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A family of meromorphically p-valent functions with positive coefficients [1]

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Abstract

In this paper we introduce a new subclass $M^n(p, \alpha, \beta)$ of meromorphically p-valent functions with positive coefficients in $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\}$ by using a linear operator D^n. The main object of this paper is to investigate the various important properties and characteristics of the class $M^n(p, \alpha, \beta)$. We also derive many results for the Hadamard products of functions belonging to the class $M^n(p, \alpha, \beta)$.

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1 Introduction

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. For functions $f(z)$ in the class Σ_p, we define a linear operator D^n by the recurrence formula

$$D^0 f(z) = f(z) \ ,$$

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$$D^1 f(z) = Df(z) = z^{-p} + \sum_{k=0}^{\infty} (p+k+1) a_k z^k = \frac{\left(z^{p+1} f(z)\right)'}{z^p},$$

$$D^2 f(z) = D(D^1 f(z)) = z^{-p} + \sum_{k=0}^{\infty} (p+k+1)^2 a_k z^k,$$

and (in general)

$$D^n f(z) = D(D^{n-1} f(z)) = z^{-p} + \sum_{k=0}^{\infty} (p+k+1)^n a_k z^k = \frac{\left(z^{p+1} D^{n-1} f(z)\right)'}{z^p}$$

$$(1.2) \qquad (f \in \Sigma_p; p, n \in N).$$

Then it is easily verified that

$$(1.3) \quad z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z) \; (f \in \Sigma_p; n \in N_0 = N \cup \{0\}; p \in N).$$

The linear operator D^n was considered, when $p = 1$, by Uralegaddi and Somanatha [14]. More recently, Aouf and Hossen [2] presented several results involving the operator D^n for $p \in N$.

Furthermore, let Σ_p^* denote the class of functions

$$(1.4) \qquad f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \qquad (a_k \geq 0; p \in N)$$

which are analytic and p-valent in U^*.

Making use of the operator D^n, we say that a function $f(z) \in \Sigma_p^*$ is in the class $M^n(p, \alpha, \beta)$ if it satisfies the following inequality :

$$(1.5) \qquad \Re\left\{ z^p D^n f(z) - \alpha z^{p+1} (D^n f(z))' \right\} > \beta$$

for some $\alpha (\alpha > 1), \beta (0 \leq \beta < p), p \in N, n \in N_0$, and for all $z \in U^*$.

We note that :

(i) $M^0(1, \alpha, \beta) = M(1, \alpha, \beta)$ (Altintas et al. [1]);

(ii) $M^0(p, \alpha, \beta) = M^*(p, \alpha, \beta) =$

(1.6)

$$= \left\{ f(z) \in \Sigma_p^* : \Re\left\{ z^p f(z) - \alpha z^{p+1} f'(z) \right\} > \beta, \alpha > 1, \;\; 0 \leq \beta < p, p \in N, z \in U^* \right\}.$$

In [11] Mogra defined the class $H_p^*(\alpha)$ as follows :

$$H_p^*(\alpha)=\left\{f(z)\in\Sigma_p^*:\Re\left\{-z^{p+1}f^{'}(z)\right\}>\alpha,0\leq\alpha<p,z\in U^*\right\}. \tag{1.7}$$

Some other subclasses of the class Σ_p^* were studied (for example) by Cho et al. ([5] and [6]), Liu [8], Liu and Srivastava ([9] and [10]), Joshi et al. [7], Raina and Srivastava [12], Aouf and Shammaky [3] and Chen et al. [4].

In this paper, we investigate the various important properties and characteristics of the class $M^n(p,\alpha,\beta)$. We also derive many results for the Hadamard products of functions belonging to the class $M^n(p,\alpha,\beta)$.

2 Coefficient inequalities and inclusion properties

We first determine a necessary and sufficient condition for a function $f(z)$, given by (1.4), to be in the class $M^n(p,\alpha,\beta)$.

Theorem 1 *Let $f(z)\in\Sigma_p^*$ be given by (1.4). Then $f(z)\in M^n(p,\alpha,\beta)$ if and only if*

$$\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_k\leq(1+\alpha p-\beta)\quad(\alpha>1;0\leq\beta<p;p\in N;n\in N_0). \tag{2.1}$$

Proof. Suppose that $f(z)\in M^n(p,\alpha,\beta)$. Then we find from (1.5) that

$$\begin{aligned}&\Re\left\{z^p(z^{-p}+\sum_{k=p}^{\infty}a_kz^k)-\alpha z^{p+1}(-pz^{-(p+1)}+\sum_{k=p}^{\infty}k(p+k+1)^na_kz^{k-1})\right\}\\ =\ &\Re\left\{1+\alpha p-\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_kz^{k+p}\right\}>\beta\end{aligned}$$

$$(z\in U^*;\alpha>1;0\leq\beta<p;p\in N;n\in N_0).$$

If we choose z to be real and let $z\to 1^-$, we get

$$1+\alpha p-\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_k\geq\beta\qquad(\alpha>1;0\leq\beta<p;p\in N;n\in N_0),$$

which is equivalent to (2.1).

Conversely, let us suppose that the inequality (2.1) holds true. Then we have

$$\begin{aligned}
&\left|z^p D^n f(z) - \alpha z^{p+1}(D^n f(z))' - (1+\alpha p)\right| \\
= &\left|-\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_k z^{k+p}\right| \\
\leq &\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_k |z|^{k+p} \\
\leq &(1+\alpha p-\beta) \quad (z \in U^*; \alpha > 1; 0 \leq \beta < p; p \in N; n \in N_0)
\end{aligned}$$

which implies that $f(z) \in M^n(p,\alpha,\beta)$.

Corollary 1 *Let the function $f(z)$ defined by (1.4) be in the class $M^n(p,\alpha,\beta)$. Then*

$$a_k \leq \frac{(1+\alpha p-\beta)}{(p+k+1)^n(\alpha k-1)} \qquad (k \geq p; p \in N; n \in N_0)\,. \tag{2.2}$$

The equality in (2.2) is attained for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(1+\alpha p-\beta)}{(p+k+1)^n(\alpha k-1)} z^k \qquad (k \geq p; p \in N; n \in N_0)\,. \tag{2.3}$$

Theorem 2 *The class $M^n(p,\alpha,\beta)$ is closed under convex linear combinations.*

Proof.

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0; j = 1,2; p \in N) \tag{2.4}$$

be in the class $M^n(p,\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = (1-t)f_1(z) + tf_2(z)\,(0 \leq t \leq 1) \tag{2.5}$$

is also in the class $M^n(p,\alpha,\beta)$. Since

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}[(1-t)a_{k,1} + ta_{k,2}]\, z^k\,(0 \leq t \leq 1)\,, \tag{2.6}$$

with the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)[(1-t)a_{k,1} + ta_{k,2}] \\
= &(1-t)\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_{k,1} + t\sum_{k=p}^{\infty}(p+k+1)^n(\alpha k-1)a_{k,2} \\
\leq &(1-t)(1+\alpha p-\beta) + t(1+\alpha p-\beta) = (1+\alpha p-\beta)
\end{aligned}$$

(2.7) $$(\alpha > 1; 0 \leq \beta < p; p \in N; n \in N_0; 0 \leq t \leq 1)\,,$$

which shows that $h(z) \in M^n(p, \alpha, \beta)$. Hence we have Theorem 2.

3 Distortion theorem

In this section we prove the following growth and distortion theorem for the class $M^n(p, \alpha, \beta)$.

Theorem 3 *If a function $f(z)$ defined by (1.4) is in the class $M^n(p, \alpha, \beta)$, then*

(3.1) $$\left(\frac{(p+m-1)!}{(p-1)!} - \frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)} \cdot \frac{p!}{(p-m)!} r^{2p}\right) r^{-(p+m)} \leq \left|f^{(m)}(z)\right| \leq$$

$$\leq \left(\frac{(p+m-1)!}{(p-1)!} + \frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)} \cdot \frac{p!}{(p-m)!} r^{2p}\right) r^{-(p+m)}$$

$$(0 < |z| = r < 1; \alpha > 1; 0 \leq \beta < p; p \in N; n \in N_0; m \in N_0; p > m)\;.$$

The result is sharp for the function $f(z)$ given by

(3.2) $$f(z) = z^{-p} + \frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)} z^p \qquad (p \in N; n \in N_0)\,.$$

Proof.

$$\frac{(2p+1)^n(\alpha p-1)}{p!} \sum_{k=p}^{\infty} k! a_k \leq \sum_{k=p}^{\infty} (p+k+1)^n(\alpha k-1) a_k \leq (1+\alpha p-\beta)\,,$$

which yields

(3.3) $$\sum_{k=p}^{\infty} k! a_k \leq \frac{(1+\alpha p-\beta)p!}{(2p+1)^n(\alpha p-1)} \qquad (p \in N; n \in N_0)\,.$$

Now, by differentiating both sides of (1.4) m times with respect to z, we have

(3.4) $$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m}$$

$$(p \in N; m \in N_0; p > m)\;,$$

and Theorem 3 follows easily from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2).

Putting (i) $m = 0$ and (ii) $m = 1$ in Theorem 3, we have the following corollaries:

Corollary 2 *If a function $f(z)$ defined by (1.4) is in the class $M^n(p,\alpha,\beta)$, then*

$$\frac{1}{r^p}-\frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)}r^p \le |f(z)| \le \frac{1}{r^p}+\frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)}r^p \tag{3.5}$$

$$\left(0<|z|=r<1;\alpha>1;0\le\beta<p;p\in N;n\in N_0\right).$$

The result is sharp for the function $f(z)$ given by (3.2).

Corollary 3 *If a function $f(z)$ defined by (1.4) is in the class $M^n(p,\alpha,\beta)$, then*

$$\frac{p}{r^{p+1}}-\frac{p(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)}r^{p-1} \le \left|f'(z)\right| \le \frac{p}{r^{p+1}}+\frac{p(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)}r^{p-1} \tag{3.6}$$

$$\left(0<|z|=r<1;\alpha>1;0\le\beta<p;p\in N;n\in N_0\right).$$

The result is sharp for the function $f(z)$ given by (3.2).

4 Radii of meromorphically p-valent starlikeness and convexity

In this section we determine the radii of meromorphically p-valent starlikeness of order $\delta\,(0\le\delta<p)$ and meromorphically p-valent convexity of order $\delta\,(0\le\delta<p)$ for functions in the class $M^n(p,\alpha,\beta)$.

Theorem 4 *Let the function $f(z)$ defined by (1.4) be in the class $M^n(p,\alpha,\beta)$. Then*

(i) $f(z)$ is meromorphically p-valent starlike of order $\delta(0\le\delta<p)$ in the disc $|z|<r_1$, that is,

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\delta \quad \left(|z|<r_1\,;\,0\le\delta<p;\,p\in N\right), \tag{4.1}$$

where

$$r_1=\inf_{k\ge p}\left\{\frac{(p-\delta)(p+k+1)^n(\alpha k-1)}{(k+2p-\delta)(1+\alpha p-\beta)}\right\}^{\frac{1}{k+p}}. \tag{4.2}$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\delta(0\le\delta<p)$ in the disc $|z|<r_2$, that is,

$$\Re\left\{-(1+\frac{zf''(z)}{f'(z)})\right\}>\delta \quad \left(|z|<r_2\,;\,0\le\delta<p;\,p\in N\right), \tag{4.3}$$

where

(4.4) $$r_2 = \inf_{k \geq p} \left\{ \frac{p(p-\delta)(p+k+1)^n(\alpha k - 1)}{k(k+2p-\delta)(1+\alpha p - \beta)} \right\}^{\frac{1}{k+p}} .$$

Each of these results is sharp for the function $f(z)$ given by (2.3).

Proof. We must show that

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \delta \quad \text{for} \quad |z| < r_1 \, , \, 0 \leq \delta < p \text{ and } p \in N \, .$$

Indeed we have

(4.5) $$\left| \frac{zf'(z)}{f(z)} + p \right| \leq \frac{\sum_{k=p}^{\infty} (k+p) a_k \, |z|^{k+p}}{1 - \sum_{k=p}^{\infty} a_k \, |z|^{k+p}} \, .$$

Thus $\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \delta$ if

(4.6) $$\sum_{k=p}^{\infty} \frac{(k+2p-\delta)}{(p-\delta)} a_k |z|^{k+p} \leq 1 \, .$$

But Theorem 1 ansures that

(4.7) $$\sum_{k=p}^{\infty} \frac{(p+k+1)^n(\alpha k - 1)}{(1+\alpha p - \beta)} a_k \leq 1 \, .$$

In view of (4.7), it follows that (4.6) will be true if

(4.8) $$\frac{(k+2p-\delta)}{(p-\delta)} \, |z|^{k+p} \leq \left\{ \frac{(p+k+1)^n(\alpha k - 1)}{(1+\alpha p - \beta)} \right\} \quad (k \geq p; p \in N; n \in N_0) \, .$$

The last inequality (4.8) leads us immediately to the disc $|z| < r_1$, where r_1 is given by (4.2).

(ii) It is sufficient to show that

(4.9) $$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq p - \delta \quad \text{for} \quad |z| < r_2, \, 0 \leq \delta < p \text{ and } p \in N \, .$$

Note that

(4.10) $$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq \frac{\sum_{k=p}^{\infty} k(k+p) a_k \, |z|^{k+p}}{p - \sum_{k=p}^{\infty} k a_k \, |z|^{k+p}} \, .$$

Thus $\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \le p - \delta$ if

$$\sum_{k=p}^{\infty} \frac{k(k+2p-\delta)}{p(p-\delta)} a_k |z|^{k+p} \le 1 \; . \tag{4.11}$$

Hence, by Theorem 1, (4.11) will be true if

$$\frac{k(k+2p-\delta)}{p(p-\delta)} |z|^{k+p} \le \left\{ \frac{(p+k+1)^n(\alpha k - 1)}{(1+\alpha p - \beta)} \right\} \quad (k \ge p; p \in N; n \in N_0) \, . \tag{4.12}$$

The last inequality (4.12) readily yields the disc $|z| < r_2$ with r_2 defined by (4.4), and the proof of Theorem 4 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (2.3).

5 Integrals transforms

In this section we consider integral transforms of functions in the class $M^n(p, \alpha, \beta)$. We shall need the following lemma in our investigation.

Lemma 1 *[11]. A function $f(z)$ defined by (1.4) is in the class $H_p^*(\alpha)(0 \le \alpha < p)$ if and only if*

$$\sum_{k=p}^{\infty} k a_k \le (p - \alpha) \, . \tag{5.1}$$

Theorem 5 *If $f(z)$ is in the class $M^n(p, \alpha, \beta)$, then the integral transforms*

$$F_{c+p-1}(z) = c \int_0^1 u^{c+p-1} f(uz) du, \qquad 0 < c < \infty \tag{5.2}$$

are in the class $H_p^*(\theta), 0 \le \theta < p$, where

$$\theta = \theta(p, \alpha, \beta, c) = p\left(1 - \frac{c(1+\alpha p - \beta)}{(2p+c)(2p+1)^n(\alpha p - 1)}\right) . \tag{5.3}$$

The result is the best possible for the function $f(z)$ given by (3.2).

Proof. Suppose $f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \in M^n(p, \alpha, \beta)$. Then we have

$$\begin{aligned} F_{c+p-1}(z) &= c \int_0^1 u^{c+p-1} f(uz) du \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{c}{p+k+c} a_k z^k \, . \end{aligned}$$

In view of Lemma 1, it is sufficient to show that

$$\sum_{k=p}^{\infty} \frac{k}{p-\theta} \cdot \frac{c}{p+k+c} a_k \leq 1 . \tag{5.4}$$

Since $f(z) \in M^n(p,\alpha,\beta)$, we have

$$\sum_{k=p}^{\infty} \frac{(p+k+1)^n(\alpha k-1)}{(1+\alpha p-\beta)} a_k \leq 1 .$$

Thus (5.4) will be satisfied if

$$\frac{kc}{(p-\theta)(p+k+c)} \leq \frac{(p+k+1)^n(\alpha k-1)}{(1+\alpha p-\beta)} \quad \text{for each } k ,$$

or

$$\theta \leq p - \frac{ck(1+\alpha p-\beta)}{(p+k+c)(p+k+1)^n(\alpha k-1)} . \tag{5.5}$$

Since the right hand side of (5.5) is an increasing function of k, putting $k=p$ in (5.5), we get

$$\theta \leq p\Big(1 - \frac{c(1+\alpha p-\beta)}{(2p+c)(2p+1)^n(\alpha p-1)}\Big) .$$

Hence the theorem.

6 Convolution properties

For the functions

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0; j=1,2; p \in N) \tag{6.1}$$

we denote by $(f_1 \otimes f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 \otimes f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} a_{k,1} a_{k,2} z^k . \tag{6.2}$$

Theorem 6 *Let the functions $f_j(z) (j=1,2)$ defined by (6.1) be in the class $M^n(p,\alpha,\beta)$. Then $(f_1 \otimes f_2)(z) \in M^n(p,\alpha,\gamma)$, where*

$$\gamma = 1 + \alpha p - \frac{(1+\alpha p-\beta)^2}{(2p+1)^n(\alpha p-1)} . \tag{6.3}$$

The result is sharp for the functions $f_j(z)(j = 1, 2)$ *given by*

$$f_j(z) = z^{-p} + \frac{(1 + \alpha p - \beta)^2}{(2p + 1)^n(\alpha p - 1)} z^p \qquad (j = 1, 2; p \in N; n \in N_0)\,. \tag{6.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [13], we need to find the largest γ such that

$$\sum_{k=p}^{\infty} \frac{(p + k + 1)^n(\alpha k - 1)}{(1 + \alpha p - \gamma)} a_{k,1} \,.\, a_{k,2} \le 1 \tag{6.5}$$

for $f_j(z) \in M^n(p, \alpha, \gamma)(j = 1, 2)$. Since $f_j(z) \in M^n(p, \alpha, \beta)((j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{(p + k + 1)^n(\alpha k - 1)}{(1 + \alpha p - \beta)} a_{k,j} \le 1 \qquad (j = 1, 2)\,. \tag{6.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{(p + k + 1)^n(\alpha k - 1)}{(1 + \alpha p - \beta)} \sqrt{a_{k,1}a_{k,2}} \le 1\,. \tag{6.7}$$

This implies that we need only to show that

$$\frac{a_{k,1}a_{k,2}}{(1 + \alpha p - \gamma)} \le \frac{\sqrt{a_{k,1}a_{k,2}}}{(1 + \alpha p - \beta)} \qquad (k \ge p) \tag{6.8}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{(1 + \alpha p - \gamma)}{(1 + \alpha p - \beta)} \qquad (k \ge p)\,. \tag{6.9}$$

Hence, by the inequality (6.7), it is sufficient to prove that

$$\frac{(1 + \alpha p - \beta)}{(p + k + 1)^n(\alpha k - 1)} \le \frac{(1 + \alpha p - \gamma)}{(1 + \alpha p - \beta)} \qquad (k \ge p)\,. \tag{6.10}$$

It follows from (6.10) that

$$\gamma \le 1 + \alpha p - \frac{(1 + \alpha p - \beta)^2}{(p + k + 1)^n(\alpha k - 1)} \qquad (k \ge p)\,. \tag{6.11}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = 1 + \alpha p - \frac{(1+\alpha p-\beta)^2}{(p+k+1)^n(\alpha k-1)} \qquad (k \geq p)\,, \tag{6.12}$$

we see that $\varphi(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq \varphi(p) = 1 + \alpha p - \frac{(1+\alpha p-\beta)^2}{(2p+1)^n(\alpha p-1)}\,, \tag{6.13}$$

which evidently completes the proof of Theorem 6.

Using arguments similar to those in the proof of Theorem 6, we obtain the following result.

Theorem 7 *Let the function $f_1(z)$ defined by (6.1) be in the class $M^n(p,\alpha,\beta)$. Suppose also that the function $f_2(z)$ defined by (6.1) is in the class $M^n(p,\alpha,\gamma)$. Then $(f_1 \otimes f_2)(z) \in M^n(p,\alpha,\xi)$, where*

$$\xi = 1 + \alpha p - \frac{(1+\alpha p-\beta)(1+\alpha p-\gamma)}{(2p+1)^n(\alpha p-1)}\,. \tag{6.14}$$

The result is sharp for the functions $f_j(z) (j=1,2)$ given by

$$f_1(z) = z^{-p} + \frac{(1+\alpha p-\beta)}{(2p+1)^n(\alpha p-1)} z^p \quad (p \in N; n \in N_0) \tag{6.15}$$

and

$$f_2(z) = z^{-p} + \frac{(1+\alpha p-\gamma)}{(2p+1)^n(\alpha p-1)} z^p \quad (p \in N; n \in N_0)\,. \tag{6.16}$$

Theorem 8 *Let the functions $f_j(z) (j=1,2)$ defined by (6.1) be in the class $M^n(p,\alpha,\beta)$. Then the function $h(z)$ defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{6.17}$$

belongs to the class $M^n(p,\alpha,\zeta)$, where

$$\zeta = 1 + \alpha p - \frac{2(1+\alpha p-\beta)^2}{(2p+1)^n(\alpha p-1)}\,. \tag{6.18}$$

The result is sharp for the functions $f_j(z) (j=1,2)$ given already by (6.4).

Proof. Noting that

(6.19)
$$\sum_{k=p}^{\infty}\frac{[(p+k+1)^n(\alpha k-1)]^2}{(1+\alpha p-\beta)^2}a_{k,j}^2 \le (\sum_{k=p}^{\infty}\frac{(p+k+1)^n(\alpha k-1)}{(1+\alpha p-\beta)}a_{k,j})^2 \le 1 \quad (j=1,2)\,,$$

for $f_j(z) \in M^n(p,\alpha,\beta)(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{1}{2}\;\frac{[(p+k+1)^n(\alpha k-1)]^2}{(1+\alpha p-\beta)^2}(a_{k,1}^2+a_{k,2}^2) \le 1\ . \tag{6.20}$$

Therefore, we have to find the largest ζ such that

$$\frac{1}{(1+\alpha p-\zeta)} \le \frac{(p+k+1)^n(\alpha k-1)}{2(1+\alpha p-\beta)^2} \quad (k \ge p)\,, \tag{6.21}$$

that is, that

$$\zeta \le 1+\alpha p - \frac{2(1+\alpha p-\beta)^2}{(p+k+1)^n(\alpha k-1)} \quad (k \ge p)\,. \tag{6.22}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = 1+\alpha p - \frac{2(1+\alpha p-\beta)^2}{(p+k+1)^n(\alpha k-1)} \quad (k \ge p)\,, \tag{6.23}$$

we observe that $\Psi(k)$ is an increasing function of k. We conclude that

$$\zeta \le \Psi(p) = 1+\alpha p - \frac{2(1+\alpha p-\beta)^2}{(2p+1)^n(\alpha p-1)}\,, \tag{6.24}$$

which completes the proof of Theorem 8.

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