delta$, such conditions are known.

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1 Introduction

Let $\mathbb{C}$ be the complex plane and $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. Let A be the class of all analytic functions in Δ, with $f(0) = f'(0) - 1 = 0$ and B the class of all analytic functions h in Δ with $h(0) = 1$. Let

$$R = \{f \in A : \Re f'(z) > 0, z \in \Delta\}$$

Functions in R are often called functions of bounded argument rotation or functions of bounded turning. It is well known that if $f \in R$, then $F \in R$, where

$$F(z) = \frac{2}{z} \int_0^z f(t) dt$$

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P.Mocanu, H.Al-Amiri, Kit C.Chan, J.Gresser and S.Suebert in [1] obtained certain conditions on $g \in A$ and $\gamma \in \mathbb{R}$ so that if $f \in R$, then $F \in R$ where

$$F(z) = \frac{\gamma+1}{g^{\gamma}(z)} \int_0^z f(t) g^{\gamma-1}(t) g'(t) dt \tag{1}$$

For $\gamma = 1$, such conditions were obtained in [2]. Let $h \in B$, $g \in A$ with $g(z)/z \neq 0$ in Δ and consider the integral operator $I : A \to A$ where $F = I(f)$ is defined by

$$F(z) = \int_0^z \frac{f(t)h(t)}{g(t)} dt \tag{2}$$

In this note we obtain sufficient conditions on h and g such that $I(R) \subset R$. If we take $h(z) = g'(z)$ for all $z \in \Delta$ we obtain the result from [1] in the case $\gamma = 0$. On the other hand we will obtain new examples of "$bounded - argument - rotation$" preserving integral operators.

2 Preliminaries

In order to prove our main result, we will need the following lemma.

Lemma 1 *[3] Let K(z) and L(z) be complex–valued functions on Δ and suppose that for $z \in \Delta$,*

$$|\Im L(z)| \leq \Re K(z) \tag{3}$$

If p is analytic in Δ, with $p(0) = 1$ and if, for $z \in \Delta$

$$\Re\{K(z) z p'(z) + L(z) p(z)\} > 0 \tag{4}$$

then $\Re p(z) > 0$ in Δ.

3 Main Result

Theorem 1 *Let $g \in A$, $g(z)/z \neq 0$ in Δ and let $h \in B$. Suppose that:*

$$\Re \frac{g(z)}{z h(z)} \geq \left| \Im \left(\frac{g(z)}{h(z)} \right)' \right| \tag{5}$$

If I is the integral operator defined in (2), then $I(R) \subset R$.

Proof. Let $f \in R$ and $F = I(f)$. If we denote

$$Q(z) = \frac{g(z)}{z h(z)}$$

we have that

$$\frac{zQ'(z)}{Q(z)} = \frac{zg'(z)}{g(z)} - 1 - \frac{zh'(z)}{h(z)}$$

And then

$$\begin{aligned} zQ'(z) + Q(z) &= \frac{g'(z)}{h(z)} - \frac{g(z)h'(z)}{h^2(z)} \\ &= \left(\frac{g(z)}{h(z)}\right)^2 \end{aligned}$$

But from(5) we have:

(6) $$\Re Q(z) \geq \left|\Im\left(Q(z) + zQ'(z)\right)\right|, z \in \Delta$$

From (2) we deduce

$$\frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)} + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}$$

and then:

$$zF''(z) = f'(z)\frac{zh(z)}{g(z)} + F(z)\left(\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right)$$

Using the above– mentioned notations we have:

$$zF''(z)Q(z) + F'(z)\left[zQ'(z) + Q(z)\right] = f'(z)$$

We denote:

$$K(z) = Q(z)$$

and

$$L(z) = zQ'(z) + Q(z)$$

If $p(z) = F'(z)$ and if $f \in R$ we have that $\Re f'(z) > 0$ for all $z \in \Delta$. Then we have, for $f \in R$:

(7) $$\Re\left[K(z)zp'(z) + L(z)p(z)\right] > 0, z \in \Delta$$

Relation (6) is equivalent to:

(8) $$|\Im L(z)| \leq \Re K(z)$$

Then, from (7) and (8), by applying Lemma 1 we deduce that $\Re F'(z) = \Re p(z) > 0$ for all $z \in \Delta$. Hence, $f \in R$ implies $F \in R$ and the theorem is proved.

4 Some Particular Cases

Corollary 1 *If $f \in R$ then $\Re f(z)/z > 0$ for all $z \in \Delta$*

Proof. We apply Theorem 1 with $h(z) = g(z)/z$ for all $z \in \Delta$ and we obtain the above well–known result.

Corollary 2 *If $f, g \in A$ and*

$$\Re g(z)/z \geq \left|\Im g'(z)\right|, z \in \Delta \tag{9}$$

then $f \in R$ implies $\Re\left[f(z)/z\right] > 0$ for all $z \in \Delta$

Proof. We apply Theorem 1 with $h(z) = 1$ for all $z \in \Delta$. Condition (9) is equivalent with (5) and the implication follows easily from Theorem 1.

Remark 1 *If, in Corollary 2 we take $g(z) = z$ for all $z \in \Delta$ we obtain Corollary 1.*

Remark 2 *If $h(z) = g'(z)$ for all $z \in \Delta$ we obtain a particular case (for $\gamma = 0$) of Theorem 2 from [1]*

Example 1 *If we let $g \in A$ with $g(z)/z \neq 0$ in Δ and $h(z) = e^{\lambda z}\frac{g(z)}{z}$, then, condition (5) becomes:*

$$\Re e^{-\lambda z} \geq \left|\Im e^{-\lambda z}(1 - \lambda z\right| \tag{10}$$

If we put $\lambda z = \zeta = \rho e^{i\theta}, \theta \in \mathbb{R}$ we obtain:

$$Ree^{-\rho e^{i\theta}} \geq \left|\Im e^{-\rho e^{i\theta}}(1 - \rho e^{i\theta})\right| \tag{11}$$

Let $t = \rho \sin\theta$. Condition (11) becomes:

$$\cos t \geq |\rho \sin(\theta - t) + \sin t| \tag{12}$$

It is easy to check that condition (12) is true for $0 < \rho \leq (\sqrt{3} - 1)/2$
Hence, by applying Theorem 1 we obtain:

Corollary 3 *If $|\lambda| \leq (\sqrt{3}-1)/2 = 0.366\ldots$ and if $I : A \to A$ is the integral operator defined by $F = I(f)$, where*

$$F(z) = \int_0^z \frac{f(t)}{t} e^{\lambda t} dt$$

then $I(R) \subset R$.

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