

## On certain classes of $\omega - p$ -valent functions <sup>1</sup>

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### Abstract

In this paper we study new subclasses of  $p$ -valent function. Coefficient inequalities, inclusion relation, and extreme points of these classes are studied. Furthermore, relevant connections of our classes with the existing ones through various choices of the parameters involved are also considered.

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## 1 Introduction

Let  $A_p$  denote the class of functions  $f(z)$  of the form

$$(1) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$

which are analytic and  $p$ -valent in the open unit disk  $U = \{z : |z| < 1\}$ .

A function  $f \in A_p$  is said to be  $p$ -valently starlike of order  $\beta$  if it satisfies the condition

$$Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad (0 \leq \beta < 1, z \in U)$$

and this class of functions are usually denoted by  $S_p^*(\beta)$ .

Also, a function  $f \in A_p$  is said to be  $p$ -valently convex of order  $\beta$  if it satisfies the condition

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta, \quad (0 \leq \beta < 1, z \in U)$$

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and this class of functions are usually denoted by  $K_p(\beta)$ .

The classes mentioned above have been severally investigated from different point of view as it could be seen in literatures.

In the recent time, Kanas and Ronning [1], precisely in 1995 introduced a subclass of our well known class of analytic functions and denote it by  $A(\omega)$  and is of the form

$$(2) \quad f(z) = (z - \omega) + \sum_{k=2}^{\infty} a_k (z - \omega)^k$$

which are normalized with  $f(\omega) = 0$  and  $f'(\omega) - 1 = 0$  and  $\omega$  is a fixed point in  $U$ . We also remark here that  $A(\omega) \subset A$  and  $S(\omega) \subset S$  the classes of analytic and univalent functions respectively.

Kanas and Ronning made use of these classes of analytic and univalent functions to define and studied the following classes of  $\omega$ -starlike and  $\omega$ -convex functions. These classes were defined as follows

$$ST(\omega) = S^*(\omega) = \left\{ f \in S(\omega) : \operatorname{Re} \frac{(z - \omega)f'(z)}{f(z)} > 0, z \in U \right\}$$

$$CV(\omega) = S^c(\omega) = \left\{ f \in S(\omega) : 1 + \operatorname{Re} \frac{(z - \omega)f''(z)}{f'(z)} > 0, z \in U \right\}$$

where  $\omega$  is a fixed point in  $U$ . They obtained several related univalent results [1].

After this, several other authors had further work in this direction, the likes of Acu and Owa [2], Oladipo [3,4], and Aouf et al [5] and they all obtained some interesting results.

The motivation for this paper is that the authors wish to extend these classes of functions to the domain of  $\omega - p$ -valent functions and see its relevant connections to our usual known classes of  $p$ -valent function.

Let  $\omega$  be a fixed point in  $U$ . We denote by  $A_p(\omega)$  the class of analytic functions of the form

$$(3) \quad f(z) = (z - \omega)^p + \sum_{k=p+1}^{\infty} a_k (z - \omega)^k$$

which are analytic and  $\omega - p$ -valent in the open unit disk.

It is easy to see that  $A_p(0) = A_p$ .

A function  $f \in A_p(\omega)$  is said to be  $\omega - p$ -valently starlike of order  $\beta$  if it satisfies the condition

$$(4) \quad \operatorname{Re} \frac{(z - \omega)f'(z)}{f(z)} > \beta, \quad (0 \leq \beta < 1, z \in U)$$

and this class of functions shall be denoted by  $S_p^*(\omega, \beta)$ .

Also a function  $f \in A_p(\omega)$  is said to be  $\omega - p$ -valently convex of order  $\beta$  if it satisfies the condition

$$(5) \quad 1 + \operatorname{Re} \frac{(z - \omega)f''(z)}{f'(z)} > \beta, \quad (0 \leq \beta < 1, z \in U)$$

and this class of functions shall be denoted by  $K_p(\omega, \beta)$ . We quickly mention here that the  $|\omega| = d$  see [1,2,3,4,14].

With the aid of  $\omega - p$ -modified Cătaş et al derivative operator the authors wish to give the following definition.

**Definition 1** Let  $\omega$  be a fixed point in  $U$ ,  $m \in N$ ,  $n \in N_0$ ,  $\lambda \geq 0$ ,  $l \geq 0$ ,  $\alpha \geq 0$ ,  $p \in N$ ,  $0 \leq \beta < p$ . Then the function  $f \in A_p(\omega)$  is said to be in the class  $T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$  if

$$(6) \quad \operatorname{Re} \left\{ \frac{I_{\omega,p}^m(\lambda, l)f(z)}{I_{\omega,p}^n(\lambda, l)f(z)} \right\} > \alpha \left| \frac{I_{\omega,p}^m(\lambda, l)f(z)}{I_{\omega,p}^n(\lambda, l)f(z)} - 1 \right| + \beta$$

where  $I_{\omega,p}^n(\lambda, l)f(z)$  is a derivative operator such that  $I_{\omega,p}^n(\lambda, l) : A_p(\omega) \rightarrow A_p(\omega)$  and it is defined as follows

$$(7) \quad I_{\omega,p}^0(\lambda, l)f(z) = f(z)$$

$$\begin{aligned} I_{\omega,p}^1(\lambda, l)f(z) &= I_{\omega,p}(\lambda, l)f(z) = I_{\omega,p}^0(\lambda, l)f(z) \left( \frac{1 - \lambda + l}{1 + l} \right) + (I_{\omega,p}^0(\lambda, l)f(z))' \frac{\lambda(z - \omega)^p}{1 + l} \\ &= \left( \frac{1 + \lambda(p - 1) + l}{1 + l} \right) (z - \omega)^p + \sum_{k=p+1}^{\infty} \left( \frac{1 + \lambda(k - 1) + l}{1 + l} \right) a_k (z - \omega)^k. \end{aligned}$$

And in general we have

$$(8) \quad \begin{aligned} I_{\omega,p}^n(\lambda, l)f(z) &= I_{\omega,p}(\lambda, l)(I_{\omega,p}^{n-1}(\lambda, l)f(z)) \\ &= \left( \frac{1 + \lambda(p - 1) + l}{1 + l} \right)^n (z - \omega)^p + \sum_{k=p+1}^{\infty} \left( \frac{1 + \lambda(k - 1) + l}{1 + l} \right)^n a_k (z - \omega)^k. \end{aligned}$$

With various choices of the parameters involved various new and existing classes and derivative operators by various authors could be derived. For example we have the following:

**Remark 1** (i) For  $\omega = 0$  we have  $f \in T_{m,n}(p, \alpha, \beta, \lambda, l, 0) = T_{m,n}(p, \alpha, \beta, \lambda, l)$  and

$$\operatorname{Re} \left\{ \frac{I_{0,p}^m(\lambda, l)f(z)}{I_{0,p}^n(\lambda, l)f(z)} \right\} > \alpha \left| \frac{I_{0,p}^m(\lambda, l)f(z)}{I_{0,p}^n(\lambda, l)f(z)} - 1 \right| + \beta$$

(ii) For  $\omega = 0, \lambda = 1, l = 0$  we have  $f \in T_{m,n}(p, \alpha, \beta, 1, 0, 0) = T_{m,n}(p, \alpha, \beta)$  and

$$\operatorname{Re} \left\{ \frac{I_{0,p}^m(1, 0)f(z)}{I_{0,p}^n(1, 0)f(z)} \right\} > \alpha \left| \frac{I_{0,p}^m(1, 0)f(z)}{I_{0,p}^n(1, 0)f(z)} - 1 \right| + \beta$$

(iii) For  $\omega = 0, \lambda = 1, l = 0$  and  $\alpha = 0$  we have  $f \in T_{m,n}(p, 0, \beta, 1, 0, 0) = T_{m,n}(p, \beta)$

$$\operatorname{Re} \left\{ \frac{I_{0,p}^m(1, 0)f(z)}{I_{0,p}^n(1, 0)f(z)} \right\} > \beta$$

(iv) For  $\omega = 0, \lambda = 1, l = 0$  and  $\alpha = 0, \beta = 0$  we have  $f \in T_{m,n}(p, 0, 0, 1, 0, 0) = T_{m,n}(p,)$

$$\operatorname{Re} \left\{ \frac{I_{0,p}^m(1, 0)f(z)}{I_{0,p}^n(1, 0)f(z)} \right\} > 0$$

(v) For  $p = 1, \lambda = 1, l = 0$  we have  $f \in T_{m,n}(1, \alpha, \beta, 1, 0, \omega) = T_{m,n}(\alpha, \beta, \omega)$

$$\operatorname{Re} \left\{ \frac{I_{\omega,1}^m(1, 0)f(z)}{I_{\omega,1}^n(1, 0)f(z)} \right\} > \alpha \left| \frac{I_{\omega,1}^m(1, 0)f(z)}{I_{\omega,1}^n(1, 0)f(z)} - 1 \right| + \beta$$

**Remark 2** (i) For  $p = 1$ , the operator  $I_{\omega,1}^n(\lambda, l) = I_{\omega}^n(\lambda, l)$  was introduced and studied by Aouf et al [5].

(ii) For  $p = 1, \lambda = 1, l = 0$ , operator  $D_{\omega}^m = I_{\omega,1}^n(1, 0)$  was introduced and studied by Acu and Owa [6].

(iii) For  $\omega = 0, p = 1$ , the operator  $I_{0,1}^n(\lambda, l) = I^n(\lambda, l)$  was introduced and studied by Cătaş et al [7].

(iv) For  $p = 1, \omega = 0, l = 0, \lambda \geq 0$ , the operator  $D_{\lambda}^n = I_{0,1}^n(\lambda, 0) = I^n(\lambda)$  was introduced and studied by Al-Oboudi [8].

(v) For  $\omega = 0, p = 1, \lambda = 1$ , the operator  $I^n(1, l)$  was recently studied by Cho and Kim [9], also by Cho and Srivastava [10].

(vi) For  $\omega = 0, p = 1, l = 0, \lambda = 1$ , the operator  $D^n = I^n(1, 0)$  was introduced and studied by Sălăgean [11].

(vii) For  $\omega = 0, p = 1, l = \lambda = 1$  the operator  $I^n(1, 1)$  was studied by Uralegaddi and Somanatha [12].

(viii) For  $\omega = 0, \lambda = 1, l = 0$  the operator  $D_p^n = I_{0,p}^n(1, 0)$  was studied by Akbulut et al [13].

Also let  $T_{m,n}^s(p, \alpha, \beta, \lambda, l, \omega)$  ( $s = 0, 1, 2, \dots$ ) be the subclass of  $A_p(\omega)$  consisting of functions  $f(z)$  which satisfy the condition

$$(9) \quad T_{m,n}^s(p, \alpha, \beta, \lambda, l, \omega) \Leftrightarrow I_{\omega,p}^s(\lambda, l)f(z) \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$$

## 2 Coefficient inequalities

**Theorem 1** If  $f(z) \in A_p(\omega)$  satisfies

$$(10) \quad \sum_{k=p+1}^{\infty} \Omega(m, n, k, l, r, d, \lambda, \alpha, \beta, p, \omega) |a_k| \leq 2(\tau^m - \beta\tau^n)$$

where

$$(11) \quad \begin{aligned} & \Omega(m, n, k, l, r, d, \lambda, \alpha, \beta, p, \omega) = \\ & = (r + d)^{k-p} [|\sigma^m - \sigma^n - \beta\sigma^n| + (\sigma^m - \sigma^n - \beta\sigma^n) + 2\alpha|\sigma^m - \sigma^n|] \end{aligned}$$

and

$$\begin{aligned} \sigma^m &= \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^m, & \sigma^n &= \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^n \\ \tau^m &= \left( \frac{1 + \lambda(p-1) + l}{1+l} \right)^m, & \tau^n &= \left( \frac{1 + \lambda(p-1) + l}{1+l} \right)^n \end{aligned}$$

$m \in N, n \in N_0, \lambda \geq 0, l \geq 0, k \geq 2, p \in N$  for some  $0 \leq \beta < p$ , then  $f \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$

**Proof.** Suppose that (10) is true for  $\beta(0 \leq \beta < p), \alpha \geq 0, l \geq 0, \lambda \geq 0, \alpha \geq 0$  and  $\omega$  is fixed in  $U$ . For  $f(z) \in A_p(\omega)$ , let us define the function  $F(z)$  by

$$F_{\omega,p}(z) = \frac{I_{\omega,p}^m(\lambda, l)f(z)}{I_{\omega,p}^n(\lambda, l)f(z)} - \alpha \left| \frac{I_{\omega,p}^m(\lambda, l)f(z)}{I_{\omega,p}^n(\lambda, l)f(z)} - 1 \right| - \beta.$$

It suffices to show that

$$\left| \frac{F_{\omega,p}(z) - 1}{F_{\omega,p}(z) + 1} \right| < 1, \quad z \in U$$

and  $\omega$  is a fixed point in  $U$ .

We note that

$$\left| \frac{F_{\omega,p}(z) - 1}{F_{\omega,p}(z) + 1} \right| =$$

$$\left| \frac{I_{\omega,p}^m(\lambda, l)f(z) - \alpha e^{i\theta} |I_{\omega,p}^m(\lambda, l)f(z) - I_{\omega,p}^n(\lambda, l)f(z)| - \beta I_{\omega,p}^n(\lambda, l)f(z) - I_{\omega,p}^n(\lambda, l)f(z)}{I_{\omega,p}^m(\lambda, l)f(z) - \alpha e^{i\theta} |I_{\omega,p}^m(\lambda, l)f(z) - I_{\omega,p}^n(\lambda, l)f(z)| - \beta I_{\omega,p}^n(\lambda, l)f(z) + I_{\omega,p}^n(\lambda, l)f(z)} \right|$$

Applying (8) to the above equation and transform the right hand side to become

$$\begin{aligned} & \left| \frac{F_{\omega,p}(z) - 1}{F_{\omega,p}(z) + 1} \right| = \\ & = \left| \frac{(\tau^m - \tau^n - \beta\tau^n) + \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n - \beta\sigma^n) a_k (z - \omega)^{k-p} - \alpha e^{i\theta} \left| (\tau^m - \tau^n) + \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n) a_k (z - \omega)^{k-p} \right|}{(\tau^m + \tau^n - \beta\tau^n) + \sum_{k=p+1}^{\infty} (\sigma^m + \sigma^n - \beta\sigma^n) a_k (z - \omega)^{k-p} - \alpha e^{i\theta} \left| (\tau^m - \tau^n) + \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n) a_k (z - \omega)^{k-p} \right|} \right| \leq \\ & \frac{(\tau^n + \beta\tau^n - \tau^m) + \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n - \beta\sigma^n| |a_k| |(z - \omega)|^{k-p} - \alpha |e|^{i\theta} (\tau^m - \tau^n) + \alpha |e|^{i\theta} \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| |a_k| |(z - \omega)|^{k-p}}{(\tau^m + \tau^n - \beta\tau^n) - \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n - \beta\sigma^n) |a_k| |(z - \omega)|^{k-p} - \alpha |e|^{i\theta} (\tau^m - \tau^n) - \alpha |e|^{i\theta} \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| |a_k| |(z - \omega)|^{k-p}} \\ & \leq \frac{(\beta\tau^n + \tau^n - \tau^m - \alpha\tau^m + \alpha\tau^n) + \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n - \beta\sigma^n| (r+d)^{k-p} |a_k| + \alpha \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| (r+d)^{k-p} |a_k|}{(\tau^m + \tau^n - \beta\tau^n - \alpha\tau^m + \alpha\tau^n) - \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n - \beta\sigma^n) (r+d)^{k-p} |a_k| - \alpha \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| (r+d)^{k-p} |a_k|} \end{aligned}$$

The last expression is bounded above by 1, if

$$\begin{aligned} & \beta\tau^n + \tau^n - \tau^m - \alpha\tau^m + \alpha\tau^n + \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n - \beta\sigma^n| (r+d)^{k-p} |a_k| + \alpha \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| (r+d)^{k-p} |a_k| \\ & \leq \tau^m + \tau^n - \beta\tau^n - \alpha\tau^m + \alpha\tau^n - \sum_{k=p+1}^{\infty} (\sigma^m - \sigma^n - \beta\sigma^n)(r+d)^{k-p} |a_k| - \alpha \sum_{k=p+1}^{\infty} |\sigma^m - \sigma^n| (r+d)^{k-p} |a_k| \end{aligned}$$

which is equivalent to our condition (10), where

$$\begin{aligned} \sigma^m &= \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^m, & \sigma^n &= \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^n \\ \tau^m &= \left( \frac{1 + \lambda(p-1) + l}{1+l} \right)^m, & \tau^n &= \left( \frac{1 + \lambda(p-1) + l}{1+l} \right)^n \end{aligned}$$

and we complete the proof of Theorem 1.

In view of Theorem 1 above we have the following.

On setting  $l = 0$  in Theorem 1 we have

**Corollary 1** *If  $f(z) \in A_p(\omega)$  satisfies*

$$\sum_{k=p+1}^{\infty} \Omega(m, n, k, 0, r, d, \lambda, \alpha, \beta, p, \omega) |a_k| \leq 2(\tau^m - \beta\tau^n)$$

where

$$\begin{aligned} & \Omega(m, n, k, 0, r, d, \lambda, \alpha, \beta, p, \omega) = \\ & = (r+d)^{k-p} [|\sigma^m - \sigma^n - \beta\sigma^n| + (\sigma^m - \sigma^n - \beta\sigma^n) + 2\alpha|\sigma^m - \sigma^n|] \end{aligned}$$

and

$$\begin{aligned} \sigma^m &= (1 + \lambda(k-1))^m, & \sigma^n &= (1 + \lambda(k-1))^n \\ \tau^m &= (1 + \lambda(p-1))^m, & \tau^n &= (1 + \lambda(p-1))^n \end{aligned}$$

for some  $0 \leq \beta < p$  and other parameter as earlier defined then  $f(z) \in T_{\omega,p}(p, \alpha, \beta, \lambda, 0, \omega)$ .

Also putting  $\lambda = 1, l = 0$  in Theorem 1 we have

**Corollary 2** *If  $f(z) \in A_p(\omega)$  satisfies*

$$\sum_{k=p+1}^{\infty} \Omega(m, n, k, 0, r, d, 1, \alpha, \beta, p, \omega) |a_k| \leq 2(p^m - \beta p^n)$$

where

$$\begin{aligned} & \Omega(m, n, k, 0, r, d, 1, \alpha, \beta, p, \omega) = \\ & = (r+d)^{k-p} [|k^m - k^n - \beta k^n| + (k^m - k^n - \beta k^n) + 2\alpha|k^m - k^n|] \end{aligned}$$

with every other parameter as earlier defined, then  $f(z) \in T_{m,n}(p, \alpha, \beta, 1, 0, \omega) \equiv T_{m,n}(p, \alpha, \beta, \omega)$ .

On setting  $m = 1, n = 0, l = 0, \lambda = 1$  in Theorem 1 we have

**Corollary 3** *If  $f(z) \in A_p(\omega)$  satisfies*

$$\sum_{k=p+1}^{\infty} \Omega(1, 0, k, 0, r, d, 1, \alpha, \beta, p, \omega) |a_k| \leq 2(p - \beta)$$

where

$$\Omega(1, 0, k, 0, r, d, 1, \alpha, \beta, p, \omega) = (r + d)^{k-p} [|k - \beta| + (k - \beta) + 2\alpha|k|]$$

then  $f(z) \in T_{1,0}(p, \alpha, \beta, 1, 0, \omega) \equiv T(p, \alpha, \beta, \omega)$ .

For  $m = 1, n = 0, l = 0, \lambda = 1, p = 1$  in Theorem 1 we have

**Corollary 4** *If  $f(z) \in A_1(\omega)$  satisfies*

$$\sum_{k=p+1}^{\infty} \Omega(1, 0, k, 0, r, d, 1, \alpha, \beta, 1, \omega) |a_k| \leq 2(1 - \beta)$$

where

$$\Omega(1, 0, k, 0, r, d, 1, \alpha, \beta, 1, \omega) = (r + d)^{k-1} [|k - \beta| + (k - \beta) + 2\alpha|k|]$$

then  $f \in T(\alpha, \beta, \omega)$ .

For various choices of the parameters involved various coefficient inequalities both new and the existing ones could be obtained.

**Theorem 2** *If  $f(z) \in A_p(\omega)$  satisfies*

$$(12) \quad \sum_{k=p+1}^{\infty} \sigma^s \Omega(m, n, k, l, r, d, \lambda, \alpha, \beta, p, \omega) |a_k| \leq 2(\tau^m - \beta\tau^n)$$

where

$$\Omega(m, n, k, l, r, d, \lambda, \alpha, \beta, p, \omega)$$

is as defined by (11) for some  $\beta(0 \leq \beta < 1, \alpha \geq 0, \lambda \geq 0, l \geq 0, m \in N, n \in N_0)$  then  $f(z) \in T_{m,n}^s(p, \alpha, \beta, \lambda, l, \omega)$

**Proof.** From

$$f(z) \in T_{m,n}^s(p, \alpha, \beta, \lambda, l, \omega) \Leftrightarrow I_{\omega,p}^s(\lambda, l)f(z) \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$$

replacing  $a_k$  by  $\sigma^s a_k$  in Theorem 1, we have the theorem.

### 3 Inclusion Relation for $T_{m,n}(p, \lambda, \alpha, \beta, l, \omega)$

In view of Theorem 1 and Theorem 2, we have

**Theorem 3**  $T_{m+1,n+1}(p, l, \lambda, \alpha, \beta, \omega) \subset T_{m,n}(p, l, \lambda, \alpha, \beta, \omega)$

**Proof.** From Theorem 1 we have that

$$(13) \quad \sum_{k=p+1}^{\infty} \Omega(m, n, k, \alpha, \beta, \lambda, l, r, d) a_k \leq \sum_{k=p+1}^{\infty} \Omega(m+1, n+1, k, \alpha, \beta, \lambda, l, r, d) a_k$$

Therefore, if  $f(z) \in T_{m+1,n+1}(p, \alpha, \beta, \lambda, l, \omega)$ , then  $f(z) \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$ . Hence we get the require result.

### 4 Extreme point

Now let us determine the extreme point for the class  $T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$ .

**Theorem 4** Let  $f_1(z) = (z - \omega)^p$  and

$$(14) \quad f_j(z) = (z - \omega)^p + \frac{2(\tau^m - \beta\tau^n)}{\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)} (z - \omega)^k, \quad k \geq 2$$

where  $\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)$  is as defined in (10). Then  $f(z) \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$ , if and only if it can be expressed in the form

$$(15) \quad f(z) = \sum_{k=p}^{\infty} \gamma_k f_k(z)$$

where

$$\gamma_k \geq 0, \quad \sum_{k=p}^{\infty} \gamma_k = 1$$

**Proof.** Suppose that

$$(16) \quad f(z) = \sum_{k=p}^{\infty} \gamma_k f_k(z) = (z - \omega)^p + \sum_{k=p+1}^{\infty} \gamma_k \frac{2(\tau^m - \beta\tau^n)}{\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)} (z - \omega)^k$$

Then

$$(17) \quad \begin{aligned} & \sum_{k=p+1}^{\infty} \Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d) \frac{2(\tau^m - \beta\tau^n)}{\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)} \gamma_k \\ &= \sum_{k=p+1}^{\infty} 2(\tau^m - \beta\tau^n) \gamma_k = 2(\tau^m - \beta\tau^n) \sum_{k=p+1}^{\infty} \gamma_k \end{aligned}$$



$$= 2(\tau^m - \beta\tau^n)(1 - \gamma_1) < 2(\tau^m - \beta\tau^n).$$

Then  $f(z) \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$  from the definition of the class  $T_{m,n}(p, l, \lambda, \alpha, \beta, \omega)$ . Conversely, suppose that  $f \in T_{m,n}(p, \alpha, \beta, \lambda, l, \omega)$ . Since

$$(18) \quad a_k \leq \frac{2(\tau^m - \beta\tau^n)}{\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)}, \quad k \geq p+1$$

we may set

$$(19) \quad \gamma_k = \frac{\Omega(m, n, p+1, \alpha, \beta, \lambda, l, r, d)}{2(\tau^m - \beta\tau^n)}$$

and

$$(20) \quad \gamma_1 = 1 - \sum_{k=p+1}^{\infty} \gamma_k$$

Then

$$(21) \quad f(z) = \sum_{k=p+1}^{\infty} \gamma_k f_k(z)$$

This complete the proof.

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