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Chord length distribution functions for an isosceles trapezium [1]

Loredana Sorrenti

Abstract

In this paper we study the geometric probability of the length of a chord for an isosceles trapezium on a Buffon grid.

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1 Introduction

We denote by $\mathfrak{R}_D$ the Buffon grid consisting of parallel lines with constant distance D. In recent years many authors have studied the geometric probability of the length of a chord, determined by a straight line of the Buffon grid on a polygon. In [1] the problem is studied for a regular hexagon, in [6] for a rectangle, in [3] for a non regular triangle, in [7] for a regular pentagon and in [8] for a regular polygon. In [2] Andrei Duma and Sebastiano Rizzo considered as test body a rectangular trapezium, which is small with respect to $\mathfrak{R}_D$, that is, with the diagonal less or equal than D and they computed the probability that the test body intersects on one of the lines of $\mathfrak{R}_D$, a chord with length greater or equal than s. In order to solve the problem for an arbitrary trapezium it is necessary to consider a great number of cases. In this paper, we study the problem considering as test body an isosceles trapezium, which we denote by $\mathcal{T}$. Let $\text{diam}(\mathcal{T})$ be the diameter of $\mathcal{T}$. We say that $\mathcal{T}$ is small with respect to $\mathfrak{R}_D$, if $\text{diam}(\mathcal{T}) \leq D$. We compute the probability $p(s)$ that $\mathcal{T}$ intersects a line segment with length greater or equal than s, $0 \leq s \leq \text{diam}(T)$, on a line of $\mathfrak{R}_D$.

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2 General considerations

We denote by $\mathcal{T}$ the trapezium with sides a, b, c, heights h and δ and with diagonals d. We may assume $b \leq a$; then because of the existing symmetries, the angle α between the two sides a and c is less or equal than $\frac{\pi}{2}$.

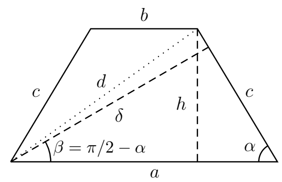

Figure 1:

There exist the following relations between the dimensions of $\mathcal{T}$: $c = \sqrt{h^2 + (\frac{a-b}{2})^2}$, $d = \sqrt{h^2 + \left(\frac{a+b}{2}\right)^2}$, $b \leq \delta \leq \min(a, d)$, $h \leq c < d$, $\mathrm{diam}(\mathcal{T}) = \max(a, d)$, $\min(a, b, c, d, \delta, h) = \min(b, h)$. We have to consider the following four situations:

(I) $\min(b, h) = b$, $\max(a, d) = a$;

(II) $\min(b, h) = b$, $\max(a, d) = d$;

(III) $\min(b, h) = h$, $\max(a, d) = a$;

(IV) $\min(b, h) = h$, $\max(a, d) = d$.

Each of the situation (I), (II), (III), (IV), includes different cases. In situation (I) we have to consider the following cases:

1. $b \leq \delta \leq h \leq c < d \leq a$,
2. $b \leq h \leq \delta \leq c < d \leq a$,
3. $b \leq h \leq c \leq \delta < d \leq a$.

In situation (II) we have to consider the following cases:

4. $b \leq \delta \leq h \leq a \leq c < d$,
5. $b \leq \delta \leq h \leq c \leq a < d$,
6. $b \leq \delta \leq a \leq h \leq c < d$,
7. $b \leq \delta \leq h \leq a \leq c < d$,
8. $b \leq h \leq \delta \leq a \leq c < d$,
9. $b \leq h \leq \delta \leq c \leq a < d$,
10. $b \leq h \leq c \leq \delta \leq a < d$.

In situation (III) we have to distinguish the following cases:

11. $h \leq b \leq \delta \leq c < d \leq a$,
12. $h \leq b \leq c \leq \delta < d \leq a$,
13. $h \leq c \leq b \leq \delta < d \leq a$.

Finally, in situation (IV), we have to distinguish the following cases:

14. $h \leq \leq \delta \leq c \leq a \leq d$,
15. $h \leq b \leq \delta \leq c \leq a \leq d$,
16. $h \leq b \leq \delta \leq a \leq c < d$,
17. $h \leq c \leq b \leq \delta \leq a \leq d$.

We note that for each one of the seventeen cases described above, it is necessary to consider six subcases, which are determined by the relations between s and a, b, c, d, δ, h. Then there are 102 subcases to distinguish. Taking into account that some of these cases coincide, we obtain 13 cases.
We consider as fundamental tile $\mathcal{F}$ of $\mathfrak{R}_D$, a boundless strip of width D and we denote by φ the angle between the direction of the straight lines of $\mathcal{F}$ and the oriented side a. Because of the existing symmetries we may assume $\varphi \in \left[0; \frac{\pi}{2}\right]$. We denote by $x_s(\varphi)$ the distance between two parallel segments with length s intersecting $\mathcal{T}$. If such two segments does not exist we set $x_s(\varphi) = 0$. In order to compute the desired probability $p(s)$, we will use the well known Stoka's formula.

$$p(s) = \frac{\int_0^{\frac{\pi}{2}} x_s(\varphi) d\varphi}{\int_0^{\frac{\pi}{2}} D d\varphi} = \frac{2}{\pi D} \int_0^{\frac{\pi}{2}} x_s(\varphi) d\varphi. \tag{1}$$

3 The case $\mathbf{s} \leq \min(\mathbf{b}, \mathbf{h})$

We note that this case is a common subcase of 1,2,3,4,. . . ,17. If $\varphi \in [0, \alpha]$ (see figure 2), then $x_s(\varphi)$ assumes the following form:

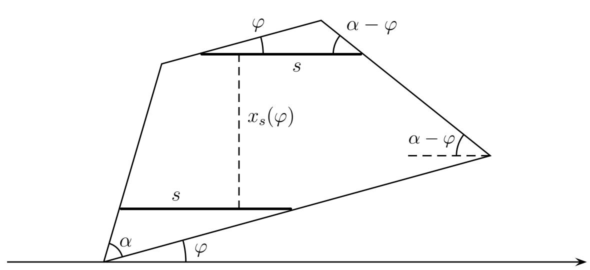

Figure 2:

$$x_s(\varphi) = a\sin\varphi + c\sin(\alpha-\varphi) - \frac{s\sin\varphi\sin(\varphi+\alpha)}{\sin\alpha} - \frac{s\sin\varphi\sin(\alpha-\varphi)}{\sin\alpha}.$$

Then

$$(2) \qquad x_s(\varphi) = a\sin\varphi + c\sin(\alpha-\varphi) - s\sin 2\varphi,$$

$$(2') \qquad \int_0^\alpha x_s(\varphi)d\varphi = (a+c)(1-\cos\alpha) - \frac{1}{2}(1-\cos 2\alpha).$$

If $\varphi \in [\alpha; \frac{\pi}{2}]$, then (see figure 3) $x_s(\varphi)$ assumes the following form:

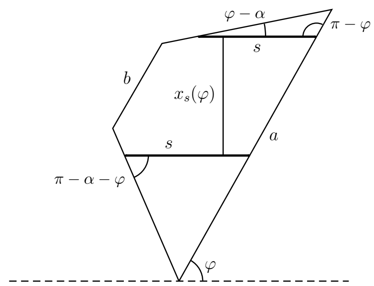

Figure 3:

$$(3) \qquad x_s(\varphi) = a\sin\varphi - \frac{s\sin(\alpha+\varphi)\sin\varphi}{\sin\alpha} - \frac{s\sin(\alpha-\varphi)\sin\varphi}{\sin\alpha},$$

$$(3') \qquad \int_\alpha^{\frac{\pi}{2}} x_s(\varphi)d\varphi = a\cos\alpha - s\left(\frac{\pi}{2}-\alpha\right)\cot\alpha - \frac{s}{2}\sin 2\alpha.$$

It follows from (2′) and (3′) that:

$$\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi = a + c(1-\cos\alpha) - s\left(\frac{\pi}{2}-\alpha\right)\cot\alpha - \frac{s}{2}(1-\cos 2\alpha + sin 2\alpha).$$

We note that $2a + 2c(1-\cos\alpha) = a + 2c + a - 2c\cos\alpha = a + 2c + b$ is the perimeter $\text{per}(\mathcal{T})$ of $\mathcal{T}$. The it follows from formula (1), that:

$$(4) \qquad p(s) = \frac{1}{\pi D}[\text{per}(\mathcal{T}) - s((\pi-2\alpha)\cot\alpha + 1 + \sin 2\alpha - \cos 2\alpha)].$$

Remark 1 If $s = 0$, we obtain from 4, the following

$$p(0) = \frac{\text{per}(\mathcal{T})}{\pi D},$$

which is a classical result.

Remark 2 If $\alpha = \frac{\pi}{2}$, it follows from 4, the following

$$p(s) = \frac{\text{per}(\mathcal{T}) - 2s}{\pi D},$$

which is M. Pettineo's formula for rectangle [6].

4 The case $\mathbf{b} \leq \mathbf{s} \leq \min(\delta, \mathbf{h})$

We note that this case is a common subcase of 1,2,3,4,...,10.
We denote by $\varphi_0 \in [0, \alpha[$ the angle defined as follows:

$$l \sin \alpha = s \sin(\alpha - \varphi_0).$$

If $\varphi \in [0, \varphi_0[$, with the notations of figure 4, the following linear equations:

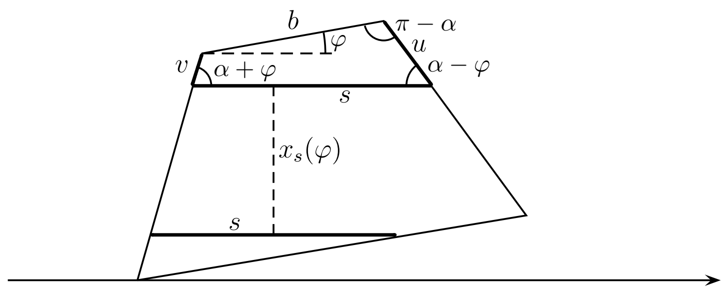

Figure 4:

$$u \sin(\alpha - \varphi) - v \sin(\alpha + \varphi) = b \sin \varphi$$
$$u \cos(\alpha - \varphi) + v \cos(\alpha + \varphi) = s - b \cos \varphi.$$

By solving the system consisting of the two linear equations above, we obtain

$$u = \frac{s \sin(\alpha + \varphi) - b \cos \alpha}{\sin 2\alpha}.$$

If $\varphi \in [0, \varphi_0[$, $x_s(\varphi)$ assumes the following form:

$$x_s(\varphi) = a \sin \varphi + c \sin(\alpha - \varphi) - \frac{s \sin \varphi \sin(\varphi + \alpha)}{\sin \alpha} - u \sin(\alpha - \varphi).$$

Then, we obtain:

$$(5) \qquad x_s(\varphi) = a \sin \varphi + c \sin(\alpha - \varphi) - \frac{b \sin(\alpha - \varphi)}{2 \sin \alpha} - \frac{s}{2 \sin 2\alpha} + \frac{1}{2} s \cos(2\varphi + \alpha) - \frac{1}{2} s \frac{\cos 2\varphi}{\sin 2\alpha}.$$

It follows that

$$(5') \int_0^\alpha x_s(\varphi d\varphi) = a(1 - \cos \varphi_0) + \frac{1}{2} b \left(\frac{\cos(\alpha - \varphi_0)}{\sin \alpha} - \cot \alpha \right) + $$
$$+ c \left(\cos(\alpha - \varphi_0) - \cos \alpha \right) - \frac{s \varphi_0}{2 \sin 2\alpha} +$$
$$+ \frac{s}{4} \left(\sin(2\varphi_0 + \alpha) - \sin \alpha - \frac{\sin 2\varphi_0}{\sin 2\alpha} \right).$$

In order to compute $x_s(\varphi)$ when $\varphi \in [\varphi_0, \alpha[$, we use formula (2) and we obtain the following:

$$(2'') \int_{\varphi_0}^{\alpha} x_s(\varphi d\varphi) = a(\cos\varphi_0 - \cos\alpha) + c(1 - \cos(\alpha - \varphi_0)) + \frac{s}{2}(cos 2\alpha - \cos 2\varphi_0).$$

Finally, it follows from (5′), (2′′) and (3′), that

$$\int_0^{\frac{\pi}{2}} x_s(\varphi d\varphi) = a + \frac{b}{2}\left(\frac{\cos(\alpha-\varphi_0)}{\sin\alpha} - \cot\alpha\right) + c(1-\cos\alpha) + -\frac{s}{2}\left((\pi - 2\alpha)\cot\alpha + \sin 2\alpha + + \frac{\varphi_0}{\sin 2\alpha} + \frac{1}{2}\sin\alpha - \frac{1}{2}\sin(2\varphi_0 + \alpha) + \frac{\sin 2\varphi_0}{2\sin 2\alpha} + \cos 2\varphi_0 - \cos 2\alpha\right),$$

and then

$$(6) \qquad p(s) = \frac{1}{\pi D}\left[2a + b\left(\frac{\cos(\alpha-\varphi_0)}{\sin\alpha} - cot\alpha\right) + 2c(1-\cos\alpha) + \sin 2\alpha - s(\pi - 2\alpha)\cot\alpha + \frac{\varphi_0}{\sin 2\alpha} + \frac{1}{2}\sin\alpha - \frac{1}{2}\sin(2\varphi_0+\alpha) + + \frac{\sin 2\varphi_0}{2\sin 2\alpha} + \cos 2\varphi_0 - \cos 2\alpha\right].$$

Remark 3 We note that if $b = s \le \min(c, \delta, h)$, then $\varphi_0 = 0$, and formula (6) coincides with formula (4).

5 The case $\mathbf{h} \le \mathbf{s} \le \min(\mathbf{b}, \mathbf{c})$

We note that this case is a common subcase of 11,12,13,14,. . . ,17.
We denote by $\varphi_1 \in [\alpha, \frac{\pi}{2}[$ the angle determined by the following relation:

$$s \sin\varphi_1 = h.$$

If $\varphi \in [0, \alpha[$ then $x_s(\varphi)$ is obtained from (2) and $\int_0^\alpha x_s(\varphi)d\varphi$ returns formula (2′). If $\varphi \in [\alpha, \varphi_1]$, then $x_s(\varphi)$ is obtained from (3) and it follows that:

$$(3'') \qquad \int_0^{\varphi_1} x_s(\varphi)d\varphi = a(\cos\alpha - \cos(\varphi_1)) + s(\varphi_1 - \alpha)\cot\alpha + + \frac{s}{2}(\sin 2\varphi_1 - \sin 2\alpha).$$

Moreover note that when $\varphi \in]\varphi_1, \frac{\pi}{2}]$, then $x_s(\varphi) = 0$. Then:

$$\int_0^{\frac{\pi}{2}} x_s(\varphi) d\varphi = a(1 - \cos\varphi_1) + c(1 - \cos\alpha) - s(\varphi_1 - \alpha)\cot\alpha + \\ -\frac{s}{2}(1 + \sin\alpha - \cos(2\alpha) - \sin 2\varphi_1),$$

and we obtain:

$$p(s) = \frac{1}{\pi D}[2a(1 - \cos\varphi_1) + 2c(1 - \cos\alpha) - 2s(\varphi_1 - \alpha)\cot\alpha + \\ -s(1 + \sin 2\alpha - \cos 2\alpha - \sin 2\varphi_1)]. \tag{7}$$

Remark 4 If $h = b = s$ we obtain φ_0=0, $\varphi_1 = \frac{\pi}{2}$ and formulas (6) and (7) returns the same result, that is the following:

$$p(b) = p(h) = \frac{1}{\pi D}[2a + 2c(1 - \cos\alpha) - 2b\left(\frac{\pi}{2} - \alpha\right)\cot\alpha + \\ -b(1 + \sin 2\alpha - \cos 2\alpha)].$$

6 The case $b \leq \delta \leq s \leq \min(a, h)$

We note that this case is a common subcase of 1,4,5,6,7.
In this case, in order to compute $x_s(\varphi)$ we need the angle φ_0, defined in section 4, and another angle φ_2, defined by the following relation

$$s\sin(\alpha + \varphi_2) = a\sin\alpha.$$

It is easy to see that $0 \leq \varphi_2 \leq \frac{\pi}{2} - \alpha \leq \pi - 2\alpha - \varphi_2 \leq \varphi_0 \leq \alpha \leq \frac{\pi}{2}$. Moreover

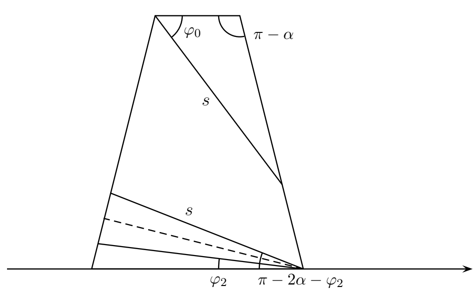

Figure 5:

$x_s(\varphi) = 0$ if $\varphi \in]\varphi_2, \pi - 2\alpha - \varphi_2[$. When $\varphi \in [0, \varphi_2] \cup [\pi - 2\alpha - \varphi_2, \varphi_0]$, $x_s(\varphi)$ is obtained from (5), then

$$(5'')\int_0^{\varphi_2} x_s(\varphi)d\varphi = a(1-\cos\varphi_2)+\left(\frac{b}{\sin 2\alpha}+c\right)(\cos(\alpha-\varphi_2)+ -\cos\alpha)-s\frac{\varphi_2}{2\sin 2\alpha}+\frac{s}{4}\left(\sin(2\varphi_2+\alpha)-\sin\alpha+ -\frac{\sin 2\varphi_2}{\sin 2\alpha}\right),$$

$$(5''')\int_{\pi-2\alpha-\varphi_2}^{\varphi_0} x_s(\varphi)d\varphi = a(-\cos(2\alpha+\varphi_2)-\cos\varphi_0)+\left(\frac{b}{2\sin\alpha}+c\right)\cdot +(\cos(\alpha-\varphi_0)+\cos(3\alpha+\varphi_2))+ -s\frac{2\alpha+\varphi_0+\varphi_2-\pi}{2\sin 2\alpha}+\frac{s}{4}\left(\sin(2\varphi_0+\alpha)+ \sin(3\alpha+2\varphi_2)-\frac{\sin 2\varphi_0+\sin(4\alpha+2\varphi_2)}{\sin 2\alpha}\right).$$

Then we obtain:

$$\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi = a(1-\cos\varphi_2-\cos(2\alpha+\varphi_2))+\frac{b}{\sin 2\alpha}(\cos(\alpha-\varphi_0)+ +\cos(3\alpha+\varphi_2)-\cos\alpha+\cos(\alpha-\varphi_2))+c\,(1-\cos\alpha+ +\cos(\alpha-\varphi_2)+\cos(3\alpha+\varphi_2))+-\frac{s}{2}\left[\frac{\varphi_2}{\sin 2\alpha}+\frac{1}{2}\sin\alpha+ -\frac{1}{2}\sin(2\varphi_2+\alpha)+\frac{\sin 2\varphi_2}{2\sin 2\alpha}+\frac{2\alpha+\varphi_0+2\varphi_2-\pi}{\sin 2\alpha}+ -\frac{1}{2}\sin(2\varphi_0+\alpha)-\frac{1}{2}\sin(3\alpha+2\varphi_2)-\cos 2\alpha+ +\frac{\sin 2\varphi_0+\sin 2\varphi_2+\sin(4\alpha+2\varphi_2)}{2\sin 2\alpha}+\cos(2\varphi_0)+ +(\pi-2\alpha)\cot\alpha+\sin 2\alpha\right] := A(\varphi_0,\varphi_2,s),$$

and then it follows that

$$p(s)=\frac{2A(\varphi_0,\varphi_2,s)}{\pi D}. \tag{8}$$

Remark 5 If $s=\delta$ we obtain $\alpha+\varphi_2=\frac{\pi}{2}$ and formulas (6) and (8) gives the same result for $p(\delta)$.

7 The case $\max(\mathbf{b},\mathbf{h})\le\mathbf{s}\le\min(\delta,\mathbf{c})$

We note that this case is a common subcase of 2,3,8,9,10,11,12,14,15,16.
Let φ_0 and φ_1 be the two angles introduced in the precedent sections. In this case, the following relation holds:

$$0\le\varphi_0\le\alpha\le\varphi_1\le\frac{\pi}{2}.$$

We note that if $\varphi \in \left]\varphi_1, \frac{\pi}{2}\right]$, then $x_s(\varphi) = 0$. When $\varphi \in [0, \varphi_0[, [\varphi_0, \alpha[, [\alpha, \varphi]$, we can compute $x_s(\varphi)$ using formulas (5), (2) and (3), respectively. Then $p(s)$ is obtained by adding formulas $(5'), (2'')$ and $(3'')$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= a(1-\cos\varphi_1) + \frac{b}{2}\left(\frac{\cos(\alpha-\varphi_0)}{\sin\alpha} - \cot\alpha\right) + \\ &+c(1-\cos\alpha) - \frac{s}{2}\left(\frac{\varphi_0}{\sin 2\alpha} + 2(\varphi_1-\alpha)\cot\alpha - \sin 2\varphi_1 + \right.\\ &+\sin 2\alpha - \cos 2\alpha + \cos 2\varphi_0 + \frac{1}{2}\sin\alpha - \frac{1}{2}\sin(2\varphi_0+\alpha) + \\ &\left.+\frac{\sin\varphi_0}{2\sin\alpha}\right) := B(\varphi_0, \varphi_1, s).\end{aligned}$$

Then, the required probability is:

$$p(s) = \frac{2B(\varphi_0, \varphi_1, s)}{\pi D}. \tag{9}$$

Remark 6 If $b \leq h = s = \delta \leq c$, we obtain $\varphi_2 = \frac{\pi}{2} - \alpha$ and $\varphi_1 = \frac{\pi}{2}$. In this special case formulas (8) and (9) gives the same result, since $A(\varphi_0, \frac{\pi}{2} - \alpha, h) = B(\varphi_0, \frac{\pi}{2}, \alpha)$.

8 The case $\mathbf{h \leq c \leq s \leq b}$

We note that this is a common subcase of 13 and 17. In order to obtain $x_s(\varphi)$ we need the angle φ_1 defined by $s\sin\varphi_1 = h$. Since $s \leq c$, then $\varphi \leq \alpha$ and then $x_s(\varphi = 0)$, if $\varphi \in \left]\varphi, \frac{\pi}{2}\right]$. If $\varphi \in [0, \varphi_1]$, we can use formula (2). Then

$$(2''') \int_0^{\varphi_1} x_s(\varphi)d\varphi = (a+c)(1-\cos\varphi_1) - \frac{s}{2}(1-\cos 2\varphi_1),$$

then

$$p(s) = \frac{2(a+c)(1-\cos\varphi_1) - s(1-\cos 2\varphi_1)}{\pi D}. \tag{10}$$

9 The case $\max(\mathbf{h}, \delta) \leq \delta \leq \min(\mathbf{a}, \mathbf{c})$

This case is a common subcase of 1,2,4,5,7,8,9,11,14,16. Let us consider the angles φ_0, φ_1 and φ_2, defined in the precedent sections (see figure 6). In this case, the following relation between these angles holds:

$$0 \leq \varphi_2 \leq \pi - 2\alpha - \varphi_2 \leq \varphi_0 \leq \alpha \leq \varphi_1 \leq \frac{\pi}{2}.$$

We note that, if $\varphi \in]\varphi_2, \pi-2\alpha-\varphi_2[\cup \left]\varphi_1, \frac{\pi}{2}\right]$, then $x_s(\varphi) = 0$. If φ belongs to $[0, \varphi_2]$, $[\pi - 2\alpha[$, $[\varphi_0, \alpha[$ and $[\alpha, \varphi_1]$, we obtain $x_s(\varphi)$ using (5), (5), (2) and (3), respectively.

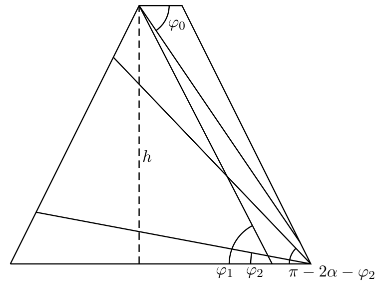

Figure 6:

Then $\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi$ is obtained by adding (5″), (5‴), (2‴) and (3″).

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi \quad = \quad & a(1-\cos\varphi_1-\cos\varphi_2-\cos(2\alpha+\varphi_2))+\\
&+\frac{b}{2\sin\alpha}(\cos(\alpha-\varphi_2)-\cos\alpha+\cos(\alpha-\varphi_2)+\\
&+\cos(3\alpha-\varphi_2))-\frac{s}{2}\,(\cos 2\varphi_0-\cos 2\alpha+\sin 2\alpha+\\
&-\sin 2\varphi_1-\frac{1}{2}\sin(2\varphi_2+\alpha)+\frac{1}{2}\sin\alpha-\frac{1}{2}\sin(2\varphi_0+\alpha)+\\
&-\frac{1}{2}\sin(3\alpha+2\varphi_2)+2(\varphi-\alpha)\cot\alpha+\frac{2\alpha+\varphi_0+\varphi_2-\pi}{\sin 2\alpha}+\\
&-\left.\frac{\sin 2\varphi_0+\sin 2\varphi_2+\sin(3\alpha+2\varphi_2)}{2\sin 2\alpha}\right):=\\
&:=C(\varphi_0,\varphi_1,\varphi_2,s).
\end{aligned}$$

By applying (1) we obtain

$$p(s) \quad = \quad \frac{2C(\varphi_0,\varphi_1,\varphi_2,s)}{\pi D}. \tag{11}$$

Remark 7 If $h \leq \delta = s \leq c$, we obtain $\varphi_2+\alpha=\frac{\pi}{2}$ and $B(\varphi_0,\varphi_1,\delta) = C\left(\varphi_0,\varphi_1,\frac{\pi}{2}-\alpha,\delta\right)$. Then, in this special case, formulas (9) and (11) coincide.

10 The case $\max(\mathbf{b},\mathbf{c}) \leq \mathbf{s} \leq \delta$

We note that this case is a common subcase of 3,10,12,13,15 and 17. In this case the following relations holds:

$$0 \leq \varphi_0 \leq \varphi_1 \leq \alpha.$$

If $\varphi \in \left]\varphi_1,\frac{\pi}{2}\right]$, then $x_s(\varphi)=0$. If $\varphi\in[0,\varphi_0[$ then $x_s(\varphi)$ is obtained using formula (5), and if $\varphi\in[\varphi_0,\varphi_1]$, then $x_s(\varphi)$ is obtained using formula (2). Then $\int_0^{\varphi_0} x_s(\varphi)d\varphi$

is given by formula (5′) and

$$(2^{IV})\int_{\varphi_0}^{\varphi_1} x_s(\varphi)d\varphi = a(\cos\varphi_0-\cos\varphi_1)+c(\cos(\alpha-\varphi_1)-\cos(\alpha-\varphi_0))+ +\frac{s}{2}(\cos 2\varphi_1-\cos 2\varphi_0).$$

It follows that

$$\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi = a(1-\cos\varphi_1)+\frac{1}{2}b\left(\frac{\cos(\alpha-\varphi_0)}{\sin\alpha}-\cot\alpha\right)+ +c(\cos(\alpha-\varphi_1)-\cos\alpha)-s\frac{\varphi_0}{2\sin 2\alpha}+\frac{s}{2}(\cos 2\varphi_1+ -\cos 2\varphi_0)+\frac{s}{4}\left(\sin(2\varphi_0+\alpha)-\sin\alpha-\frac{\sin 2\varphi_0}{\sin 2\alpha}\right):= := D(\varphi_0,\varphi_1,s).$$

Then

$$(12)\qquad p(s) = \frac{2D(\varphi_0,\varphi_1,s)}{\pi D}.$$

Remark 8 If $b \le c = s \le \delta$, then $\alpha = \varphi_1$ and $B(\varphi_0,\alpha,c) = D(\varphi_0,\alpha,c)$. Then formulas (9) and (12) return the same value for $p(c)$.

11 The case $\mathbf{b \le \delta \le a \le s \le h}$

This is a subcase of 6. We note that, in this case, since $s \le c$ we obtain $\varphi_0 < \alpha$. We consider the angle $\varphi_3 \le \varphi_0$ determined by the following relation (see figure 7)

$$s\sin(\alpha+\varphi_3) = a\sin\alpha.$$

We note that if $\varphi \in [0,\varphi_3[$, then $x_s(\varphi)=0$. If φ belongs to $[\varphi_3,\varphi_0[$, $[\varphi_0,\alpha[$, $\left[\alpha,\frac{\pi}{2}\right]$, then in order to compute $x_s(\varphi)$ we use (5), (2), (3) respectively. In particular,

$$(5^{IV})\int_{\varphi_3}^{\varphi_0} x_s(\varphi)d\varphi = a(\cos\varphi_3-\cos\varphi_0)+\frac{b}{2\sin\alpha}(\cos(\alpha-\varphi_0)-\cos(\alpha-\varphi_3)+ +c(\cos(\alpha-\varphi_0)-\cos(\alpha-\varphi_3))-\frac{s}{2}\cdot\frac{\varphi_0-\varphi_3}{\sin 2\alpha}+ +\frac{s}{4}(\sin(2\varphi_0+\alpha)-\sin(2\varphi_3+\alpha)-\frac{\sin 2\varphi_0-\sin 2\varphi_3}{\sin 2\alpha})\Big).$$

In order to compute $\int_{\varphi_0}^{\alpha} x_s(\varphi)d\varphi$, we use (2) and we obtain (2″) and in order to compute $\int_{\alpha}^{\frac{\pi}{2}} x_s(\varphi)d\varphi$ we use (3) and we obtain (3′). Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= a\cos\varphi_3 + \frac{b}{\sin 2\alpha}(\cos(\alpha-2\varphi_0)-\cos(\alpha-\varphi_3)) + \\ &+c(1-\cos(\alpha-\varphi_3)) - s\left(\left(\frac{\pi}{2}-\alpha\right)\cot\alpha + \frac{\varphi_0-\varphi_3}{2\sin\alpha}\right) + \\ &-\frac{s}{4\sin 2\alpha}(\sin 2\varphi_0 - \sin 2\varphi_3) - \frac{s}{2}(\sin 2\alpha + \cos 2\varphi_0 + \\ &-\cos 2\alpha + \frac{1}{2}\sin(2\varphi_3+\alpha)) := E(\varphi_0,\varphi_3).\end{aligned}$$

Then in this case we obtain the following formula for computing the desired probability:

$$p(s) = \frac{2E(\varphi_0,\varphi_3)}{\pi D}. \tag{13}$$

Remark 9 If $b \leq \delta \leq a = s \leq h$, we obtain $\varphi_2 = 0$ and $\varphi_3 = \pi - 2\alpha$. Then, in this case, formulas (8) and (13) gives the same result for $p(a)$.

12 The case $\max(\mathbf{c}, \delta, \mathbf{h}) \leq \mathbf{s} \leq \min(\mathbf{a}, \mathbf{d})$

This is a subcase of 1,2,3,8,9,10,11,12,13,14,15,17. The relation between the angles φ_0, φ_1, φ_2 are the following:

$$0 \leq \varphi_2 \leq \pi - 2\alpha - \varphi_2 \leq \varphi_0 \leq \varphi_1 \leq \alpha.$$

If $\varphi \in]\varphi_2, \pi-2\alpha-\varphi_2[\cup]\varphi_1, \frac{\pi}{2}]$, then $x_s(\varphi) = 0$. If φ belongs to $[0,\varphi_2]$, $[\pi-2\alpha-\varphi_2,\varphi_0[$, $[\varphi_0,\varphi_1]$, then we compute $x_s(\varphi)$ using (5), (5) and (2) respectively, and we obtain (5″), (5‴) and (2^{IV}). Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= a(1-\cos\varphi_1-\cos\varphi_2-\cos(2\alpha+\varphi_2)) + \\ &+\frac{b}{\sin\alpha}(\cos(\alpha-\varphi_0)+\cos(\alpha-\varphi_2)-\cos\alpha + \\ &\cos(3\alpha+\varphi_2)) + c(\cos(\alpha-\varphi_2)-\cos\alpha+\cos(3\alpha+\varphi_2) + \\ &+\cos(\alpha-\varphi_1)) - \frac{s}{2\sin 2\alpha}(2\alpha+\varphi_0+2\varphi_2-\pi) + \\ &+\frac{s}{2}(\cos 2\varphi_1 - \cos 2\varphi_0) + \frac{s}{4}\left[\sin(2\varphi_0+\alpha) - \sin\alpha + \right. \\ &+\sin(2\varphi_2+\alpha) + \sin(3\alpha+2\varphi_2) + \\ &\left.-\frac{\sin 2\varphi_0 + \sin 2\varphi_2 + \sin(4\alpha+2\varphi_2)}{\sin 2\alpha}\right] =: F(\varphi_0,\varphi_1,\varphi_2;s).\end{aligned}$$

Then, in this case we obtain the following formula for the desired probability

$$p(s) = \frac{2F(\varphi_0,\varphi_1,\varphi_2)}{\pi D}. \tag{14}$$

13 The case $\max(\mathbf{a},\mathbf{h}) \leq s \leq c$

This case is a subcase of 4,6,7,8,16. It is easy to see (figure 7) that the following

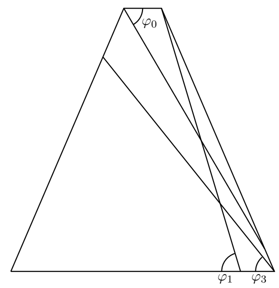

Figure 7:

relations holds:

$$\varphi_3 \leq \varphi_0 \leq \alpha \leq \varphi_1.$$

If $\varphi \in [0,\varphi_3[\cup]\varphi_1,\frac{\pi}{2}]$, then $x_s(\varphi)=0$. If φ belongs to $[\varphi_3,\varphi_0[$, $[\varphi_0,\alpha[$, $[\alpha,\varphi_1]$ we compute $x_s(\varphi)$ using (5), (2), (3), respectively and we obtain (5^{IV}), $(2'')$ and $(3'')$. Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= a(\cos\varphi_3-\cos\alpha)+\frac{b}{2\sin\alpha}(\cos(\alpha-\varphi_0)+\\ &-\cos(\alpha-\varphi_3)+\cos\alpha-\cos\varphi_1)+c(1-\cos(\alpha-\varphi_3))+\\ &-\frac{s}{2}\left(\frac{\varphi_0-\varphi_1}{\sin 2\alpha}+2(\varphi_1-\alpha)\cot\alpha\right)+\frac{s}{2}(\sin 2\varphi_1+\\ &-\sin 2\alpha+\cos 2\alpha-\cos 2\varphi_0)+\frac{s}{4}\left[\sin(2\varphi_0+\alpha)+\right.\\ &\left.-\sin(2\varphi_3+\alpha)+\frac{\sin 2\varphi_3-\sin 2\varphi_0}{\sin 2\alpha}\right] =: G(\varphi_0,\varphi_1,\varphi_3).\end{aligned}$$

Then

$$p(s)=\frac{2G(\varphi_0,\varphi_1,\varphi_3)}{\pi D}. \tag{15}$$

Remark 10 If $h \leq a = s = c$ then $\varphi_1 = a$, $\varphi_2 = 0$, $\varphi_3 = \pi - 2\alpha$ and $F(\varphi_0,\alpha,0) = G(\varphi_0,\alpha,\pi-2\alpha)$. Then, in this case, (14) and (15) return the same formula for $p(a)=p(c)$.

14 The case $\mathbf{d} \le \mathbf{s} \le \mathbf{a}$

This a subcase of 1,2,3,11, 12,13. We note that if $\varphi \in \left]\varphi_3, \frac{\pi}{2}\right]$, then $x_s(\varphi) = 0$. If $\varphi \in [0, \varphi_3]$, then we can compute $x_s(\varphi)$ using (5). Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= \int_0^{\varphi_3} x_s(\varphi)d\varphi = a(1-\cos\varphi_3) + \frac{1}{2}b\left(\frac{\cos(\alpha-\varphi_3)}{\sin\alpha} - \cot\alpha\right) \\ &\quad + c(\cos(\alpha-\varphi_3) - \cos\alpha - s\frac{\varphi_3}{2\sin 2\alpha}) + \frac{s}{4}\,(\sin(2\varphi_3+\alpha)+ \\ &\quad -\sin\alpha - \frac{\sin 2\varphi_3}{\sin 2\alpha}\right) =: H(\varphi_3; s).\end{aligned}$$

Then

$$p(s) = \frac{2H(\varphi_3)}{\pi D}. \tag{16}$$

15 The case $\max(\mathbf{a}, \mathbf{c}) \le s \le d$

This is a subcase of 4,5,6,7,8,9,10,14,15,16,17. In this case, it is easy to see (figure 8) that the following relations holds: $\varphi_3 \le \varphi_0 \le \varphi_1 \le \alpha$. We note that if $\varphi \in$

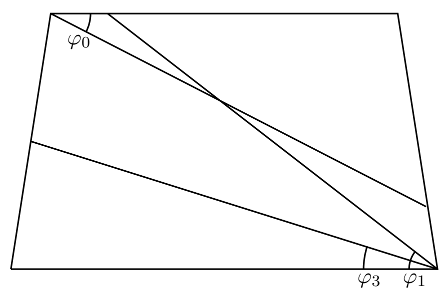

Figure 8:

$[0, \varphi_3[\cup]\varphi_1, \frac{\pi}{2}[$, then $x_s(\varphi) = 0$. If $\varphi \in [\varphi_3, \varphi_0[$ we compute $x_s(\varphi)$ using (5) and then $\int_{\varphi_3}^{\varphi_0} x_s(\varphi)d\varphi$ is given by (5^{IV}). If $\varphi \in [\varphi_0, \varphi_1]$ we compute $x_s(\varphi)$ using (2) and then $\int_{\varphi_3}^{\varphi_0} x_s(\varphi)d\varphi$ is given by (2^{IV}). Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x_s(\varphi)d\varphi &= \int_{\varphi_3}^{\varphi_0} x_s(\varphi)d\varphi + \int_{\varphi_0}^{\varphi_1} x_s(\varphi)d\varphi = \\ &= a(\cos(\varphi_3 - \cos\varphi_1)) + \frac{b}{\sin 2\alpha}\,(\cos(\alpha-\varphi_0) - \cos(\alpha-\varphi_3)+ \\ &\quad + c(\cos(\alpha-\varphi_1) - \cos(\alpha-\varphi_3)) - s\frac{\varphi_0-\varphi_3}{2\sin 2\alpha} + \\ &\quad + \frac{s}{2}(\cos 2\varphi_1 - \cos 2\varphi_0) + \frac{s}{4}(\sin(2\varphi_0+\alpha) - \sin(2\varphi_3+\alpha) + \\ &\quad - \frac{\sin 2\varphi_0 - \sin 2\varphi_3}{\sin 2\alpha}\right) =: I(\varphi_0, \varphi_1, \varphi_3; s).\end{aligned}$$

It follows that

$$p(s) = \frac{2I(\varphi_0, \varphi_1, \varphi_3)}{\pi D}. \tag{17}$$

Remark 11 In 11 and 14, the following special case may be considered: $c \leq s = a \leq d$. If this condition holds, then $\varphi_2 = 0$ and $\varphi_3 = \pi - 2\alpha$. Then, in this case $F(\varphi_0, \varphi_1, 0) = I(\varphi_0, \varphi_1, \pi - 2\alpha)$ and formulas (14) and (17) coincide.
The special case $a = d$ is common to (13) and (14). Since $s \leq a$, then in (13),

$$\lim_{s \to a^-} \varphi_3(s) = 0.$$

In (14), since $s \leq d$,

$$\lim_{s \to d^-} \varphi_3(s) = \beta = \lim_{s \to d^-} \varphi_1(s) = \lim_{s \to d^-} \varphi_0(s),$$

where β is the angle between a and d. It follows that

$$\lim_{s \to a^-} H(\varphi_3) = 0 = \lim_{s \to d^-} I(\varphi_0, \varphi_1, \varphi_3).$$

Then

$$H(0) = 0 = I(\beta, \beta, \beta).$$

16 The distribution function of the chord in $\mathcal{T}$

As an application of the results obtained in the precedent sections, we are able to compute the *distribution function of the chord in* $\mathcal{T}$. Let us consider the function $F : [0, \max(a, d)] \to [0, +\infty[$, which assigns to each s, $0 \leq s \leq \max(a, d)$ the conditioned probability that a straight line intersecting $\mathcal{T}$, determines on $\mathcal{T}$ a chord with length less or equal than s. This function is called *distribution function of the chord in* $\mathcal{T}$. In order to compute F we consider a lattice $\mathfrak{R}_D$, with $D > \max(a, d)$. With the introduced notations, we have:

$$F(s) = 1 - \frac{p(s)}{p(0)}. \tag{18}$$

We denote by $f := F'$ the derivative of F. We note that f is the density of the distribution function of the chord in $\mathcal{T}$. When s is small, that is, $s \leq \min(b, h)$, we obtain (see formula (4)):

$$\begin{aligned} F(s) &= s \cdot \frac{(\pi - 2\alpha)\cot\alpha + 1 + \sin 2\alpha - \cos 2\alpha}{\mathrm{per}(\mathcal{T})}, \\ f(s) &= \frac{(\pi - 2\alpha)\cot\alpha + 1 + \sin 2\alpha - \cos 2\alpha}{\mathrm{per}(\mathcal{T})}. \end{aligned}$$

If $s > \min(b, h)$ the distribution function F depends on s, since $\varphi_0, \varphi_1, \varphi_2$ and φ_3 are function of s. In such cases F is not a linear function of s, since $\sin\varphi_j$ and $\cos\varphi_j$, for $0 \leq j \leq 3$ are not linear functions of s. Then if $s > \min(b, h)$, f is not a constant function.

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Loredana Sorrenti
University of Reggio Calabria
Faculty of Engineering
DIMET
Via Graziella (Feo di Vito), 89100, Reggio Calabria, Italy
e-mail: loredana.sorrenti@unirc.it