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Quadrature based three-step iterative method for non-linear equations [1]

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Abstract

In this paper, we present three-step quadrature based iterative method for solving non-linear equations. The convergence analysis of the method is discussed. It is established that the new method has convergence order eight. Numerical tests show that the new method is comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement of the previously known results in the literature.

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1 Introduction

Let us consider a single variable non-linear equation

$$f(x) = 0. \tag{1}$$

Finding zeros of a single variable nonlinear equation (1) efficiently, is an interesting and very old problem in numerical analysis and has many applications in applied sciences.

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In recent years, researchers have developed many iterative methods for solving equation (1). These methods can be classified as one-step, two-step and three-step methods, see[$1-14$]. These methods have been proposed using Taylor series, decomposition techniques, error analysis and quadrature rules, etc. Abbasbandy[2], Chun[4] and Grau[8] have proposed many two-step and three-step methods.

In this paper, we present three-step quadrature based iterative method for solving non-linear equations. We prove that the new method has order of convergence eight. The method and its algorithm is described in section 2. The convergence analysis of the method is discussed in section 3. Finally, in section 4, the method is tested on numerical examples given in the literature. It was noted that the new method is comparable with the well known existing methods and in many cases gives better results. Our results can be considered as an improvement and refinement of the previously known results in the literature.

2 The Iterative Method

Weerakoon and Fernando [13] ,Gyurhan Nedzhibov [12] and M. Frontini and E. Sormani [$6-7$] have proposed various methods by the approximation of the indefinite integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt, \tag{2}$$

using Newton Cotes formulae of order zero and one. We approximate, here however the integral (2) by rectangular rule at a generic point $\frac{x+z_n}{2}$ with the end-points x and z_n . We thus have:

$$\int_{z_n}^{x} f'(t)dt = (x - z_n) f'\left(\frac{x+z_n}{2}\right),$$

this gives

$$-f(z_n) = (x - z_n) f'(\frac{x+z_n}{2}). \tag{3}$$

From (3), we have:

$$x - z_n = -\frac{f(z_n)}{f'(\frac{x+z_n}{2})} \tag{4}$$

Therefore, we have:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(\frac{x^*+z_n}{2})} \tag{5}$$

For a generic point $w_n = \dfrac{x^* + z_n}{2}$, consider the Ostrowski's method and the Newton's method:

$$x^* = y_n - \frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)}, \tag{6}$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{7}$$

This formulation allows to suggest many one-step, two-step and three-step methods. We however define the following three-step iterative method:

Algorithm 2.1 For a given initial guess x_0, find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{8}$$

$$w_n = y_n - \frac{1}{2}\left[\frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)} + \frac{f(y_n)}{f'(y_n)}\right], \tag{9}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(w_n)}. \tag{10}$$

where z_n is defined by (7).

Algorithm 2.1 can further be modified by using an approximation for $f^{'}(y_n)$ with the help of Taylor's expansion.

Let y_n be defined by (8). If we use Taylor expansion of $f^{'}(y_n)$:

$$f^{'}(y_n) \simeq f^{'}(x_n) + f^{''}(x_n)(y_n - x_n),$$

(where the higher derivatives are neglected) in combination with Taylor approximation of $f(y_n)$:

$$f(y_n) \simeq f(x_n) + f'(x_n)(y_n - x_n) + \frac{1}{2}f^{''}(x_n)(y_n - x_n)^2,$$

we can remove the second derivative and approximate $f^{'}(y_n)$ as:

$$f^{'}(y_n) \simeq 2\left[\frac{f(y_n)-f(x_n)}{y_n-x_n}\right]-f'(x_n). \tag{11}$$

then Algorithm 2.1 can be written in the form of the following algorithm:

Algorithm 2.2 For a given initial guess x_o,find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{12}$$

$$w_n = y_n - \frac{1}{2}\left[\frac{(x_n-y_n)f(y_n)}{f(x_n)-2f(y_n)} + \frac{f(y_n)}{2\left[\frac{f(y_n)-f(x_n)}{y_n-x_n}\right]-f'(x_n)}\right], \tag{13}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(w_n)}, \tag{14}$$

where z_n is defined by (7).

We will compare this method with the Ostrowski's method, Grau's method and seventh order method defined in [1] by Jisheng Kou et al. The algorithms of these methods are given below:

Algorithm 2.3 For a given initial guess x_0, find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{15}$$

$$x_{n+1} = y_n - \frac{(x_n-y_n)f(y_n)}{f(x_n)-2f(y_n)}. \tag{16}$$

Algorithm 2.4 For a given initial guess x_0, find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{17}$$

$$\mu = \frac{x_n-y_n}{f(x_n)-2f(y_n)}, \tag{18}$$

$$z_n = y_n - \mu f(y_n), \tag{19}$$

$$x_{n+1} = z_n - \mu f(z_n). \tag{20}$$

Algorithm 2.5 For a given initial guess x_0, find the approximate solution by the iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{21}$$

$$z_n = y_n - \frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)}, \tag{22}$$

$$x_{n+1} = z_n - \left[\left(1 + \frac{f(y_n)}{f(x_n) - 2f(y_n)}\right)^2 + \frac{f(z_n)}{f(y_n)}\right]\frac{f(z_n)}{f'(x_n)} \tag{23}$$

3 Convergence Analysis

Let us now discuss the convergence analysis of the algorithm 2.2 discussed above.

Theorem 1 *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, then the algorithm 2.2 has eighth order convergence.*

Proof. Let α be a simple zero of f and $x_n = \alpha + e_n$. By Taylor's expansion, we have:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8) + O\left(e_n^9\right), \tag{24}$$

$$f'(x_n) = f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7) + O(e_n^9). \tag{25}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3, ...\text{and } e_n = x_n - \alpha. \tag{26}$$

Using (24) and (25), we have

$$(27)\quad \frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + (6c_3^2 - 4c_5 + 8c_2^4 + 10c_2c_4 - 20c_3c_2^2)e_n^5 + (-5c_6 + 13c_2c_5 - 33c_2c_3^2 - 16c_2^5 + 52c_3c_2^3 + 17c_4c_3 - 28c_4c_2^2)e_n^6 + (-32c_2^6 + c_7 - 8c_3c_5 + 24c_2^2c_5 - 8c_2c_6 - 56c_2^3c_4 - 90c_2^2c_3^2 + 52c_2c_4c_3 - 4c_4^2 + 9c_3^3 + 112c_2^4c_3)e_n^7 + (33c_3^2c_4 - 54c_3^3c_2 + 16c_4^2c_2 - 9c_4c_5 + 96c_3^2c_2^3 - 84c_3c_4c_2^2 + 32c_3c_2c_5 - 2c_7c_2 - 32c_3c_2^5 - 8c_5c_2^3 - 9c_3c_6 + 16c_4c_2^4 + 4c_6c_2^2)e_n^8 + O(e_n^9).$$

Using (27) in (12), we thus have:

$$(28)\quad y_n = \alpha + c_2e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + (4c_5 - 10c_2c_4 + 20c_3c_2^2 - 8c_2^4 - 6c_3^2)e_n^5 + (28c_4c_2^2 + 33c_2c_3^2 + 5c_6 - 52c_3c_2^3 - 17c_4c_3 - 13c_2c_5 + 16c_2^5)e_n^6 + (-c_7 - 52c_2c_4c_3 + 4c_4^2 - 9c_3^3 + 56c_2^3c_4 + 8c_2c_6 - 24c_2^2c_5 + 90c_2^2c_3^2 + 32c_2^6 + 8c_3c_5 - 112c_2^4c_3)e_n^7 + (32c_3c_2^5 + 54c_3^3c_2 - 33c_3^2c_4 + 84c_3c_4c_2^2 + 9c_3c_6 - 32c_3c_2c_5 + 2c_7c_2 - 4c_6c_2^2 + 8c_5c_2^3 - 16c_4c_2^4 - 16c_4^2c_2 + 9c_4c_5 - 96c_3^2c_2^3)e_n^8 + O\left(e_n^9\right).$$

By Taylor's series, we have:

$$f(y_n) = (y_n - \alpha)\, f'(\alpha) + \frac{1}{2!}(y_n - \alpha)^2 f''(\alpha) + \ldots \quad .$$

Using (28) in the above relation and on simplifying, we have:

$$(29)\quad f(y_n) = f'(\alpha)(c_2e_n^2 + 2\left(c3 - c_2^2\right)e_n^3 + \left(-7c_2c_3 + 3c_4 + 5c_2^3\right)e_n^4 + (24c_3c_2^2 - 12c_2^4 + 4c_5 - 10c_2c_4 - 6c_3^2)e_n^5 + (37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5 + 34c_4c_2^2 + 5c_6 - 17c_4c_3 - 13c_2c_5)e_n^6 + (-40c_2c_4c_3 + 56c_2^2c_3^2 - 34c_2^4c_3 + 24c_2^3c_4 - 16c_2^2c_5 - c_7 + 4c_4^2 - 9c_3^3 + 8c_2c_6 + 8c_3c_5)e_n^7 + (-23c_3c_4c_2^2 - 16c_3c_2c_5 - 33c_3^2c_4 + 42c_3^3c_2 - 7c_4^2c_2 + 9c_4c_5 + 78c_3^2c_2^3 + 2c_7c_2 - 216c_3c_2^5 - 34c_5c_2^3 + 9c_3c_6 + 105c_4c_2^4 + 6c_6c_2^2 + 80c_2^7)e_n^8) + O(e_n^9).$$

Using $(24), (25), (28)$ and (29) in (11), we have:

$$(30)\, f'(y_n) = f'(\alpha)(1+(2c_2^2-c_3)e_n^2+(-4c_2^3-2c_4+6c_2c_3)e_n^3+(-3c_5 -16c_3c_2^2+4c_3^2+8c_2c_4+8c_2^4)e_n^4+(-22c_4c_2^2+10c_2c_5 +40c_3c_2^3-16c_2^5+10c_4c_3-18c_2c_3^2 -4c_6)e_n^5+(-5c_7+6c_4^2-48c_2c_4c_3+58c_2^3c_4-28c_2^2c_5 +62c_2^2c_3^2+32c_2^6+12c_2c_6-4c_3^3-96c_2^4c_3+12c_3c_5)e_n^6+(64c_2^7-6c_8 +108c_4c_2^4-32c_3^3c_2+14c_6c_2^2+14c_4c_5+244c_3^2c_2^3-46c_5c_2^3 +14c_3c_6-104c_3c_4c_2^2-256c_3c_2^5-14c_3^2c_4)e_n^7+(870c_2^2c_3^3-14c_6c_2^3 -528c_2c_3^2c_4+2c_8c_2+48c_2c_5c_4+8c_5^2 +768c_3c_2^6+16c_6c_4-1504c_2^4c_3^2-128c_2^8+1050c_3c_2^3c_4-288c_4c_2^5- 126c_4^2c_2^2-50c_3^4+2c_3c_7+44c_5c_3^2+88c_6c_2c_3-348c_2^2c_5c_3- 2c_7c_2^2+16c_4^2c_3+86c_2^4c_5)e_n^8)+O(e_n^9).$$

Using $(28), (29)$ and (30) in (7), we have:

$$(31) z_n = \alpha+(-c_2c_3+c_2^3)e_n^4+\left(-2c_3^2+8c_3c_2^2-2c_2c_4-4c_2^4\right)e_n^5+(10c_2^5 +18c_2c_3^2-7c_4c_3+12c_4c_2^2-30c_3c_2^3-3c_2c_5)e_n^6+(-4c_2c_6 +80c_2^4c_3-40c_2^3c_4+16c_2^2c_5+52c_2c_4c_3-10c_3c_5-80c_2^2c_3^2 +12c_3^3-20c_2^6-6c_4^2)e_n^7+(252c_3^2c_2^3+37c_4^2c_2+68c_3c_2c_5 +50c_3^2c_4-17c_4c_5-178c_3c_2^5-209c_3c_4c_2^2+101c_4c_2^4-51c_5c_2^3 +20c_6c_2^2-5c_7c_2-13c_3c_6-91c_3^3c_2+36c_2^7)e_n^8+O(e_n^9).$$

By Taylor's series, we have:

$$f(z_n) = (z_n-\alpha)\, f'(\alpha)+\frac{1}{2!}\,(z_n-\alpha)^2\, f''(\alpha)+ \ldots \quad .$$

Using (31) in the above relation and on simplifying, we have:

$$(32)\;\; f(z_n) = f'(\alpha)(c_2(-c_3+c_2^2)e_n^4+(8c_3c_2^2-2c_2c_4-4c_2^4-2c_3^2)e_n^5+(-30c_3c_2^3 +18c_2c_3^2+10c_2^5-3c_2c_5+12c_4c_2^2-7c_4c_3)e_n^6+(-4c_2c_6+80c_2^4c_3 -40c_2^3c_4+16c_2^2c_5+52c_2c_4c_3-10c_3c_5-80c_2^2c_3^2+12c_3^3-20c_2^6 -6c_4^2)e_n^7+(253c_3^2c_2^3+37c_4^2c_2+68c_3c_2c_5+50c_3^2c_4-17c_4c_5 -180c_3c_2^5-209c_3c_4c_2^2+101c_4c_2^4-51c_5c_2^3+20c_6c_2^2-5c_7c_2 -13c_3c_6-91c_3^3c_2+37c_2^7)e_n^8)+O(e_n^9).$$

Using (24), (27), (28) and (29) in (13), we have:

$$(33)\quad w_n = \alpha+(-c_2c_3+c_2^3)e_n^4-2c_3^2+8c_3c_2^2-2c_2c_4-4c_2^4)e_n^5+(10c_2^5+18c_2c_3^2 -7c_4c_3+12c_4c_2^2-30c_3c_2^3-3c_2c_5)e_n^6+(-4c_2c_6+80c_2^4c_3-40c_2^3c_4 +16c_2^2c_5+52c_2c_4c_3-10c_3c_5-80c_2^2c_3^2+12c_3^3-20c_2^6-6c_4^2)e_n^7 +(4c_2^7+50c_3^2c_4-137c_3c_4c_2^2+44c_3^2c_2^3-\frac{3}{2}c_7c_2-13c_3c_6+53c_3c_2c_5 -\frac{155}{2}c_3^3c_2-21c_5c_2^3+37c_4c_2^4-58c_3c_2^5+8c_6c_2^2+29c_4^2c_2-17c_4c_5)e_n^8 +O(e_n^9).$$

By Taylor's series, we have:

$$(34)\quad f'(w_n) = f'(\alpha)(1+(-2c_3c_2^2+2c_2^4)e_n^4+(-4c_2c_3^2-8c_2^5+16c_3c_2^3-4c_4c_2^2)e_n^5 +(-14c_2c_4c_3+24c_2^3c_4+20c_2^6+36c_2^2c_3^2-60c_2^4c_3-6c_2^2c_5)e_n^6 +(32c_5c_2^3-160c_3^2c_2^3-80c_4c_2^4+24c_3^3c_2+104c_3c_4c_2^2-40c_2^7- 20c_3c_2c_5-8c_6c_2^2-12c_4^2c_2+160c_3c_2^5)e_n^7+(282c_3^2c_2^4-34c_2c_5c_4 +106c_2^2c_5c_3-3c_7c_2^2+74c_4c_2^5-152c_2^2c_3^3+100c_4c_3^2c_2+58c_4^2c_2^2 -113c_2^6c_3-274c_4c_3c_2^3+8c_2^8+16c_6c_2^3-42c_2^4c_5-26c_6c_2c_3)e_n^8) +O(e_n^9).$$

Using (31), (32) and (34) in (14), we have:

$$(35)\qquad x_{n+1} = \alpha + (c_2^7 + c_3^2c_2^3 - 2c_3c_2^5)e_n^8 + O(e_n^9),$$

or

$$(36)\qquad e_{n+1} = (c_2^7 + c_3^2c_2^3 - 2c_3c_2^5)e_n^8 + O(e_n^9).$$

Thus, we observe that the algorithm 2.2 has eighth order convergence.

4 Numerical examples

We consider here some numerical examples to demonstrate the performance of the new developed three-step iterative method, namely algorithm 2.2. We compare the methods defined in J.Kou et al. (algorithm 2.3 (G_4), algorithm 2.4 (G_6), and algorithm 2.5 (G_7) and the new developed three-step method algorithm 2.2 (MN) in this paper. All the computations are performed using Maple 10.0. We take $\in = 10^{-15}$ as tolerance.

The following criteria is used for estimating the zero:

(i) $\delta = |x_{n+1} - x_n| < \in$

(ii) $|f(x(_n))| < \in$

The following examples of J.Kou et al. [1] are used for numerical testing:

Example	**Exact Zero**
$f_1 = x^3 + 4x^2 - 15,$	$\alpha = 1.6319808055660636,$
$f_2 = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$	$\alpha = -1.207647827130919,$
$f_3 = \sin(x) - \frac{1}{2}x,$	$\alpha = 1.8954942670339809,$
$f_4 = 10xe^{-x^2} - 1,$	$\alpha = 1.67963061042845,$
$f_5 = \cos(x) - x,$	$\alpha = 0.73908513321516067,$
$f_6 = \sin^2(x) - x^2 + 1,$	$\alpha = 1.4044916482153411,$
$f_7 = e^{-x} + \cos(x),$	$\alpha = 1.74613953040801241765.$

For convergence criteria, it was required that δ, the distance between two consecutive iterates was less than 10^{-15}, n represents the number of iterations and $f(x_n)$, the absolute value of the function. All the values are computed with 350 significant digits. The numerical comparison is given in Table 4.1.

	n	$f(x_n)$
$f_1, x_0 = 2$		
G_4	3	1.03e-228
G_6	3	4.46e-179
G_7	3	1.06e-274
MN	3	1.00e-348
$f_2, x_0 = -1$		
G_4	3	8.82e-223
G_6	3	2.54e-155
G_7	3	1.20e-264
MN	3	2.79e-259
$f_3, x_0 = 2$		
G_4	3	5.12e-313
G_6	3	8.44e-252
G_7	3	3.00e-320
MN	3	3.00e-350

	n	$f(x_n)$
$f_4, x_0 = 1.8$		
G_4	3	1.16e-236
G_6	3	9.37e-187
G_7	3	1.34e-281
MN	3	0
$f_5, x_0 = 1$		
G_4	3	7.05e-296
G_6	3	4.12e-237
G_7	3	0
MN	3	0
$f_6, x_0 = 1.6$		
G_4	3	3.26e-226
G_6	3	7.54e-178
G_7	3	6.26e-271
MN	3	1.00e-349
$f_7, x_0 = 2$		
G_4	3	1.05e-279
G_6	3	1.58e-223
G_7	3	3.00e-320
MN	3	3.00e-350

Table 4.1.

5 CONCLUSION

From Table 4.1, we observe that our three-step iterative method is comparable with the methods defined in the paper of Jisheng Kou et al. [1] and in many cases gives better results in terms of the function evaluation $f(x_n)$. Moreover the computational efficiency of the algorithm 2.2 i.e. $8^{\frac{1}{5}} \simeq 1.515717$ is better than the efficiency of most of the other methods defined in the literature.

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