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Certain subclasses of meromorphic functions using convolution and subordination [1]

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Abstract

In this paper, we define certain subclasses of meromorphic functions using convolution and differential subordination and investigate various inclusion properties of these subclasses.

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1 Introduction

Let M denote the class of functions f defined by

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the punctured unit disc $E=\{z:0<|z|<1\}$, and having a simple pole at the origin with residue unity. Let Σ will denote the subclass of M consisting of univalent functions.

Given two function f and g, which are anlytic in the open unit disc Δ, the function f is said to be subordinate to g, written as $f\prec g$ or $f(z)\prec g(z)$ $(z\in\Delta)$, if there exists a Schwarz function $w(z)$ analytic in Δ, with $w(0)=0$ and $|w(z)|<1$ and such that $f(z)=g(w(z))$.

If g is univalent in Δ, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta)$.

The convolution (or Hadamard Product) of two power series $f(z)$, given by (1) and $g(z)=\frac{1}{z}+\sum_{n=0}^{\infty}b_nz^n$ is defined by

$$(f*g)(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nb_nz^n.$$

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In this paper, unless otherwise stated $g \in M$ be arbitrary but a fixed function and $\mathcal{H}$ be the class of all functions h which are analytic and univalent in Δ and for which $h(\Delta)$ is convex with $h(0) = 1$ and $Re\{h(z)\} > 0$ $(z \in \Delta)$.

Let $M_g^s(h)$ denote the class of functions $f \in M$ such that

$$-\frac{z(g * f)'(z)}{(g * f)(z)} \prec h(z)$$

where $(g * f)(z) \neq 0$ for $z \in \Delta$.

Let $M_g^k(h)$ denote the class of functions $f \in M$ such that

$$-\left\{1 + \frac{z(g * f)''(z)}{(g * f)'(z)}\right\} \prec h(z)$$

where $(g * f)'(z) \neq 0$ for $z \in \Delta$.

Let $M_g^c(h)$ denote the class of functions $f \in M$ such that

$$-\frac{z(g * f)'(z)}{(g * \phi)(z)} \prec h(z),$$

for some $\phi \in M_g^s(h)$.

Now, we define our new subclasses of meromorphic functions using convolution and subordination.

Definition 1 *Let $M_g(\alpha; h)$ denote the class of functions $f \in M$ such that*

$$-\frac{\alpha z(z(g * f)'(z))' + (1 - \alpha)z(g * f)'(z)}{\alpha z(g * f)'(z) + (1 - \alpha)(g * f)(z)} \prec h(z)$$

where $(g * f)'(z)(g * f)(z) \neq 0$, $\alpha \geq 0$.

Definition 2 *Let $MR_g(\alpha; h)$ denote the class of functions $f \in M$ such that*

$$-\frac{\alpha z(z(g * f)'(z))' + (1 - \alpha)z(g * f)'(z)}{\alpha z(g * \phi)'(z) + (1 - \alpha)(g * \phi)(z)} \prec h(z)$$

for some $\phi \in M_g(\alpha; h)$ and $\alpha \geq 0$.

We note that the class $M_g(0; h) = M_g^s(h)$, $M_g(1; h) = M_g^k(h)$. For $\alpha = 1$, $g(z) = \frac{1}{z(1-z)}$ and $h(z) = \frac{1+(1-2\beta)z}{1-z}$, the class $M_g(\alpha; h)$ reduces to well known class of meromorphic convex univalent function of order β $(0 \leq \beta < 1)$ in Δ.

Also $M_{\frac{1}{z(1-z)}}\left(0, \frac{1+(1-2\beta)z}{1-z}\right)$ is the well known class of meromorphic starlike univalent function of order β $(0 \leq \beta < 1)$ in Δ.

2 Preliminary Result

In proving our main results, we need the following lemmas.

Lemma 1 *[2] Let $\beta, \gamma \in \mathcal{C}$ and $h \in H(\Delta)$ be convex univalent in Δ with $h(0) = 1$ and $Re(\beta h(z) + \gamma) > 0$ in Δ, and let*

$$p(z) = 1 + p_1 z + \cdots \in H(\Delta).$$

Then $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.

Lemma 2 *[3] Let $\beta, \gamma \in \mathcal{C}$, $h \in H(\Delta)$ be convex univalent in Δ with $h(0) = 1$ and $Re(\beta h(z) + \gamma) > 0$ in Δ. Let $q \in H(\Delta)$ with $q(0) = 1$ and $q(z) \prec h(z)$ in Δ. If $p(z) = 1 + p_1 z + \cdots \in H(\Delta)$ then $p(z) + \dfrac{zp'(z)}{\beta q(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.*

Lemma 3 *[1, p. 248] Let $\phi \in K$, the class of convex univalent functions, $g \in S^*$ and $F \in H(\Delta)$ such that $ReF > 0$. Then $\frac{(\phi * Fg)}{(\phi * g)}$ lies in the convex hull of $F(E)$.*

Several inclusion properties for the subclasses of analytic functions using convolution and subordination were discussed in [3, 4, 5, 6]. Motivated by the above mentioned work, in this paper, we investigate several inclusion properties of the classes $M_g(\alpha; h)$ and $MR_g(\alpha; h)$. Some applications involving integral operators are also considered.

3 Inclusion Results

Theorem 1 *Let $h \in \mathcal{H}$ with $\max\limits_{z \in \Delta} (Re\{h(z)\}) < \frac{1}{\alpha} - 1$, $(0 \leq \alpha < 1)$. Then $M_g(\alpha; h) \subseteq M_g(0, h) = M_g^s(h)$.*

Proof. Let $f \in M_g(\alpha; h)$ and $p(z) = -\dfrac{z(g*f)'(z)}{(g*f)(z)}$. Then

$$\begin{aligned}
&\alpha z(z(g*f)'(z))' + (1-\alpha) z(g*f)'(z) \\
&\quad = -\alpha z((g*f)(z)p'(z) + (g*f)'(z)p(z)) - (1-\alpha)p(z)(g*f)(z) \\
&\quad = -(\alpha zp'(z) + p(z)((1-\alpha) - \alpha p(z)))(g*f)(z),
\end{aligned}$$

and

$$\alpha z(g*f)'(z) + (1-\alpha)(g*f)'(z) = (-\alpha p(z) + (1-\alpha))(g*f)(z).$$

Hence

$$\begin{aligned}
&-\frac{\alpha z(z(g*f)'(z))' + (1-\alpha)z(g*f)'(z)}{\alpha z(g*f)(z) + (1-\alpha)(g*f)(z)} \\
&= \frac{\alpha zp'(z) + p(z)((1-\alpha) - \alpha p(z))}{(1-\alpha) - \alpha p(z)} \\
&= p(z) + \frac{zp'(z)}{-p(z) + (\alpha^{-1} - 1)}.
\end{aligned}$$

Since $f \in M_g(\alpha; h)$, we conclude by Lemma 1 that $-p(z) \prec h(z)$, for $Re(-h(z) + \frac{1}{\alpha} - 1) > 0$ in Δ, which implies $f \in M_g^s(h)$.

Theorem 2 *Let $f \in M_g(\alpha; h)$. Then for $0 \le \alpha < 1$, we have $F \in M_g(\alpha; h)$, where*

$$F(z) = \frac{z^{1-1/\alpha}}{\alpha} \int_0^z t^{\frac{1}{\alpha-2}} f(t)dt. \tag{2}$$

Proof. From (2), we get

$$\alpha z F'(z) + (1-\alpha)F(z) = f(z).$$

This, by convolution with $g(z)$, gives

$$\alpha(g * zF')(z) + (1-\alpha)(g * F)(z) = (g * f)(z).$$

Using the fact that $(g * zF')(z) = z(g * F)'(z)$, we have

$$\alpha z(g * F)'(z) + (1-\alpha)(g * F)(z) = (g * f)(z). \tag{3}$$

Taking logarithmic derivative with respect to z and multiplying by $-z$, we get

$$-\frac{\alpha z(z(g * F)'(z))' + (1-\alpha)z(g * F)'(z)}{\alpha z(g * F)'(z) + (1-\alpha)(g * F)(z)} = -\frac{z(g * f)'(z)}{(g * f)(z)}. \tag{4}$$

Since $f \in M_g(\alpha; h) \subseteq M_g^s(h)$ (by Theorem 1), we have

$$-\frac{z(g * f)'(z)}{(g * f)(z)} \prec h(z),$$

which implies $F \in M_g(\alpha; h)$.

Theorem 3 *$f \in M_g(\alpha; h)$ iff $\psi = \lambda z f' + (1-\lambda) f \in M_g^s(h)$.*

Proof. Using the fact that

$$z(g * f)' = g * zf', \qquad z^2(g * f)'' = g * z^2 f''$$

and the property of Hadamard product, the proof follows immediately.

Theorem 4 *If $\phi \in K$, the set of all convex function and $f \in M_g(\alpha; h)$ then $\phi * f \in M_g(\alpha; h)$.*

Proof. From Theorem 3, if $\phi \in K$ and $f \in M_g(\alpha; h)$ then
$\psi = \alpha z f' + (1-\alpha) f \in M_g^s(h)$.

Let $F = -\dfrac{\alpha z(z(g*f)'(z))' + (1-\alpha)z(g*f)'(z)}{\alpha z(g*f)'(z) + (1-\alpha)(g*f)(z)}$, so that $F \prec h$.
Now,

$$\begin{aligned}
&-\frac{\alpha z(z(g*\phi*f)'(z))' + (1-\alpha)z(g*\phi*f)'(z)}{\alpha z(g*\phi*f)'(z) + (1-\alpha)(g*\phi*f)(z)} \\
&\quad = -\frac{\alpha z(z(\phi*(g*f))'(z))' + (1-\alpha)z(\phi*(g*f))'(z)}{\alpha z(\phi*(g*f))'(z) + (1-\alpha)(\phi*(g*f))(z)} \\
&\quad = -\frac{(\phi*[\alpha z(z(g*f)'(z))' + (1-\alpha)z(g*f)'])(z)}{(\phi*[\alpha z(g*f)'(z) + (1-\alpha)(g*f)(z)} \\
&\quad = \frac{(\phi*[F\alpha z(g*f)' + (1-\alpha)(f*g)])(z)}{(\phi*[\alpha z(g*f)'(z) + (1-\alpha)(f*g)])(z)} \\
&\quad = \frac{\phi*F(\psi*g)}{\phi*(\psi*g)}.
\end{aligned}$$

Since $\psi \in M_g^s(h)$, $g*\psi \in MS^*$. Thus, from Lemma 3, we have $\phi*f \in M_g(\alpha;h)$.

Theorem 5 *$M_g(\alpha;h) \subseteq M_{\phi*g}(\alpha;h)$ for every $\phi \in K$.*

Proof. Let $f \in M_g(\alpha;h)$, then by Theorem 4, we have
$\phi*f \in M_g(\alpha;h)$. Hence

$$-\frac{\alpha z(z(g*(\phi*f))'(z))' + (1-\alpha)z(g*(\phi*f))'(z)}{\alpha z(g*(\phi*f))'(z) + (1-\alpha)(g*(\phi*f))(z)} \prec h(z)$$

in Δ, which implies $f \in M_{\phi*g}(\alpha;h)$.

Next, we prove an inclusion relation for the class $MR_g(\alpha;h)$.

Theorem 6 *Let $h \in \mathcal{H}$ with $\max_{z\in\Delta}(Re\{h(z)\}) < \frac{1}{\alpha} - 1$, $0 \le \alpha < 1$. Then, $MR_g(\alpha:h) \subset MR_g(0;h) = M_g^C(h)$.*

Proof. Let $f \in M_g(\alpha;h)$. By setting

$$p(z) = -\frac{z(g*f)'(z)}{(g*\phi)(z)} \quad \text{and} \quad q(z) = -\frac{z(g*\phi)'(z)}{(g*\phi)(z)},$$

we have

$$\begin{aligned}
&-\frac{\alpha z(z(g*f)'(z))' + (1-\alpha)z(g*f)'(z)}{\alpha z(g*\phi)'(z) + (1-\alpha)(g*\phi)(z)} \\
&\quad = -\frac{\alpha z\left(-p'(z) - p(z)\frac{(g*\phi)'(z)}{(g*\phi)(z)}\right) - (1-\alpha)p(z)}{\alpha z\frac{(g*\phi)'(z)}{(g*\phi)(z)} + (1-\alpha)} \\
&\quad = \frac{\alpha z p'(z) + p(z)\left(\alpha z\frac{(g*\phi)'(z)}{(g*\phi)(z)}\right) + (1-\alpha)p(z)}{\alpha z\frac{(g*\phi)'(z)}{(g*\phi)(z)} + (1-\alpha)} \\
&\quad = p(z) + \frac{zp'(z)}{-q(z) + \left(\frac{1}{\alpha} - 1\right)} \prec h(z),
\end{aligned}$$

since $f \in MR_g(\alpha; h)$. Here, by Theorem 6, $q(z) \prec h(z)$.

Now, by an application of Lemma 2, we have $p(z) \prec h(z)$, which completes Theorem 6.

Theorem 7 *If $f \in MR_g(\alpha; h)$ then $F \in MR_g(\alpha; h)$ where F is as defined in (2), for $0 \leq \alpha < 1$.*

Proof. Differentiating F as defined in (2) with respect to z, we have

$$\alpha z F'(z) + (1-\alpha)F(z) = f(z).$$

This on convolution with $g(z)$ yields

$$\alpha z (g * F)'(z) + (1-\alpha)(g * F)(z) = (g * f)(z),$$

where we used the identity $(g * zF')(z) = z(g * F)'(z)$.
Again differentiating with respect to z we get

$$-\left[\alpha z (z(g * F)'(z))' + (1-\alpha) z (g * F)'(z)\right] = -z(g * f)'(z). \tag{5}$$

Since $f \in MR_g(\alpha; h)$, there exist a $\phi \in M_g(\alpha; h)$ such that

$$\frac{\alpha z (z(g * f)'(z))' + (1-\alpha) z (g * f)'(z)}{\alpha z (g * \phi)'(z) + (1-\alpha)(g * \phi)(z)} \prec h(z).$$

Then, by Theorem 2, ψ defined by

$$\psi(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha - 2}} \phi(t) dt \tag{6}$$

is in $M_g(\alpha; h)$.

Differentiating (4) with respect to z and convoluting the result with $g(z)$ yields after simplification

$$\alpha z (g * \psi)'(z) + (1-\alpha)(g * \psi)(z) = (g * \phi)(z) \tag{7}$$

Finally (3) and (5) together yield,

$$-\frac{\alpha z (z(g * F)'(z))' + (1-\alpha)(g * F)'(z)}{\alpha z (z(g * \psi)'(z))' + (1-\alpha) z (g * \psi)'(z)} = -\frac{z(g * f)'(z)}{(g * \psi)(z)}.$$

Since, $f \in MR_g(\alpha; h)$, then by Theorem 6, we have

$$-\frac{z(g * f)'(z)}{(g * \psi)(z)} \prec h(z).$$

Thus, we get $F \in MR_g(\alpha; h)$.

Finally, we define a new class $MI_g(\alpha; h)$.

Definition 3 *Let $MI_g(\alpha; h)$ denote the class of functions $f \in M$ such that*

$$J_g(\alpha; f(z), \phi(z)) = \frac{\alpha(z(g*f)'(z))'}{(g*\phi)'(z)} + (1-\alpha)\frac{z(g*f)'(z)}{(g*\phi)(z)} \prec h(z)$$

for some $\phi \in M_g^s(h)$ and $\alpha \geq 0$ in Δ.

Remark 1 *When $\alpha = 0$, $MI_g(\alpha; h)$ coincides with $M_g^c(\alpha; h)$. In particular if $\phi(z)$ is coincide with $f(z)$ then $MI_g(\alpha, h)$ is nothing but $M_g^s(h)$.*

Theorem 8

1. *If $f \in MI_g(\alpha; h)$ then $f \in MI_g(0, h) = M_g^c(\alpha; h)$ for $\alpha < 0$.*
2. *for $\alpha < \beta \leq 0$, $MI_g(\alpha; h) \subset MI_g(\beta; h)$.*

Proof.

1. Let $f \in MI_g(\alpha; h)$.
   By setting
   $$p(z) = -\frac{z(g*f)'(z)}{(g*\phi)(z)} \quad \text{and} \quad q(z) = -\frac{z(g*\phi)'(z)}{(g*\phi)(z)},$$
   we have
   $$\begin{aligned} -&\left(\frac{\alpha(z(g*f)'(z))'}{(g*\phi)'(z)} + (1-\alpha)\frac{z(g*f)'(z)}{(g*\phi)(z)}\right) \\ &= \alpha\left(\frac{zp'(z)}{q(z)} + p(z)\right) + (1-\alpha)p(z) \\ &= p(z) + \frac{zp'(z)}{q(z)}. \end{aligned}$$
   Since $f \in MI_g(\alpha; h)$, we have $p(z) + \frac{\alpha zp'(z)}{q(z)} \prec h(z)$ and $q(z) \prec h(z)$. Now an application of Lemma 2 yields that $p(z) \prec h(z)$, which implies $f \in M_g^c(\alpha; h)$.
2. If $\beta = 0$, then this part reduces to part (i). Therefore, we assume that $\beta \neq 0$.
   Suppose $f \in MI_g(\alpha; h)$. Then $J_g(\alpha; f(z), \phi(z)) \prec h(z)$.
   Let z_1 be arbitrary point in Δ. Then
   $$J_g(\alpha; f(z_1), \phi(z_1)) \in h(\Delta).$$
   Also, by part (i), we have $-\frac{z(g*f)'(z)}{(g*\phi)(z)} \prec h(z)$ and so
   $-\frac{z_1(g*f)'(z_1)}{(g*\phi)(z_1)} \in h(\Delta)$.

Now,

$$J_g(\beta; f(z), \phi(z))$$
$$= \left(1 - \frac{\beta}{\alpha}\right)\left(-\frac{z(g*f)'(z)}{(g*\phi)(z)}\right) + \frac{\beta}{\alpha}\left(J_g(\alpha; f(z), \phi(z))\right).$$

Since $0 < \frac{\beta}{\alpha} < 1$ and $h(\Delta)$ is convex, we have

$$J_g(\beta; f(z_1), \phi(z_1)) \in h(\Delta).$$

Therefore $J_g(\beta; f(z), \phi(z)) \prec h(z)$, that is $f \in MI_g(\beta; f(z), \phi(z))$, which implies

$$MI_g(\alpha; h) \subseteq MI_g(\beta; h).$$

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