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Note on the strong law of large numbers in a Hilbert space [1]

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Abstract

We study the Kolmogorov's strong law of large numbers for the sums of Hilbert valued random variables under the condition $E\|X_1\| < \infty$ and the weaker assumption that the random variables is only pairwise independent identically distributed. In addition, the strong law of large numbers for r-dimensional arrays is also obtained.

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1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = X_1 + \cdots + X_n$. Law of large numbers has always been of importance in probability. If $\{X_n, n \geq 1\}$ is independent, identically distributed random variables, the most famous strong law of large numbers was proved by Kolmogorov

$$E|X_1| < \infty, EX_1 = \mu, \quad \text{if and only if} \quad \frac{S_n}{n} \to \mu, \quad a.s. \tag{1}$$

When $\{X_n, n \geq 1\}$ is pairwise independent identically distributed random variables with finite mean μ, then the weak law of large numbers can be obtained (see, [4, Theorem 5.2.2]),

$$\frac{S_n}{n} \xrightarrow{P} \mu.$$

In the last two decades, the study of pairwise independent random variables has been of increasing and renewed interest and the main reason of this is perhaps the

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surprising simple proof by Eyemadi of Kolmogorov's strong law of large numbers. Etemadi [9] presented a proof for the strong law of large numbers of pairwise independent identically distributed random variables, which w as very elementary, in the sense that it did not use Kolmogorov's inequality.

Using Etemadi's idea, Csörgő et al [5, 6] derived a variant of Kolmogorov's strong law of large numbers and a convergence theorem for non-identically distributed pairwise independent random variables. In [2], Marcinkiewicz's strong law of large numbers for pairwise independent identically distributed random variables was showed under the assumption that the higher moment of X_1 exists. Wang and Bhaskara Rao [18] extend Jamison et al's [11] convergence theorem for weighted sums of independent identically distributed random variables to pairwise independent identically distributed case. Choi and Sung [3] also obtained a convergence theorem for weighted sums of pairwise independent identically distributed random variables and Kolmogorov's strong law of large numbers for pairwise independent identically distributed random variables from their result as a corollary. For the further development for the law of large numbers, the readers are referred to the survey paper of Sung [16].

It's also worth mentioning that there are some examples of pairwise independent identically distributed random variables which the central limit theorem fails, see [12, 1, 7, 8].

Many classical strong law of large numbers can be extended to the finite dimensional space, but some problems arise when we deal with Banach space valued random variables. However, it is not very difficult to generalize the law of large numbers for random variables to Hilbert space valued random variables, since in this case variance of the sum of independent random variables is equal to the sum of the variances. For example, if the variance of independent (uncorrelated) Hilbert valued random variable satisfies some conditions (see [17]), or we assume that the random sequence is independent identically distributed Banach valued random variables (see [13]), then the Kolmogorov's strong law of large numbers holds. Up to the knowledge of the authors, in the Hilbert case, the study on the Kolmogorov's strong law of large numbers for the pairwise independent identically distributed Hilbert valued random variables remains virgin. In the present note, the corresponding results are shown.

2 Main results

Let H be a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H.

Theorem 1 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent, identically distributed, Hilbert valued random variables with $E\|X_1\| < \infty$. Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = EX_1, \quad a.s.$$

Proof. Let

$$Y_i = X_i 1_{\{\|X_i\| \le i\}}, \quad S_n^* = \sum_{i=1}^n Y_i,$$

where 1_A denotes the indicator function of the event A. For $\varepsilon > 0$, let $k_n = [\alpha^n]$, $\alpha > 1$, where $[a]$ denotes the integral part of a. Let $\{e_k, k \ge 1\}$ be an orthonormal basis in the Hilbert space H. Then, by Parseval's identity and independence, we have

$$\begin{aligned} E\|S_{k_n}^* - ES_{k_n}^*\|^2 &= \sum_{i=1}^\infty E\langle S_{k_n}^* - ES_{k_n}^*, e_i\rangle^2 \\ &= \sum_{i=1}^\infty E\left(\sum_{j=1}^{k_n} \langle Y_j - EY_j, e_i\rangle\right)^2 \\ &= \sum_{i=1}^\infty \sum_{j=1}^{k_n} E\langle Y_j - EY_j, e_i\rangle^2 = \sum_{j=1}^{k_n} E\|Y_j - EY_j\|^2. \end{aligned} \tag{2}$$

Furthermore, by Markov's inequality and (2), we have

$$\begin{aligned} \sum_{n=1}^\infty P\left(\frac{\|S_{k_n}^* - ES_{k_n}^*\|}{k_n} > \varepsilon\right) &\le c\sum_{n=1}^\infty \frac{1}{k_n^2} E\|S_{k_n}^* - ES_{k_n}^*\|^2 \\ &= c\sum_{n=1}^\infty \frac{1}{k_n^2} \sum_{j=1}^{k_n} E\|Y_j - EY_j\|^2 \\ &\le c\sum_{n=1}^\infty \frac{1}{k_n^2} \sum_{j=1}^{k_n} E\|Y_j\|^2 \\ &\le c\sum_{j=1}^\infty \frac{1}{j^2} E\|Y_j\|^2 \\ &= c\sum_{j=1}^\infty \frac{1}{j^2} \sum_{i=0}^{j-1} E\left(\|X_1\|^2 1_{\{i \le \|X_1\| \le i+1\}}\right) \\ &\le c\sum_{i=0}^\infty \frac{1}{i+1} E\left(\|X_1\|^2 1_{\{i \le \|X_1\| \le i+1\}}\right) \\ &\le c\sum_{i=0}^\infty E\left(\|X_1\| 1_{\{i \le \|X_1\| \le i+1\}}\right) = cE\|X_1\| < \infty, \end{aligned} \tag{3}$$

where c is an unimportant positive constant which is allowed to change in the consecutive inequalities. From the above inequality and the Borel-Cantelli lemma, we have

$$\lim_{n\to\infty} \frac{S_{k_n}^* - ES_{k_n}^*}{k_n} = 0, \quad a.s.$$

Since $\lim_{n\to\infty} EY_n = EX_1$, then $\lim_{n\to\infty} EY_{k_n} = EX_1$. Thus

$$\begin{aligned}\left\|\frac{ES^*_{k_n}}{k_n} - EX_1\right\| =&\frac{1}{k_n}\left\|\sum_{i=1}^{k_n} E(Y_i - X_1)\right\| \\ \leq&\frac{1}{k_n}\sum_{i=1}^{k_n} E\left\|Y_i - X_1\right\| \to 0\end{aligned}$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} \frac{S^*_{k_n}}{k_n} = EX_1, \quad a.s. \tag{4}$$

Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty} P(Y_n \neq X_n) =&\sum_{n=1}^{\infty} P(\|X_n\| \geq n) \\ =&\sum_{n=1}^{\infty}\sum_{i=n}^{\infty} E1_{\{i\leq\|X_1\|\leq i+1\}} \\ =&\sum_{i=1}^{\infty} iE1_{\{i\leq\|X_1\|\leq i+1\}} \\ \leq&\sum_{i=1}^{\infty} E\left(\|X_1\|1_{\{i\leq\|X_1\|\leq i+1\}}\right) \leq E\|X_1\| < \infty.\end{aligned} \tag{5}$$

Hence by the Borel-Cantelli lemma,

$$\lim_{n\to\infty} \frac{S_{k_n}}{k_n} = EX_1, \quad a.s. \tag{6}$$

Let us observe that for any $n \in \mathbb{N}$, there exists $k(n) \in \mathbb{N}$, such that

$$k_{k(n)-1} = [\alpha^{k(n)-1}] < n \leq [\alpha^{k(n)}] = k_{k(n)}.$$

Therefore, we get

$$\begin{aligned}\left\|\frac{S_n}{n} - \frac{S_{k_{k(n)}}}{k_{k(n)}}\right\| =&\frac{1}{k_{k(n)}}\left\|\frac{k_{k(n)}S_n}{n} - S_{k_{k(n)}}\right\| \\ \leq&\frac{1}{k_{k(n)}}\left(\frac{k_{k(n)}}{n} - 1\right)\|S_{k_{k(n)}}\| + \frac{1}{n}\sum_{i=n+1}^{k_{k(n)}} \|X_i\| \\ \leq&(\alpha - 1)\frac{1}{k_{k(n)}}\|S_{k_{k(n)}}\| + (\alpha - 1)\left\|X_1\right\|\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{S_n}{n}-\frac{S_{k_{k(n)-1}}}{k_{k(n)-1}}\right\| =&\frac{1}{k_{k(n)-1}}\left\|\frac{k_{k(n)-1}S_n}{n}-S_{k_{k(n)-1}}\right\|\\ \le&\frac{1}{k_{k(n)-1}}\left(1-\frac{k_{k(n)-1}}{n}\right)\|S_{k_{k(n)-1}}\|+\frac{1}{n}\sum_{i=k_{k(n)-1}+1}^{n}\|X_i\|\\ \le&\left(1-\frac{1}{\alpha}\right)\frac{1}{k_{k(n)-1}}\|S_{k_{k(n)-1}}\|+\left(1-\frac{1}{\alpha}\right)\|X_1\|.\end{aligned}$$

Since α may be arbitrary close to 1, we conclude that

$$\lim_{n\to\infty}\frac{S_n}{n}=EX_1, \quad a.s.$$

Theorem 2 *Let $\{X_{mn}\}$ be a double sequence of pairwise independent, identically distributed, Hilbert valued random variables with $E\left(\|X_{11}\|\log^+\|X_{11}\|\right)<\infty$. Then*

$$\lim_{(m,n)\to\infty}\frac{S_{mn}}{mn}=EX_{11}, \quad a.s.$$

Proof. The proof of the case when either m or n is fixed and the other one goes to infinity can be obtained immediately from Theorem 1, hence it suffices to consider the case when both m and n tend to infinity. We shall follow the proof of Theorem 1. Define

$$Y_{ij}=X_{ij}1_{\{\|X_{ij}\|\le ij\}}, \quad S^*_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}Y_{ij}.$$

Let $k_m=[\alpha^m]$ and $l_n=[\alpha^n]$, $\alpha>1$. Let d_k be the number of divisor of k, i.e., the cardinality of $\{(i,j):ij=k\}$. Then the right hand side modified version of (3) gives us,

$$\begin{aligned}&c\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k_m^2l_n^2}\sum_{i=1}^{k_m}\sum_{j=1}^{l_n}E\|Y_{ij}-EY_{ij}\|^2\\ \le&c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E\|Y_{ij}\|^2}{(ij)^2}=c\sum_{k=1}^{\infty}\frac{d_k}{k^2}\sum_{i=0}^{k-1}E\left(\|X_{11}\|^21_{\{i\le\|X_{11}\|\le i+1\}}\right)\\ =&c\sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty}\frac{d_k}{k^2}\right)E\left(\|X_{11}\|^21_{\{i\le\|X_{11}\|\le i+1\}}\right)\\ \le&c\sum_{i=0}^{\infty}\frac{\log i}{i+1}E\left(\|X_{11}\|^21_{\{i\le\|X_{11}\|\le i+1\}}\right)\\ \le&c\sum_{i=0}^{\infty}E\left(\|X_{11}\|\log^+\|X_{11}\|1_{\{i\le\|X_{11}\|\le i+1\}}\right)\\ =&cE\left(\|X_{11}\|\log^+\|X_{11}\|\right)<\infty,\end{aligned} \tag{7}$$

where we use $\sum_{k=i+1}^{\infty}\frac{d_k}{k^2}=O\left(\frac{\log i}{i+1}\right)$ (c.f. Hardy and Wright [10] and Smythe [15]). Moreover, (5) adapted to this case becomes,

$$
\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P(Y_{ij}\neq X_{ij}) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P(\|X_{ij}\|\geq ij)\\
&= \sum_{k=1}^{\infty}d_kP(\|X_{11}\|\geq k)\\
&= \sum_{k=1}^{\infty}d_k\sum_{i=k}^{\infty}E1_{\{i\leq\|X_{11}\|\leq i+1\}}\\
&= \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i}d_k\right)E1_{\{i\leq\|X_{11}\|\leq i+1\}}\\
&\leq c\sum_{i=1}^{\infty}i\log iE1_{\{i\leq\|X_{11}\|\leq i+1\}}\\
&\leq cE\left(\|X_{11}\|\log^+\|X_{11}\|\right)<\infty,
\end{aligned}
\tag{8}
$$

where we use the fact $\sum_{k=1}^{n}d_k=O(n\log n)$. The rest of the proof follows similarly.

Remark 1 *Theorem 2 is called the strong law of large numbers for 2-dimensional arrays of random variables. The result can be generalized to r-dimensional array of random variables immediately and the sufficient condition becomes correspondingly*

$$E\left(\|X_{11}\|\left(\log^+\|X_{11}\|\right)^{r-1}\right)<\infty.$$

Remark 2 *For the i.i.d. case, Smythe [15] obtained, by martingale approach, the strong law of large numbers for r-dimensional arrays under the condition*

$$E\left(|X_{11}|\left(\log^+|X_{11}|\right)^{r-1}\right)<\infty.$$

For the case of B-valued i.i.d. random variables, Padgett and Taylor [14] proved the strong law of large numbers for r-dimensional arrays under the condition

$$E\left(\|X_{11}\|\left(\log^+\|X_{11}\|\right)^{r-1}\right)<\infty.$$

Thus our results generalize the know works from i.i.d. to pairwise independent identically distributed case.

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