General Mathematics Vol. 19, No. 3 (2011), 25–30

A Generalized Integral Operator Associated with Functions of Bounded Boundary Rotation [1]

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Abstract

In this paper, we define the subclass $\mathcal{V}_k^\lambda(\beta, \delta, n)$ of analytic functions by using the generalized Al-Oboudi differential operator. We determine certain properties of the integral operator $I_n(f_1, \cdots, f_m)$ for the functions belonging to the class $\mathcal{V}_k^\lambda(\beta, \delta, n)$.

2000 Mathematics Subject Classification: 30C45.
Key words and phrases: Bounded boundary rotation, Analytic functions, Convex functions, Generalized integral operator.

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

defined in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ containing univalent functions defined in $\mathcal{U}$. Let $\mathcal{P}_k^\lambda(\beta)$ denote the class of analytic functions $p(z)$ defined in $\mathcal{U}$ satisfying the following properties

i. $p(0) = 1$.

ii. $\displaystyle\int_0^{2\pi} \left| \frac{\Re e^{i\lambda} p(z) - \beta \cos\lambda}{1-\beta} \right| d\theta \le k\pi \cos\lambda$

[1] *Received 03 September, 2009*
Accepted for publication (in revised form) 10 February, 2011

where $k \geq 2$, λ is real, $|\lambda| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $z = re^{i\theta}$, $0 \leq r < 1$.
Let $\mathcal{V}_k^\lambda(\beta)$ [7] denote the class of functions $f(z)$ analytic in $\mathcal{U}$ satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_k^\lambda(\beta)$$

where k, λ and β are as above.
For $\beta = 0$ we get the class $\mathcal{V}_k^\lambda$ of functions with bounded boundary rotation studied by Moulis [6].
Any function $f(z) \in \mathcal{V}_k^\lambda(\beta)$ if and only if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta\cos\lambda, \qquad |z| < \frac{k - \sqrt{k^2 - 4}}{2}.$$

A function $f \in \mathcal{U}$ with the normalization properties $f(0) = f'(0) - 1 = 0$ is said to be in the class $\mathcal{U}_k^\lambda(\beta)$ if $\dfrac{zf'(z)}{f(z)} \in \mathcal{P}_k^\lambda(\beta)$.
For $f \in \mathcal{A}$, *Sălăgean* [10] introduced the differential operator $D^n : A \longrightarrow A$, $n \in \mathbb{N}$ defined as

$$D^0 f(z) = f(z), \;\; D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)).$$

Al-Oboudi [2] generalized this operator by considering $D_\delta^n : A \longrightarrow A$, $n \in \mathbb{N}$, $\delta > 0$ defined by

$$D_\delta^0 f(z) = f(z)$$

$$D_\delta^1 = (1 - \delta) f(z) + \delta z f'(z) = D_\delta f(z)$$

$$D_\delta^n = D(D_\delta^{n-1} f(z)).$$

From the above definition, if f is of the form (1), we have

$$D_\delta^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \qquad n \in \mathbb{N}_0, \tag{2}$$

with $D_\delta^n f(0) = 0$.
Let $\mathcal{V}_k^\lambda(\beta, \delta, n)$ denote the class of functions $f(z)$ analytic in $\mathcal{U}$ with the normalization properties $f(0) = f'(0) - 1 = 0$ and

$$\frac{z(D_\delta^n f(z))'}{D_\delta^n f(z)} \in \mathcal{P}_k^\lambda(\beta)$$

where $k \geq 2$, λ is real, $|\lambda| < \dfrac{\pi}{2}$, $0 \leq \beta < 1$, $z = re^{i\theta}$, $0 \leq r < 1$.
For $\delta = 1$, $n = 1$, we get the class $\mathcal{V}_k^\lambda(\beta)$ studied by Moulis [7].

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Any function $f(z) \in \mathcal{V}_k^\lambda(\beta, \delta, n)$ if and only if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{z(D_\delta^n f(z))'}{D_\delta^n f(z)}\right)\right\} > \beta\cos\lambda, \qquad |z| < \frac{k - \sqrt{k^2-4}}{2}.$$

Let $n, m \in \mathbb{N}_0$ and $\alpha_i > 0$, $1 \le i \le m$. We define the integral operator $I_n : A^n \longrightarrow A$

$$I_n(f_1, \cdots, f_m)(z) = \int_0^z \left(\frac{D_\delta^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{D_\delta^n f_m(t)}{t}\right)^{\alpha_m} dt, \ z \in \mathcal{U}, \tag{3}$$

where $f_i \in \mathcal{A}$ and D_δ^n is the Al-Oboudi differential operator .

For parametric values of $n = 0$, $\delta = 1$ we have the integral operator

$$I_0(f_1, \cdots, f_m)(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_m(t)}{t}\right)^{\alpha_m} dt,$$

introduced in [4].

If $n = 0$, $\delta = 1$, $m = 1$, $\alpha_1 = \cdots = \alpha_m = 0$ and $D^0 f_1(z) = D^0 f(z) = f(z) \in \mathcal{A}$, we have the integral operator of Alexander $I_0(f)(z) = \int_0^z \frac{f(t)}{t} dt$ introduced in [1].

For $n = 0$, $\delta = 1$, $m = 1$, $\alpha_1 = \alpha \in [0, 1]$, $\alpha_2 = \cdots = \alpha_m = 0$ and $D^0 f_1(z) = D^0 f(z) = f(z) \in \mathcal{S}$, we have the integral operator $I(f)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$ studied in [9].

If $\alpha_i \in \mathbb{C}$ for $1 \le i \le m$, then we have the integral operator $I_n(f_1, \cdots, f_m)$ studied in [8].

2 Main Result

Theorem 1 *Let* $f_i \in \mathcal{V}_k^\lambda(\beta_i, \delta, n)$ *for* $1 \le i \le m$ *with* $0 \le \beta_i < 1$, *and* $n \in \mathbb{N}_0$, *also let* $\alpha_i > 0$, $1 \le i \le m$. *If* $\sum_{i=1}^m \alpha_i(1 - \beta_i) \le 1$, *then* $I_n(f_1, \cdots, f_m) \in \mathcal{V}_k^\lambda(\gamma)$, *with* $\gamma = 1 + \sum_{i=1}^m \alpha_i(\beta_i - 1)$.

Proof: From (2), for $1 \le i \le m$, we have

$$\frac{D_\delta^n f_i(z)}{z} = 1 + \sum_{j=2}^\infty [1 + (j-1)\delta]^n a_j z^{j-1}, \ n \in \mathbb{N}_0$$

and

$$\frac{D_\delta^n f_i(z)}{z} \neq 0, \ \ \forall z \in \mathcal{U}.$$

Consider,

$$I_n(f_i,\cdots,f_m)(z) = \int_0^z \left(\frac{D_\delta^n f_1(t)}{t}\right)^{\alpha_1} ... \left(\frac{D_\delta^n f_m(t)}{t}\right)^{\alpha_m} dt.$$

On successive differentiation of $I_n(f_1,\cdots,f_m)$, we get

$$I_n(f_1,\cdots,f_m)'(z) = \left(\frac{D_\delta^n f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{D_\delta^n f_m(z)}{z}\right)^{\alpha_m}$$

$$I_n(f_1,\cdots,f_m)''(z) =$$

$$\sum_{i=1}^m \alpha_i \left(\frac{D_\delta^n f_i(z)}{z}\right)^{\alpha_i-1} \frac{z(D_\delta^n f_i(z))' - D_\delta^n f_i(z)}{z^2} \prod_{j=1,\, j\neq i}^m \left(\frac{D_\delta^n f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{I_n(f_1,\cdots,f_m)''(z)}{I_n(f_1,\cdots,f_m)'(z)} = \sum_{i=1}^m \alpha_i \left[\frac{z(D_\delta^n f_i(z))}{D_\delta^n f_i(z)} - \frac{1}{z}\right].$$

Thus we obtain,

$$\frac{zI_n(f_1,\cdots,f_m)''(z)}{I_n(f_1,\cdots,f_m)'(z)} + 1 = \sum_{i=1}^m \alpha_i \left[\frac{z(D_\delta^n f_i(z))}{D_\delta^n f_i(z)}\right] - \sum_{i=1}^m \alpha_i + 1.$$

This relation is equivalent to

$$\Re\left\{e^{i\lambda}\left(\frac{zI_n(f_1,\cdots,f_m)''(z)}{I_n(f_1,\cdots,f_m)'(z)} + 1\right)\right\} = \sum_{i=1}^m \Re e^{i\lambda}\left\{\alpha_i \frac{z(D_\delta^n f_i(z))'}{D_\delta^n f_i(z)} - \sum_{i=1}^m \alpha_i\right\} + 1.$$

Since $f_i \in \mathcal{V}_\alpha^\lambda(\beta_i,\delta,n)$, we get

$$\Re\left\{e^{i\lambda}\left(\frac{zI_n(f_1,\cdots,f_m)''(z)}{I_n(f_1,\cdots,f_m)'(z)} + 1\right)\right\} > \sum_{i=1}^m \Re e^{i\lambda}\alpha_i - \sum_{i=1}^m \alpha_i + 1$$

$$1 + \sum_{i=1}^m \alpha_i(\beta_i - 1).$$

Hence $I_n(f_1,\cdots,f_m)(z) \in \mathcal{V}_k^\lambda(\gamma)$, where $\gamma = 1 + \sum_{i=1}^m \alpha_i(\beta_i - 1)$.

Corollary 1 *For parametric values* $n = 0,\ \delta = 1,\ \ k = 2,\ \lambda = 0,$ *we get the following result [3].*
Let $\alpha_i,\ i \in \{1, 2, ..., n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, 2, ..., n\}$ *and* $\sum_{i=1}^{n} \alpha_i \leq n + 1.$ *We suppose that the functions* $f_i,$
$i \in \{1, 2, ..., n\}$ *are the starlike functions of order* $\frac{1}{\alpha_i},\ i \in \{1, 2, ..., n\},$ *that is* $f_i \in \mathcal{S}^*\left(\frac{1}{\alpha_i}\right)$ *for all*
$i \in \{1, 2, ..., n\}.$ *Then the integral operator defined in (3) is convex.*

For $\beta_1 = \beta_2 = \cdots, = \beta_m = \beta,\ \delta = 1,$ and $n = 0,$ similarly we prove the following theorem.

Theorem 2 *Let* α_i *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, 2, ..., n\}$ *with* $\sum_{i=1}^{m} \alpha_i \leq 1.$ *We suppose that the function* $f_i \in \mathcal{V}_k^\lambda(\beta, 1, 0).$ *Then the integral operator defined in (3) belongs to* $\mathcal{V}_k^\lambda(\gamma),$ *where* $\gamma = 1 - \sum_{i=1}^{m} \alpha_i.$

Corollary 2 *For parametric values* $n = 0,\ \delta = 1,\ k = 2,\ \lambda = 0,$ *we get the following result [3].*
Let $\alpha_i,\ i \in \{1, 2, ..., n\}$ *be real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1, 2, ..., n\}$ *and* $\sum_{i=1}^{n} \alpha_i \leq 1.$ *We suppose that the functions* $f_i,$
$i \in \{1, 2, ..., n\}$ *are the starlike functions. Then the integral operator defined in (3) is convex by order* $1 - \sum_{i=1}^{n} \alpha_i.$

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Maths, 17, 1915, 12-22.

[2] AL-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Sci., 25-28, 2004, 1429-1436.

[3] D. Breaz, N. Breaz, *Some convexity properties for a general integral operator*, Journal of Inequalities in Pure and Applied Mathematics, vol. 7, no. 5, article 177, 2006.

[4] D. Breaz, N. Breaz, *Two integral operators*, Studia Univ. Babes-Bolyai Math., 47(3), 2002, 13-19.

[5] D. Breaz, S. Owa, N. Breaz, *A new integral univalent operator*, Acta Univ. Apulensis Math. Inform., 16, 2008, 11-16.

[6] E. J. Moulis, *A generalization of univalent functions with bounded boundary rotation*, Trans. Am. Math. Soc., 174, 369-381.

[7] E. J. Moulis, *Generaliztion of the Robertson functions*, Pacific J. Math., 81, 169-174.

[8] S. Bulut, *Sufficient conditions for univalence of an integral operator defined by Al-Oboudi differential operator*, J. Inequal. Appl., 2008, art. id 957042.

[9] S. S. Miller, P. T. Mocanu, M. O. Reade, *Starlike integral operator*, Pacific J. Math., 79(1), 1978, 157-168.

[10] G. S. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Mathe., Springer - Verlag, 2013, 1983, 362-372.

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