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On strong mixing property of cellular automata with respect to Markov measures [1]

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Abstract

In this paper we study mixing properties of one-dimensional linear cellular automata over the ring $\mathbb{Z}_m$, a particular class of dynamical systems, determined by right (left) permutative local rule F with respect to the uniform Markov measure induced by doubly stochastic matrix $P = p_{(i,j)}$ and the probability vector π. We prove that one-dimensional linear cellular automata associated to right (resp. left) permutative local rule $F = F[l, r]$, $0 < l < r$ (resp. $l < r < 0$), is strong mixing with respect to the uniform Markov measure. We also show that $\mathbb{Z} \times \mathbb{N}$-actions generated by the one-dimensional linear cellular automata determined by bipermutative local rule $F = F[l, r]$ and shift map is strong mixing with respect to the uniform Markov measure.

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1 Introduction

Cellular automata (CAs for short), discovered by Ulam (1952) and von Neumann (1951), have been systematically studied by Hedlund from a purely

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mathematical point of view [8]. Hedlund's paper started investigation of current problems in symbolic dynamics. The study of CAs from the point of view of the ergodic theory has received remarkable attention in the last few years ([4], [5], [6], [9], [11]), because CAs have been widely investigated in a number of disciplines (e.g., mathematics, physics, computer science, and so on).

It is well known that there are several notions of mixing (i.e. weak mixing, strong mixing, mildly mixing, completely mixing and so on) of measure preserving transformation on probability space in ergodic theory. It is important to know how these notions are related with each other. If the measure preserving transformation T on a probability space $(X,\ \mathcal{B},\ \mu)$ is strong mixing then it is both weak mixing and ergodic, that is, strong mixing is a stronger property than weak mixing and ergodic (see [13] and [15] for the details). The last decade (see. e.g [12, 13, 15]), a lot papers are devoted to this subject.

Shirvani and Rogers in [14] have proved that all onto CAs are invariant and strong mixing with respect to the Haar measure. In [13], Shereshevsky has studied some strong ergodic properties of the natural extension of a measure theoretic endomorphism. He has answered some questions raised in [14]. He has also defined n-th iteration of a permutative CA and shown that if the local rule F is right (left) permutative, then its n-th iteration also is right (left) permutative.

In [10], it is proved that weak-mixing of f implies transitivity of the natural extension of f, further, if f is mixing or weakly mixing, then chaoticity of f (individual chaos) implies chaoticity of the natural extension of f (collective chaos) and if X is a closed interval then the natural extension of f is chaotic (in the sense of Devaney) if and only if f is weakly mixing. In [6], it is studied a new definition of strong topological chaos for discrete time dynamical systems which fulfills the informal intuition of chaotic behavior considering the class of additive CAs.

In [11], Mass and Martinez have studied the dynamics of Markov measures by a particular linear CA. They have reviewed some results on the evolution of probability measures under CA acting on a fullshift.

In [9], Kleveland has proved that left (right) permutative CA $T_{F[l,r]}$ is strong mixing with respect to product measure defined by normalized Haar-measure, and some of the endomorphisms on the space of bi-infinite sequences over a finite set even k-mixing with respect to product measure. In [9], it has been generalized the mixing results. Cattaneo *et al.* [6] have studied the dynamical behavior of D-dimensional CAs. They have shown how to construct ergodic D-dimensional linear CAs over $\mathbf{Z}_m$.

In [2], the author has investigated some ergodic properties of $\mathbb{Z}^2$-actions generated by invertible linear CAs and shift transformation without considering the natural extension. Also in [1], the author has proved that $\mathbb{Z}\times\mathbb{N}$-actions generated by the bipermutative linear CAs and shift map is strong mixing with respect to uniform Bernoulli measure.

Some ergodic properties of Markov and Bernoulli shifts have been investigated in [7] and [15].

In this paper we study strong mixing property of one-dimensional linear CA associated to right (left) permutative local rule with respect to a uniform Markov measure induced by doubly stochastic matrix $P = p_{(i,j)}$ such that $p_{(i,j)} = \frac{1}{m}$ for all pair (i, j) in the point of view of ergodic theory. We prove that one-dimensional linear CA associated to right (left) permutative local rule $F = F[l, r]$, $0 < l < r$ (resp. $l < r < 0$), is strong mixing with respect to the uniform Markov measure. We also show that $\mathbb{Z} \times \mathbb{N}$-actions generated by the one-dimensional linear CAs determined by bipermutative local rule $F = F[l, r]$, $l < 0 < r$, and shift map is strong mixing with respect to the uniform Markov measure. We generalize some results of Shereshevsky [13] for the uniform Markov measure.

The rest of the paper is organized as follows. In Section 2 we establish the basic formulation of problem to state our main theorems and we give some necessary notations and definitions. In Section 3 we prove our main theorems and some results.

2 Preliminaries

(1) *Cellular Automata*: Let $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ $(m \geq 2)$ be a ring and $\mathbb{Z}_m^{\mathbb{Z}}$ be the space of all doubly-infinite sequences $x = (x_n)_{n=-\infty}^{\infty}$, $x_n \in \mathbb{Z}_m$. The shift $\sigma : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(\sigma x)_i = x_{i+1}$ is a homeomorphism of compact metric space $\mathbb{Z}_m^{\mathbb{Z}}$. A CA is a map $T : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined for $x = (x_i)_{i\in\mathbb{Z}}$, $i \in \mathbb{Z}$ by

$$(1) \qquad (T_{F[l,r]}x)_n = F(x_{l+n}, \ldots, x_{n+r}) = \sum_{i=l}^{r} \lambda_i x_{i+n} (mod\ m),$$

where $F : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ is a given local rule or map.

In this paper, we consider one-dimensional linear CA $T_{F[l,\ r]}$ determined by linear local rule F. Throughout the paper, we will use the notation $T_{F[l,r]}$ for linear CA-map defined in Eq. (1) to emphasize the local rule F and the numbers l and r.

The notion of permutative CA was first introduced by Hedlund in [8]. If the linear local rule $F : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ is given in Eq. (1), then it is permutative in the jth variable if and only if $gcd(\lambda_j, m) = 1$, where gcd denotes the greatest common divisor. A local rule F is said to be right (respectively, left) permutative, if $gcd(\lambda_r, m) = 1$ (respectively, $gcd(\lambda_l, m) = 1$). It is said that F is bipermutative if it is both left and right permutative.

Example 1 *Consider the local rule $F : \mathbb{Z}_3^3 \to \mathbb{Z}_3$ as follows:*

$$F(x_{-1},\ x_0,\ x_1) = (2x_{-1} + 2x_0 + x_1)\ (mod\ 3),$$

then F is both left and right permutative.

Let now $l,\ r \in \mathbb{Z}, l \leq r$ be given and let $F : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ be a fixed local rule.

Lemma 1 *([13], Lemma 1.5). If the local rule $F : \mathbb{Z}_m^u \to \mathbb{Z}_m$ is right (left) permutative, then so is its k-th iteration $F^k : \mathbb{Z}_m^{k(u-1)+1} \to \mathbb{Z}_m$ for each integer $k \geq 1$.*

Lemma 2 *([13], Lemma 1.6). The kth iteration $T_{F[l,\ r]}^k$ of CA $T_{F[l,\ r]}$ generated by the linear local rule F coincides with the CA $T_{F^k[kl,\ kr]}$.*

In [4], it was determined the properties of endomorphisms and automorphisms of the shift dynamical system.
(2) *Markov Measure*: Let $P = (p_{(i,j)})$ denote a $m \times m$ stochastic matrix ($p_{(i,j)} \geq 0$, $\sum_{j=0}^{m-1} p_{(i,j)} = 1$) with entries $p_{(i,j)}$ and let $\pi = \{\pi_0, \pi_1, \ldots, \pi_{m-1}\}$ be its left eigenvector. It is well known that $\pi P = P$. A pair π, P defines a set function $\mu_{\pi P}$ on the cylinders of $\mathbb{Z}_m^{\mathbb{Z}}$. Recall that it is defined the associated Markov measure by defined as follows:

$$\mu_{\pi P}({}_0[i_0, \ldots, i_k]_k) = \pi_{i_0} p_{(i_0,i_1)} \cdots p_{(i_{k-1},i_k)} \tag{2}$$

(see [7, 15] for the details).

A Markov measure on $\mathbb{Z}_m^{\mathbb{Z}}$ is uniform, if measure of any one-dimensional cylinder is equal to $\frac{1}{m}$, where m is the cardinality of $\mathbb{Z}_m$. A doubly stochastic matrix is a matrix P such that P and P^{tr} (transpose) are both stochastic. If a matrix P is a doubly stochastic then corresponding Markov measure is a uniform measure. If the cardinality of $\mathbb{Z}_m$ is equal to m, then, any doubly stochastic matrix P of $m \times m$ size will generate uniform Markov measure.

In this paper, we consider the uniform Markov measures induced by doubly stochastic matrices $P = (p_{(i,j)})$ such that

$$p_{(i,j)} = \frac{1}{m} \text{ for all pairs } (i,j). \tag{3}$$

3 Formulation of the problem and results

Theorem 1 *([15], Theorem 1.17) Let $(X,\mathcal{B}, \mu)$ be a measure space and let $\mathcal{A}$ be a semi-algebra that generates $\mathcal{B}$. Let $T : X \to X$ be a measure-preserving transformation. Then*

(i) T is ergodic iff $\forall A, B \in \mathcal{A}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B),$$

(ii) T is weak-mixing iff $\forall$ $A, B \in \mathcal{A}$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left|\mu(T^{-i}A \cap B) - \mu(A)\mu(B)\right| = 0$$

and

(iii) T is strongly-mixing iff $\forall A, B \in \mathcal{A}$

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B). \tag{4}$$

Firstly, let us consider the following one-dimensional linear CA

$$(T_{F[l,\, r]}x)_n = F(x_{l+n}, \ldots, x_{n+r}) = \sum_{i=l}^{r} \lambda_i x_{i+n} (mod\ m), \tag{5}$$

where $0 < l < r$ (or $l < r < 0$).

Lemma 3 *Suppose that the CA $T_{F[l,r]}$ is defined as in* (5) *and assume that F is right (resp. left) permutative and $0 < l < r$ (resp. $l < r < 0$) and Markov measure $\mu_{\pi P}$ be defined as in* (3)*, then linear CA $T_{F[l,r]}$ is the uniform Markov measure-preserving transformation.*

Proof. Let $A = {}_u[a_0^{(0)}, \ldots, a_v^{(0)}]_{u+v}$ be a cylinder set, where $u \leq v$, then we have;

$$\begin{aligned}
\mu_{\pi P}(T_{F[l,\ r]}^{-1}(A)) &= \mu_{\pi P}(\bigcup_{a_l^{(1)},\ldots,a_{r+v}^{(1)}} ({}_{l+u}[a_l^{(1)}, \ldots, a_{r+v}^{(1)}]_{r+u+v})) \\
&= \sum_{a_l^{(1)},\ldots,a_{r+v}^{(1)}} \mu_{\pi P}({}_{l+u}[a_l^{(1)}, \ldots, a_{r+v}^{(1)}]_{r+u+v})) \\
&= m^{(r-l)} \pi_{(a_l^{(1)})} p_{(a_l^{(1)}, a_{l+1}^{(1)})} \ldots p_{(a_{r+v-1}^{(1)}, a_{r+v}^{(1)})} \\
&= \mu_{\pi P}(A).
\end{aligned}$$

This completes the proof.

The following theorem generalizes the notion of strong mixing with respect to the Markov measure (see [9] and [13] for details).

Theorem 2 *Suppose that at least one of the following conditions is satisfied:*
(RP) $0 < l < r$ and the local rule $F : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ is right permutative;
(LP) $l < r < 0$ and the local rule $F : \mathbb{Z}_m^{r-l+1} \to \mathbb{Z}_m$ is left permutative.
Then $T_{F[l,\ r]}$ is strong mixing with respect to the uniform Markov measure satisfying (3)*.*

Proof. Let us firstly consider left permutative local rule F as follows;

$$F(x_l, \ldots, x_r) = \sum_{i=l}^{r} \lambda_i x_i (mod\ m), \tag{6}$$

where $\lambda_i \in \mathbb{Z}_m$ and $0 < l < r$. If we take $l < r < 0$, then similarly we can prove the Theorem 2.

Let $A =_u [a_0, \ldots, a_v]_{u+v}$ and $B =_y [b_0^{(o)}, \ldots, b_z^{(o)}]_{y+z}$ be two cylinder sets. Then we can observe that

$$A \cap T_{F[l,\ r]}^{-n} B =$$

$$\bigcup_{x_{v+1},\ldots,x_{nl-1}} \bigcup_{b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}} ({}_u[a_0,\ldots,a_v,x_{v+1},\ldots,x_{nl-1},b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}]_{(nr+y+z)}),$$

where $F(b^{(n)}_{i+l},\ldots,b^{(n)}_{i+r}) = \sum_{k=l}^{r}\lambda_k b^{(n)}_{i+k}(\text{mod } m){=}b^{(n-1)}_i$ for all $i=(n-1)l,\ldots,(n-1)r+z$. For brevity assume that $\mu_{\pi P}=\mu$. Thus for $nl > u+v-y$ we have

$$\begin{aligned}
&\mu(A\cap T^{-n}_{F[l,\ r]}B) = \\
&= \mu\Big(\bigcup_{x_{v+1},\ldots,x_{nl-1}} \bigcup_{b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}} ({}_u[a_0,\ldots,a_v,x_{v+1},\ldots,x_{nl-1},b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}]_{(nr+y+z)})\Big) \\
&= \mu(A)\Big(\sum_{b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}} \sum_{x_{v+1},\ldots,x_{nl-1}} p_{(a_v,x_{v+1})}\cdots p_{(x_{nl-1},b^{(n)}_{nl})} \times \\
&\quad p_{(b^{(n)}_{nl},b^{(n)}_{nl+1})}\cdots p_{(b^{(n)}_{nr+z-1},b^{(n)}_{nr+z})}\Big) \\
&= \mu(A)\sum_{b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}} p^{(nl+y-u-v)}_{(a_v,b^{(n)}_{nl})} p_{(b^{(n)}_{nl},b^{(n)}_{nl+1})}\cdots p_{(b^{(n)}_{nr+z-1},b^{(n)}_{nr+z})} \\
&= \mu(A)\sum_{b^{(n)}_{nl},\ldots,b^{(n)}_{nr+z}} \pi_{b^{(n)}_{nl}} p_{(b^{(n)}_{nl},b^{(n)}_{nl+1})}\cdots p_{(b^{(n)}_{nr+z-1},b^{(n)}_{nr+z})} \\
&= \mu(A)\mu(B).
\end{aligned}$$

however we know that $p^{(nl+y-u-v)}_{(a_v,\ b^{(n)}_{nl})} \to \pi_{b^{(n)}_{nl}}$ as $n\to\infty$ (by putting P in terms of Jordan forms) and so we know that

$$\lim_{n\to\infty}\mu(A\cap T^{-n}_{F[l,\ r]}B) = \mu(A)\mu(B).$$

Thus proof is completed.

Lemma 4 *Let the local rule F be defined as equation* (5)*. Then we have*

$$((T_{F[l,r]}\circ\sigma^{-s})x^{(n)})_i = \sum_{k=l}^{r}\lambda_k x^{(n)}_{k+s+i}(mod\ m) = x^{(n-1)}_{i+s},$$

where $x^{(n)}_i$ is the ith coordinate of $x^{(n)}\in\mathbb{Z}_m^{\mathbb{Z}}$ and for the sake of appropriateness we assume that σ is left shift.

Lemma 5 *Suppose that CA $T_{F[l,r]}$ is given as in* (5) *and let Markov measure $\mu_{\pi P}$ be defined as in* (3)*, then $\Phi = T_{F[l,r]}\circ\sigma$ is a uniform Markov measure preserving transformation, that is, $\mu_{\pi P}(\Phi^{-1}(A)) = \mu_{\pi P}(A)$.*

Proof. Let us consider the following cylinder set

$$A = \{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_u = i_0^{(0)}, \ldots, x_{u+v} = i_v^{(0)}\} =_u [i_0^{(0)}, \ldots, i_v^{(0)}]_{u+v},$$

where $i_0^{(0)}, \ldots, i_v^{(0)} \in \mathbb{Z}_m$.

The first preimage of the cylinder $A =_u [i_0^{(0)}, \ldots, i_v^{(0)}]_{u+v}$ under the $\Phi = T_{F[l,r]} \circ \sigma$ consists of the union of the following cylinder sets;

$$\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_{u+l+1} = i_{l+1}^{(1)}, \ldots, x_{u+v+r+1} = i_{v+r+1}^{(1)};\ i_{l+1}^{(1)}, \ldots, i_{v+r+1}^{(1)} \in \mathbb{Z}_m\},$$

where

$$F(x_{l+n}^{(1)}, \ldots, x_{r+n}^{(1)}) = \sum_{k=l}^{r} \lambda_k x_{k+n}^{(1)} (\text{mod } m) = x_n^{(0)}.$$

Now let us calculate the uniform Markov measure of $(T_{F[l,r]} \circ \sigma)^{-1}(A)$;

$$\begin{aligned}
\mu_{\pi P}((T_{F[l,r]} \circ \sigma)^{-1}(A)) &= \mu_{\pi P}\Big(\bigcup_{i_{l+1}^{(1)}, \ldots, i_{v+r+1}^{(1)}} ({}_{u+l+1}[i_{l+1}^{(1)}, \ldots, i_{v+r+1}^{(1)}]_{u+v+r+1})\Big) \\
&= \sum_{i_{l+1}^{(1)}, \ldots, i_{v+r+1}^{(1)}} \mu_{\pi P}({}_{u+l+1}[i_{l+1}^{(1)}, \ldots, i_{v+r+1}^{(1)}]_{u+v+r+1}) \\
&= m^{(r-l)} \pi_{(i_{l+1}^{(1)})} p_{(i_{l+1}^{(1)}, i_{l+2}^{(1)})} \ldots p_{(i_{v+r}^{(1)}, i_{v+r+1}^{(1)})} \\
&= \mu_{\pi P}(A).
\end{aligned}$$

Before giving the proof of main theorem, we must describe whether $\mathbb{Z} \times \mathbb{N}$-actions $\Phi^{(t,s)} = T_{F[l,r]}^t \circ \sigma^s$ is the uniform Markov measure-preserving transformation.

Notice that $\Phi^{(t,s)} = T_{F[l,r]}^t \circ \sigma^s = T_{F^t[tl,tr]} \circ \sigma^s = T_{F^t[tl-s,tr-s]} = \sigma^s \circ T_{F^t[tl,tr]}$, $t, s \in \mathbb{N}$.

Lemma 6 *Let CA $T_{F[l,r]}$ be given as in (5) and Markov measure $\mu_{\pi P}$ be as in (3) then $\Phi^{(t,s)} = T_{F[l,r]}^t \circ \sigma^s$ is a uniform Markov measure-preserving transformation.*

Proof. Similar to Lemma 5 let us consider the cylinder

$$A = \{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_u = i_0^{(0)}, \ldots, x_{u+v} = i_v^{(0)}\} =_u [i_0^{(0)}, \ldots, i_v^{(0)}]_{u+v},$$

where $i_0^{(0)}, \ldots, i_v^{(0)} \in \mathbb{Z}_m$.

The first preimage of the cylinder $A =_u [i_0^{(0)}, \ldots, i_v^{(0)}]_{u+v}$ under the $\Phi^{(t,\ s)}$ consists of the union of the following cylinder sets;

$$\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_{u+lt+s} = i_{lt+s}^{(t)}, \ldots, x_{u+v+rt+s} = i_{v+rt+s}^{(t)};\ i_{lt+s}^{(t)}, \ldots, i_{v+rt+s}^{(t)} \in \mathbb{Z}_m\},$$

where $F(x_{l+n}^{(t)}, \ldots, x_{r+n}^{(t)}) = \sum_{k=l}^{r} \lambda_k x_{k+n}^{(t)} (\text{mod } m) = x_n^{(t-1)}$. Now let us calculate the uniform Markov measure of $(\Phi^{(t,s)})^{-1}A$;

$$\begin{aligned}
\mu_{\pi P}(\Phi^{(-t,-s)}(A)) &= \mu_{\pi P}(\bigcup_{i_{lt+s}^{(t)}, \ldots, i_{v+rt+s}^{(t)}} ({}_{u+lt+s}[i_{lt+s}^{(t)}, \ldots, i_{v+rt+s}^{(t)}]_{u+v+rt+s})) \\
&= \sum_{i_{lt+s}^{(t)}, \ldots, i_{v+rt+s}^{(t)}} \mu_{\pi P}({}_{u+lt+s}[i_{lt+s}^{(t)}, \ldots, i_{v+rt+s}^{(t)}]_{u+v+rt+s}) \\
&= m^{t(r-l)} \pi_{(i_{lt+s}^{(t)})} p_{(i_{lt+s}^{(t)}, i_{lt+s+1}^{(t)})} \ldots p_{(i_{v+rt+s-1}^{(t)}, i_{v+rt+s}^{(t)})} \\
&= \mu_{\pi P}(A).
\end{aligned}$$

The proof is completed.

In this section, the main result is the following theorem. This theorem contains an analogous result for the $\mathbb{Z}^2$-action generated by the shift and a linear CA.

Theorem 3 *Suppose that the Markov measure $\mu_{\pi P}$ is as in* (3)*. Let CA $T_{F[l,r]}$ be given as* (5) *and F be bipermutative with $l < 0 < r$, then $\mathbb{Z} \times \mathbb{N}$-actions $\Phi^{(s,t)}$ is strong mixing with respect to the uniform Markov measure $\mu_{\pi P}$.*

Proof. Let $A =_u [a_0^{(o)}, \ldots, a_v^{(o)}]_{u+v}$ and $B =_y [b_0, \ldots, b_z]_{y+z}$ be two cylinder sets. Then we can observe that

$$(T_{F[l,r]}^t \circ \sigma^s)^{-n} A = \bigcup_{a_{nlt}^{(n)}, \ldots, a_{v+nrt}^{(n)}} ({}_{u+n(tl+s)}[a_{n(tl+s)}^{(n)}, \ldots, a_{v+n(tl+s)}^{(n)}]_{u+v+n(rt+s)}),$$

where $F(x_{i+l}^{(n)}, \ldots, x_{i+r}^{(n)}) = \sum_{k=l}^{r} \lambda_k x_{i+k}^{(n)} (\text{mod } m) = x_i^{(n-1)}$. Thus from Lemma 5 for $n(tl+s) > y+z-u$ we have

$$\begin{aligned}
&\mu(B \cap (\Phi^{(s,t)})^{-n} A) = \\
&\mu(\bigcup_{x_{z+1},\ldots,x_{n(tl+s)-1}} \bigcup_{a^{(n)}_{n(tl+s)},\ldots,a^{(n)}_{v+n(tr+s)}} \times \\
&({}_y[b_0,\ldots,b_z,x_{z+1},\ldots,x_{n(tl+s)-1},a^{(n)}_{n(tl+s)},\ldots,a^{(n)}_{v+n(tr+s)}]_{y+u+v+n(rt+s)})) \\
= \ &\mu(B)(\sum_{a^{(n)}_{n(tl+s)},\ldots,a^{(n)}_{v+n(tr+s)}} \sum_{x_{z+1},\ldots,x_{n(tl+s)-1}} p_{(b_z,\ x_{z+1})} \cdots p_{(x_{n(tl+s)-1},\ a^{(n)}_{n(tl+s)})} \times \\
&p_{(a^{(n)}_{n(tl+s)},a^{(n)}_{n(tl+s)+1})} \cdots p_{(a^{(n)}_{n(tr+s)+v-1},a^{(n)}_{n(tr+s)+v})}) \\
= \ &\mu(B) \sum_{a^{(n)}_{n(tl+s)},\ldots,a^{(n)}_{v+n(tr+s)}} p^{(n(tl+s)+u-y-z)}_{(b_z,\ a^{(n)}_{n(tl+s)})} \times \\
&p_{(a^{(n)}_{n(tl+s)},a^{(n)}_{n(tl+s)+1})} \cdots p_{(a^{(n)}_{n(tr+s)+v-1},a^{(n)}_{n(tr+s)+v})} \\
= \ &\mu(B) \sum_{a^{(n)}_{n(tl+s)},\ldots,a^{(n)}_{v+n(tr+s)}} \pi_{a^{(n)}_{n(tl+s)}} p_{(a^{(n)}_{n(tl+s)},a^{(n)}_{n(tl+s)+1})} \cdots p_{(a^{(n)}_{n(tr+s)+v-1},a^{(n)}_{n(tr+s)+v})} \\
= \ &\mu(A)\mu(B),
\end{aligned}$$

where we know that $p^{(n(tl+s)+u-y-z)}_{(b_z,a^{(n)}_{n(tl+s)})} \to \pi_{a^{(n)}_{n(tl+s)}}$ as $n \to \infty$ (by putting P in terms of Jordan forms) and so we know that

$$\lim_{n\to\infty} \mu(A \cap (T^s_{F[l,r]} \circ \sigma^t)^{-n} B) = \mu(A)\mu(B).$$

4 Conclusions

In this paper we study strong mixing property of a certain class of linear CA with respect to uniform Markov measure induced by doubly stochastic matrix $P = (p_{(i,j)})$ such that $p_{(i,j)} = \frac{1}{m}$ for all pair (i,j). We prove that a 1-dimensional linear CA satisfying certain additional conditions preserves this uniform Markov measure and is strongly mixing with respect to any such measure. We also generalize an analogous result for the $\mathbb{Z}^2$-action generated by the shift and a 1-dimensional linear CA.

Are there any D-dimensional CA, strong mixing with respect to the other Markov measures?

If Φ is a strong mixing transformation on the probability space $(X, \mathcal{B}, \mu_{\pi P})$, then it is clear that it is necessary weak mixing (and thus also ergodic). So, from theorem 2, $\mathbb{Z} \times \mathbb{N}$-action Φ is both weak mixing and ergodic with respect to uniform Markov measure $\mu_{\pi P}$.

One can prove that the Markov symbolic dynamic system is k-mixing $(k \geq 1)$.

We think that our results will also give a possibility of proving certain mixing properties for a complete formal classification of invertible multi-dimensional CA defined on alphabets of composite cardinality (or the other finite rings) with respect to uniform Markov measure. In [3], Akın and Şiap have investigated invertible CA over the Galois rings. Thus, similar computations and explorations of CA's over different rings remain to be of interest.

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