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A Note on a Generalized Integral Operator [1]

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Abstract

In this note, we introduce the new subclass $\mathcal{SP}_k^\lambda(\beta)$ of analytic functions and certain properties of a generalized integral operator are studied.

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1 Introduction

Let $\mathcal{P}_k^\lambda(\beta)$ denote the class of analytic functions $p(z)$ defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ with the following properties:

i. $p(0) = 1$

ii. $\displaystyle\int_0^{2\pi} \left| \frac{\Re\{e^{i\lambda}p(z) - \beta\cos\lambda\}}{1-\beta} \right| d\theta \leq k\pi\cos\lambda$

where, $k \geq 2,$ λ real, $|\lambda| < \frac{\pi}{2},$ $0 \leq \beta < 1$ and $z = re^{i\theta}$ for $0 \leq r < 1$.
Let $\mathcal{V}_k^\lambda(\beta)$ [5] denote the class of analytic functions f defined in $\mathcal{U}$ satisfying the normalization properties $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\beta), \quad (z \in \mathcal{U})$$

where, k, λ and β are as above. For $\beta = 0$ we get the class $\mathcal{V}_k^\lambda$ of functions with bounded boundary rotation studied by Moulis [4]. Any function $f(z) \in \mathcal{V}_k^\lambda(\beta)$ if and only if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta\cos\lambda, \quad \text{for} \quad |z| < \frac{k - \sqrt{k^2-4}}{2}.$$

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A function f defined in $\mathcal{U}$ satisfying the normalization properties $f(0) = f'(0) - 1 = 0$ is said to be in the class $\mathcal{U}_k^\lambda(\beta)$ if

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_k^\lambda(\beta), \quad (z \in \mathcal{U}).$$

From the definition of the above classes it follows that $f(z) \in \mathcal{V}_k^\lambda(\beta)$ if and only if $zf'(z) \in \mathcal{U}_k^\lambda(\beta)$.
Let $\mathcal{SP}_k^\lambda(\beta)$ be the class of normalized functions f such that

$$\frac{zf'(z)}{f(z)} - \left|\frac{zf'(z)}{f(z)} - 1\right| \in \mathcal{P}_k^\lambda(\beta), \quad (z \in \mathcal{U}) \tag{1}$$

where, β is a real number with $0 \leq \beta < 1$. Now we consider the integral operator $F_n(z)$ [2], defined by

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} ... \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{2}$$

and we study its property.

Remark 1 *We observe that for $n = 1$ and $\alpha_1 = 1$, we obtain the integral operator of Alexander [1], $F(z) = \int_0^z \frac{f(t)}{t} dt$.*

2 Main Result

Theorem 1 *Let $\alpha_i > 0$ $i \in \{1, 2, ..., n\}$ and β_i be real numbers with the property $-1 \leq \beta_i \leq 1$ and $f_i \in \mathcal{SP}_k^\lambda(\beta_i)$ for $i \in \{1, 2, ..., n\}$. If $0 < \sum_{i=1}^n \alpha_i(1 - \beta_i) \leq 1$, then the integral operator $F_n \in \mathcal{V}_k^\lambda(\gamma)$, where $\gamma = 1 + \sum_{i=1}^n \alpha_i(\beta_i - 1)$.*

Proof. *From (2), we have,*

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} .... \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

$$F_n''(z) = \sum_{i=1}^n \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{zf_i'(z) - f_i(z)}{z^2}\right) \prod_{j=1, j\neq i}^n \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{F_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right) = \sum_{i=1}^n \alpha_i \left(\frac{f_i'(z)}{f_i(z)} - \frac{1}{z}\right)$$

so that

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \left(\sum_{i=1}^{n} \frac{zf_i'(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i$$

which is equivalent to

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\sum_{i=1}^{n} \frac{zf_i'(z)}{f_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1.$$

That is,

$$\Re \left\{ 1 + \frac{zF_n''(z)}{F_n'(z)} \right\} = \sum_{i=1}^{n} \alpha_i \Re \left\{ \frac{zf_i'(z)}{f_i(z)} - \beta_i \right\} + \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{i=1}^{n} \alpha_i + 1.$$

Since, $f_i \in \mathcal{SP}_k^{\lambda}(\beta_i)$ *for* $i \in \{1, 2, ..., n\}$

$$\Re \left\{ 1 + \frac{zF_n''(z)}{F_n'(z)} \right\} > \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - i \right| + \sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1 > \sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1.$$

Therefore, $F_n \in \mathcal{V}_k^{\lambda}(\gamma)$, *where* $\gamma = 1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$.

Corollary 1 *For parametric values* $k = 2$, $\lambda = 0$, *we get Theorem 1 in [3] which reads as:*
Let $\alpha_i > 0$, *for* $i \in \{1, 2, ..., n\}$, β_i *be real numbers with the properties* $0 \le \beta_i < 1$ *and* $f_i \in \mathcal{S}_p(\beta_i)$ *for* $i \in \{1, 2, ..., n\}$. *If* $0 < \sum_{i=1}^{n} \alpha_i \le 1$, *then the integral operator* F_n *is convex by the order* $1 + \sum_{i=1}^{n} \alpha_i(\beta_i - 1)$.

Corollary 2 *For parametric values* $k = 2$, $\lambda = 0 = \beta$, *we get Theorem 2.5 in [2], stated as:*
Let α_i, $i \in \{1, 2, ..., n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, 2, ..., n\}$, $\sum_{i=1}^{n} \alpha_i \le 1$ *and* $1 - \sum_{i=1}^{n} \alpha_i \in [0, 1)$. *We consider the functions* $f_i \in \mathcal{SP}$, *for* $i \in \{1, 2, ..., n\}$. *In these conditions, the integral operator defined in* F_n *is convex by* $1 - \sum_{i=1}^{n} \alpha_i$ *order.*

Theorem 2 *Let* α_i, $i \in \{1, 2, ..., n\}$ *be real and positive and* $f_i \in \mathcal{SP}_k^{\lambda}(\beta)$ *for* $i \in \{1, 2, ..., n\}$. *If* $0 < \sum_{i=1}^{n} \alpha_i \le \frac{1}{1 - \beta}$, *then the integral operator* $F_n \in \mathcal{V}_k^{\lambda}(\gamma_1)$, *where*

$\gamma_1 = (\beta-1)\sum_{i=1}^{n}\alpha_i + 1.$

Proof. *Using (2), we have,*

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \ldots \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

so that

$$F_n''(z) = \sum_{i=1}^{n}\alpha_i\left(\frac{f_i(z)}{z}\right)^{\alpha_i-1}\left(\frac{zf_i'(z)-f_i(z)}{z^2}\right)\prod_{j=1,\, j\neq i}^{n}\left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$1+\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\left(\sum_{i=1}^{n}\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^{n}\alpha_i + 1.$$

For $f_i \in \mathcal{SP}_k^{\lambda}(\beta),\ \ i \in \{1,2,...,n\}$ *we obtain,*

$$\Re\left\{1+\frac{zF_n''(z)}{F_n'(z)}\right\} > (\beta-1)\sum_{i=1}^{n}\alpha_i + 1.$$

Since, $\beta \le 1$ *the above inequality implies*

$$0 < \sum_{i=1}^{n}\alpha_i \le \frac{1}{1-\beta}.$$

Therefore, $F_n \in \mathcal{V}_k^{\lambda}(\gamma_1)$ *where* $\gamma_1 = (\beta-1)\sum_{i=1}^{n}\alpha_i + 1.$

Corollary 3 *For parametric values* $k=2,\ \lambda=0,$ *we get Theorem 2 in [3], stated as.*
Let $\alpha_i,$ *for* $i \in \{1,2,...,n\},$ β_i *be real positive numbers and* $f_i \in \mathcal{S}_p(\beta)$
$0 \le \beta < 1$ *and for* $i \in \{1,2,...,n\}$. *If* $0 < \sum_{i=1}^{n}\alpha_i \le \frac{1}{1-\beta},$ *then the integral operator*
F_n *is convex by the order* $(\beta-1)\sum_{i=1}^{n}\alpha_i + 1.$

By taking $n=1$, we obtain the following Corollary.

Corollary 4 *Let* γ *be the real number,* $\gamma > 0$. *We suppose that the functions* $f \in \mathcal{SP}_k^{\lambda}(\beta)$ *and* $0 < \gamma \le \frac{1}{1-\beta}$. *With these conditions, the integral operator*
$F_1(z) \in \mathcal{V}_k^{\lambda}((\beta-1)\gamma+1)$.

Corollary 5 *Let $f \in \mathcal{SP}_k^\lambda(\beta)$ and consisting the integral operator of Alexander,*

$$F(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

Then, $F \in \mathcal{V}_k^\lambda(\beta)$.

Since $F(z) = \int_0^z \frac{f(t)}{t}\, dt$ we have,

$$1 + \frac{zF''(z)}{F'(z)} = \frac{zf'(z)}{f(z)}.$$

That is,

$$\Re\left\{1 + \frac{zF''(z)}{F'(z)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} + \beta > \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta > \beta.$$

This implies that the Alexander operator belongs to $\mathcal{V}_k^\lambda(\beta)$.

Corollary 6 *By taking $k = 2$, $\lambda = 0 = \beta$, we get the Theorem 2.8 in [2], which reads as:*
We suppose that $f \in \mathcal{SP}$. In this condition, the integral operator of Alexander, defined by $F_1(z) = \int_0^z \frac{f(t)}{t}\, dt$ is convex.

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