General Mathematics Vol. 18, No. 4 (2010), 43–51

A generalized family of quadrature based iterative methods [1]

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Abstract

In this paper, we present a family of iterative methods for solving non-linear equations as an application of integral inequalities. Thus, we give a new application of such inequalities other than their natural applications in Numerical integration and Special means. The family of two-step iterative methods presented in this paper recaptures many previous quadrature based iterative methods.

2010 Mathematics Subject Classification: 26D15; 26A18; 26C10; 34A34.

Key words and phrases: Integral inequalities, Quadrature based iterative methods, Two-step methods, Non-linear equations.

1 Introduction

Let us consider the equation

$$f(x) = 0, \tag{1}$$

where f is a real valued univariate non-linear function.

Locating zeros of such functions has been given much attention from several decades due to its importance in applied sciences. Newton's method is

[1] *Received 25 November, 2008*

Accepted for publication (in revised form) 23 May, 2010

the most widely used quadratically convergent iterative method in solving such problems; yet in the recent past many other efficient iterative methods for solving non-linear equations have appeared in the literature by the use of Taylor's series, interpolating polynomials, decomposition techniques and quadrature formulae.

The connection of quadrature formulae and iterative methods has already been established by Weerakoon and Fernando in [17] by using the indefinite integral representation of Newton's method [5] to obtain quadrature based iterative methods. The trend continued with the publication of the papers by Nedzhibov [10], Hasanov et al. [9] and Frontini and Sormani [8]. However, this domain is addressed only for classical quadrature rules e.g., trapezoid, mid-point, Simpson's, etc. Ujević in [15, 16], however, adopted a quite different approach by using specially derived quadrature rule, infact the equivalence of two quadrature rules to re-establish this connection and to obtain quadrature based iterative predictor-corrector type methods for solving non-linear equations.

The applications of mathematical inequalities, particularly inequalities of Ostrowski, Grüss and Čebyšev type have already been explored by S. S. Dragomir, N.S. Barnett, P. Cerone, Th. M. Rassias and S. Wang, etc., in Numerical integration, Special means and Probability theory, see e.g., [2, 4, 6, 7]. We, however, by using the approach of Weerakoon and Fernando [17] give some new applications of such inequalities to obtain iterative methods for solving non-linear equations. We, in this paper, thus establish the fact that the specially derived quadrature rules developed in the sense of inequalities may be applied to develop many other iterative methods.

Moreover, it is shown that the family of two-step iterative methods thus established has third-order convergence and it recaptures many previously presented quadrature based iterative methods.

2 A generalized family of two-step iterative methods

Consider the following family of quadrature rules derived in the sense of inequalities [1]:

Theorem 1 *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be mapping differentiable in the interior $Int\ I$ of I, and let $a, b \in Int\ I$, $a < b$. If there exists some constants $\gamma, \Gamma \in \mathbb{R}$, such that $\gamma \le f'(t) \le \Gamma$, $\forall\ t \in [a,b]$ and $f' \in L_1(a,b)$, then we have:*

$$(2)\quad \left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}\left(1-h^2\right)(b-a)(S-\gamma)$$

and

$$(3)\quad \left|(1-h)\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}\left(1-h^2\right)(b-a)(\Gamma-S)$$

where $S = \frac{f(b)-f(a)}{b-a}$, $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$.

Moreover, in [18], we have derived the following inequality:

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function whose first derivative $f' \in L_2[a,b]$ and $\gamma \le f'(t) \le \Gamma$ almost everywhere t on (a,b). Then, we have the inequality:*

$$(4)\quad \left|(1-h)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]+h\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\le \left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \times \left[\frac{1}{b-a}\left\|f'\right\|_2^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}$$

$$\le \frac{1}{2}(\Gamma-\gamma)\left[\frac{(b-a)^2}{12}\left(3h^2-3h+1\right)+h(1-h)\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$.

Remark 1 *It may be noted that for* $x = \frac{a+b}{2}$ *and for* $h \in [0,1]$ *the left hand sides of* (1), (3) *and* (4) *give the following family of quadrature rule:*

$$\int_a^b f(t)\,dt = (b-a)\left[(1-h)\,f\left(\frac{a+b}{2}\right) + h\frac{f(a)+f(b)}{2}\right] + R(f)\,, \tag{5}$$

which is a combination of mid-point and trapezoid rule.

We proceed with the indefinite integral representation of Newton's method [5]:

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \tag{6}$$

Now approximating the integral in (6) with the quadrature rule (5), we obtain:

$$\int_{x_n}^{x} f'(t)\,dt = (x-x_n)\left[(1-h)\,f'\left(\frac{x_n+x}{2}\right) + h\frac{f'(x_n)+f'(x)}{2}\right]. \tag{7}$$

Using the approximation (7) in (6) implies

$$-2f(x_n) = (x-x_n)\left[2(1-h)\,f'\left(\frac{x_n+x}{2}\right) + h\left(f'(x_n)+f'(x)\right)\right]$$

which finally results into the following implicit method:

$$x = x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+x}{2}\right) + h\left(f'(x_n)+f'(x)\right)}.$$

This implies

$$x_{n+1} = x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+y_n}{2}\right) + h\left(f'(x_n)+f'(y_n)\right)}, \tag{8}$$

where y_n is some explicit method.

If we choose y_n as Newton's method in (8), then we have the following two-step method:

$$\begin{aligned} x_{n+1} &= x_n - \frac{2f(x_n)}{2(1-h)\,f'\left(\frac{x_n+y_n}{2}\right) + h\left(f'(x_n)+f'(y_n)\right)}, \\ y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{9}$$

or

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{10}$$

$$z_n = x_n - \frac{f(x_n)}{2f'(x_n)}, \tag{11}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{2(1-h)f'(z_n) + h(f'(x_n) + f'(y_n))}. \tag{12}$$

We, now, compute the order of convergence of algorithm (9) using Maple 7.0 and is given in the form of the following theorem:

Theorem 3 *Let $w \in I$ be a simple zero of sufficiently differentiable function $f : I \sqsubseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to w, then the algorithm (9) is cubically convergent for all $h \in [0,1]$.*

Proof. Let w be a simple zero of f and $x_n = w + e_n$ with an error e_n. By Taylor's expansion, we have:

$$f(x_n) = f^{'}(w)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6\right) + O(e_n^7) \tag{13}$$

$$f^{'}(x_n) = f^{'}(w)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5\right) + O(e_n^6), \tag{14}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(w)}{f^{'}(w)}, k = 2, 3, ...\text{and } e_n = x_n - w. \tag{15}$$

Using (13) and (14), we have

$$\frac{f(x_n)}{f^{'}(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 3c_4 - 4c_2^3\right)e_n^4 + O(e_n^5). \tag{16}$$

Using (16) in (10), we obtain

$$y_n = w + c_2e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + O\left(e_n^5\right). \tag{17}$$

Expanding $f(y_n)$ and $f^{'}(y_n)$ by Taylor's series about w, we have:

$$f(y_n) = f(w) + (y_n - w)f^{'}(w) + \frac{(y_n - w)^2}{2!}f^{''}(w) + \frac{(y_n - w)^3}{3!}f^{'''}(w) + ...$$

$$(18)\quad f(y_n) = f'(w)\left(c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 3c_4 + 5c_2^3)e_n^4\right) + O\left(e_n^5\right).$$

$$f'(y_n) = f'(w) + (y_n - w)\, f''(w) + \frac{(y_n - w)^2}{2!} f'''(w) + ...$$

$$(19)\qquad \begin{aligned} f'(y_n) &= f'(w)\,(1 + 2c_2^2e_n^2 + \left(-4c_2^3 + 4c_2c_3\right)e_n^3 \\ &\quad +(-11c_3c_2^2 + 8c_2^4 + 6c_2c_4)e_n^4) + O(e_n^5). \end{aligned}$$

Also by using (16) in (11), we have

$$(20)\qquad \begin{aligned} z_n &= w + \frac{1}{2}e_n + \frac{1}{2}c_2e_n^2 + \left(-c_2^2 + c_3\right)e_n^3 \\ &\quad + \left(\frac{3}{2}c_4 - \frac{7}{2}c_2c_3 + 2c_2^3\right)e_n^4 + O\left(e_n^5\right). \end{aligned}$$

In the similar manner, expanding $f(z_n)$ and $f'(z_n)$ by Taylor's series about w, we have:

$$(21)\qquad \begin{aligned} f(z_n) &= f'(w)\left(\frac{1}{2}e_n + \frac{3}{4}c_2e_n^2 + \left(-\frac{1}{2}c_2^2 + \frac{9}{8}c_3\right)e_n^3\right. \\ &\quad + \left(\frac{5}{4}c_2^3 - \frac{17}{8}c_2c_3 + \frac{25}{16}c_4\right)e_n^4 \\ &\quad \left. + \left(-3c_2^4 + \frac{57}{8}c_3c_2^2 - \frac{9}{4}c_3^2 - \frac{13}{4}c_2c_4 + \frac{65}{32}c_5\right)e_n^5\right) + O(e_n^6). \end{aligned}$$

$$(22)\qquad \begin{aligned} f'(z_n) &= f'(w)(1 + c_2e_n + \left(c_2^2 + \frac{3}{4}c_3\right)e_n^2 + (-2c_2^3 + \frac{7}{2}c_2c_3 + \frac{1}{2}c_4)e_n^3 \\ &\quad + \left(\frac{9}{2}c_2c_4 + c_2^4 - \frac{37}{4}c_2^2c_3 + 3c_3^2 + \frac{5}{16}c_5\right)e_n^4) + O\left(e_n^5\right). \end{aligned}$$

Using (14), (19) and (22) in

$$(23)\qquad \begin{aligned} \frac{2f(x_n)}{2\,(1-h)\,f'(z_n) + h\,(f'(x_n) + f'(y_n))} &= e_n + (\frac{1}{4}\,(1 - 3h)\,c_3 - c_2^2)e_n^3 \\ +(3c_2^3 + \frac{3}{4}\,(3h - 5)\,c_2c_3 + \frac{1}{2}\,(1 - 3h)\,c_4)e_n^4 + O\left(e_n^5\right). \end{aligned}$$

Therefore, by using (23) in (12), we have:

$$x_{n+1} = w + (c_2^2 - \frac{1}{4}\,(1 - 3h)\,c_3)e_n^3 + O\left(e_n^4\right).$$

Hence, we obtain:

$$e_{n+1} = (c_2^2 - \frac{1}{4}(1-3h)\,c_3)e_n^3 + O\left(e_n^4\right).$$

Thus, we observe that the method is cubically convergent for all $h \in [0,1]$.

Remark 2 *It is clear from Theorem 3 that algorithm (9) is cubically convergent and*

1. *For* $h = 1$, *it recaptures the trapezoid Newton's method given by Weerakoon and Fernando in [17].*

2. *For* $h = 0$, *it recaptures the midpoint Newton's method given by Özban in [13] and by Frontini et. al. in [8].*

3. *For* $h = \frac{1}{3}$, *it recaptures the Simpson Newton's method given by Hasanov et. al. in [9].*

4. *For* $h = \frac{1}{2}$, *it recaptures the averaged trapezoid mid-point Newton's method given by Nedzhibov in [10] and latter by Noor in [11].*

Remark 3 *The computational efficiency of the algorithm* (9) *is less than the computational efficiency of the Newton's method except for the cases for which* $h = 0$ *and* $h = 1$*. However, the implicit method* (8) *can be used in combination with the other known methods to increase the convergence order and computational efficiency.*

References

[1] A. Rafiq, F. Zafar, *New bounds for the first inequality of Ostrowski-Grüss type and applications in numerical integration*, Nonlinear Funct. Anal. Appl., 12 (1), 2007, 75-85.

[2] N. S. Barnett, P. Cerone, S. S. Dragomir, *Inequalities for random variables over a finite interval*, Nova Science Publishers, in press, Preprint.

[3] R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS publishing company, Boston USA, 2001.

[4] S. S. Dragomir, Th. M. Rassias, *Ostrowski type inequalities and applications in numerical integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[5] J. E. Dennis, R. B. Schnable, *Numerical methods of unconstrained optimization and non-linear equations*, Prentice Hall, 1983.

[6] S. S. Dragomir, S. Wang, *Applications of Ostrowski inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett. 11, 1998, 105–109.

[7] S. S. Dragomir, S. Wang, *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. 33, 1997, 16–20.

[8] M. Frontini, E. Sormani, *Some variants of Newton's method with third-order convergence*, Appl. Math. Comput. 140, 2003, 419-426.

[9] V. I. Hasanov, I. G. Ivanov, G. Nedzhibov, *A new modification of Newton's method*, Applications of Mathematics in Engineering, 27, 2002, 278-286.

[10] G. Nedzhibov, *On a few iterative methods for solving nonlinear equations*, Application of Mathematics in Engineering and Economics, in: Proceedings of the XXVIII Summer School Sozopol 2002, Heron Press, Sofia, 2002.

[11] M. A. Noor, *New iterative methods for nonlinear equations*, Appl. Math. Comput., 2006, in press.

[12] A.M. Ostrowski, *Solution of equations in Euclidean and Banach space*, Academic Press, New York, 1973.

[13] A. Y. Özban, *Some new variants of Newton's method*, Appl. Math. Lett.,17, 2004, 677-682.

[14] J. F. Traub, *Iterative methods for solution of equations*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[15] N. Ujević, *A method for solving non-linear equations*, Appl. Math. Comput. 174, 2006, 1416-1426.

[16] N. Ujević, *An iterative method for solving non-linear equations*, J. of Comput. and Appl. Math. 201, 2007, 208 – 216.

[17] S. Weerakoon, T. G. I. Fernando, *A variant of Newton's method with accelerated third order convergence*, Appl. Math. Lett. 13, 2000, 87-93.

[18] F. Zafar, N. A. Mir, *A generalization of Ostrowski-Grüss type inequality for first differentiable mappings*, Tamsui Oxford J. Math. Sci., 26(1), 2010, 61-76.

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