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Riemannian Submersion from $\mathbb{S}^7-$ Sphere [1]

Hakan Mete Taştan

Abstract

In this paper, we construct an almost quaternion structure which is integrable in the horizontal bundle of the Riemannian submersion $\pi : \mathbb{S}^7 \to \mathbb{S}^4(\frac{1}{2})$.

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1 Preliminaries

Let M and B be smooth Riemannian manifolds. A Riemannian submersion $\pi : M \to B$ is a mapping of M onto B satisfying the following axioms;

S1. π has maximal rank; that is, each derivative map π_* of π is onto. Hence, for each $q \in B$, $\pi^{-1}(q)$ is a submanifold of M of dimension $dimM - dimB$ where the submanifolds $\pi^{-1}(q)$ are called $fibers$ of M. A vector field on M is called *vertical* if it is tangent to a fiber and *horizontal* if orthogonal to in the fiber.

S2. π_* preserves lengths of horizontal vectors; that is, the isomorphism

$$\pi_{*p} : ker(\pi_{*p})^{\perp} \to T_qB$$

is an isometry, where T_qB is tangent space of B at q and $p \in \pi^{-1}(q)$.

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For a Riemannian submersion $\pi : M \to B$, let $\mathcal{V}$ and $\mathcal{H}$ denote the projections of the tangent spaces of M onto the subspaces of vertical and horizontal vectors, respectively. The letters U, V, W will always denote vertical vector fields, and X, Y, Z horizontal vector fields. Following O'Neill [8] we define the tensor T of type $(1, 2)$ for arbitrary vector fields E and F by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

where $\mathcal{V}E, \mathcal{H}E$, etc, denote the vertical and horizontal projections of the vector field E. We denote the set of all vector fields on M by $\mathcal{X}(M)$, the set of vertical vector fields by $\mathcal{V}\mathcal{X}(M)$ and the set of horizontal vector fields by $\mathcal{H}\mathcal{X}(M)$. O'Neill has described the following three properties of the tensor T:

(1) T_E is a skew-symmetric linear operator on the tangent space of M and reverses horizontal and vertical subspaces.

(2) $T_E = T_{\mathcal{V}E}$, that is, T is vertical.

(3) For vertical vector fields V and W, T is symmetric, i.e., $T_V W = T_W V$.

In fact, along a fiber, T is the second fundamental form of the fiber provided we restrict ourselves to vertical vector fields.

Now, we simply dualize the definition of T by reversing $\mathcal{V}$ and $\mathcal{H}$ define the integrability tensor A as follows.

For arbitrary vector fields E and F,

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

(1′) A_E is a skew-symmetric operator on $\mathcal{X}(M)$ reversing the horizontal and vertical subspaces.

(2′) $A_E = A_{\mathcal{H}E}$, that is, A is horizontal.

(3′) For horizontal vector fields X, Y the tensor A is alternating, ie., $A_X Y = -A_Y X$.

Definition 1 *A basic vector field is a horizontal vector field X which is π related to a vector field X_* on B, i.e., $\pi_*(X_p) = X_{*\pi(p)}$ for all $p \in M$.*

Lemma 1 *If X and Y are basic vector fields on M which are π-related to X_* and Y_* respectively on B. Then*

1. $< X, Y > = < X_*, Y_* >^* \circ\pi$, where $<,>$ is the metric on M, and $<,>^*$ the metric on B.

2. $\mathcal{H}[X, Y]$ is basic and is π-related to $[X_*, Y_*]$

3. $\mathcal{H}\nabla_X Y$ is basic and is π-related to $\nabla^*_{X_*} Y_*$, where ∇^* is the Riemannian connection on B.

Lemma 2 *Let X and Y be horizontal vector fields, V and W be vertical vector fields. Then each of the following holds:*

1. $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$.

2. $\nabla_V W = T_V W + \hat{\nabla}_V W$, *where* $\hat{\nabla}$ *denotes the Riemannian connection along a fiber with respect to the induced metric.*

3. a) $\nabla_V X = \mathcal{H}\nabla_V X + T_V X$,

b) If X *is basic,* $\mathcal{H}\nabla_V X = A_X V$.

4. $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$.

5. $\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y$.

The proofs of these results are found in O'Neill [8] and R.H. Escobales [1].

Let $\mathcal{R}$ denote the curvature tensor of M, and $\mathcal{R}^*$ the curvature tensor of B. Since there is no danger of ambiguity, we will denote the horizontal lift of $\mathcal{R}^*$ by $\mathcal{R}^*$ as well. Following O'Neill [8] we set $< \mathcal{R}^*_{h_1 h_2} h_3, h_4 >=< \mathcal{R}^*_{h_{1^*} h_{2^*}} h_{3^*}, h_{4^*} >^*$ where h_i are horizontal vectors and $\pi_*(h_i) = h_{i^*}$.

For E and F, linearly independent vectors, we denote the tangent plane spanned by these two vectors by $P_E F$. In general, if X is a horizontal vector, then $\pi_* X$ is denoted by X_*. K, K_* and $\hat{K}$ will denote the sectional curvature of M, B and the fiber $\pi^{-1}(q)$, respectively.

Theorem 1 *Let* $\pi : M \to B$ *be a Riemannian submersion.*

(a) Then for horizontal vector fields X, Y, Z *and* H

$$< \mathcal{R}^*_{XY} Z, H >=$$

$$< \mathcal{R}_{XY} Z, H > +2 < A_X Y, A_Z H > - < A_Y Z, A_X H > - < A_Z X, A_Y H >,$$

(b) If X *and* Y *are horizontal, and* V *and* W *are vertical vector fields, then*

$$< \mathcal{R}_{XV} Y, W >=< (\nabla_X T)_V W, Y > + < (\nabla_V A)_X Y, W > - < T_V X, T_W Y >$$

$$+ < A_X V, A_Y W > .$$

Corollary 1 *Let* $\pi : M \to B$ *be a Riemannian submersion. Then for linearly independent horizontal vector fields* X *and* Y *and linearly independent vertical vector fields* V *and* W *we obtain the following relations:*

$$(a) \qquad K(P_V W) = \hat{K}(P_V W) - \frac{< T_V V, T_W W > - < T_V W, T_V W >}{< V, V >< W, W > - < V, W >^2},$$

$$(b) \qquad K(P_X V) = \frac{< (\nabla_X T)_V V, X > + < A_X V, A_X V > - < T_V X, T_V X >}{< X, X >< V, V >},$$

$$(c) \qquad K_*(P_{X_*} Y_*) = K(P_X Y) + \frac{3 < A_X Y, A_X Y >}{< X, X >< Y, Y > - < X, Y >^2}.$$

Corollary 2 *Let* $\pi : M \to B$ *be a Riemannian submersion with totally geodesic fibers. Then for orthonormal horizontal vector fields* X *and* Y *and a vertical vector field* V *of unit length*

(a) $K(P_X V) =< A_X V, A_X V >$,

(b) $K_*(P_{X_*} Y_*) = K(P_X Y) + 3 < A_X Y, A_X Y >$.

Proof. Immediate from Corollary 1.

Corollary 3 *Let* $\pi : M \to B$ *be a Riemannian submersion with totally geodesic fibers. If* X *and* Y *are horizontal, and* V *and* W *are vertical, then;*

$$< (\nabla_V A)_X Y, W > + < (\nabla_W A)_X Y, V >= 0.$$

Proof. See R.H. Escobales [2].

Definition 2 *Let* M *be an* n*-dimensional manifold with a 3-dimensional vector bundle* N *consisting of tensors of type* $(1,1)$ *over* M *such that in any coordinate neighborhood* $\mathcal{U}$ *of* M*, there is a local basis* $\{F, G, H\}$ *of* N *with*

$$F^2 = -I, \quad G^2 = -I, \quad H^2 = -I,$$

(1)

$$GH = -HG = F, \quad HF = -FH = G, \quad FG = -GF = H,$$

where, I *denotes the identity tensor of type* $(1,1)$ *in* M*.*

Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle N in $\mathcal{U}$. The bundle N is called an *almost quaternion structure* in M, and M with N an *almost quaternion manifold*, which will be denoted by (M, N). An almost quaternion manifold M is of dimension $n = 4m (m \geq 1)$.

Definition 3 *Let* (M, N) *be an almost quaternion manifold with a canonical local basis of* N *in a coordinate neighborhood* $\mathcal{U}$*. We now assume that there exists a system of coordinates* (x^l) *in each* $\mathcal{U}$ *with respect to which* F, G *and* H *have components of the form*

(2) $$F = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix}$$

where E denotes the identity $(m \times m)$-matrix. In such a case, the given almost quaternion structure N is said to be integrable.

2 Main Results

Let us consider the following Riemannian submersion

$$\pi : \mathbb{S}^7 \to \mathbb{S}^4(\frac{1}{2})$$

where $\mathbb{S}^7$ is 7-dimensional unit sphere and $\mathbb{S}^4(\frac{1}{2})$ is 4-dimensional sphere with radius $\frac{1}{2}$. The existence of π is guaranteed by [1, Theorem 3.5]. Also, in [1] Escobales shows that this Riemannian submersion π has connected totally geodesic fibers and also the fibers are equal to the 3-dimensional unit sphere $\mathbb{S}^3$.The fibers $\mathbb{S}^3$ are both totally geodesic and totally umbilical submanifolds of $\mathbb{S}^7$.

Let $\{U, V, W\}$ be an orthonormal basis field on the fiber $\mathbb{S}^3$, i.e. $\{U, V, W\} \in \mathcal{VX}(\mathbb{S}^7)$. Since $\mathbb{S}^3$ is both totally geodesic and totally umbilical submanifold of $\mathbb{S}^7$ we can assume that orthonormal vertical vector fields U, V and W are geodesic vector fields. We fix these vector fields and define three mappings i, j and k on $\mathcal{HX}(\mathbb{S}^7)$ as follows

$$i, j, k : \mathcal{HX}(\mathbb{S}^7) \to \mathcal{HX}(\mathbb{S}^7)$$

are defined by $i(X) = A_X U$, $j(X) = A_X V$, and $k(X) = A_X W$ for any horizontal vector field X. We see that these mappings are all tensor fields of type $(1, 1)$.

Lemma 3 *The tensors i, j and k which are defined above are skew-symmetric with respect to metric on $\mathbb{S}^7$.*

Proof. We will only prove one since the proofs of others are similar. For any horizontal vector fields X and Y,

(3)
$$0 = Y < X, U > = < \nabla_Y X, U > + < X, \nabla_Y U > = < A_Y X, U > + < X, A_Y U > \ ,$$

where we use Lemma 2-(5). Interchanging X and Y we have

(4)
$$0 = < A_X Y, U > + < Y, A_X U > \ .$$

Since $A_X Y = -A_Y X$, (3) and (4) yields
$< A_Y U, X > + < A_X U, Y > = 0$, that is, $< i(X), Y > = - < i(Y), X >$. Hence the proof completes.

Lemma 4 *The tensors i, j and k are all isometries with respect to the metric on $\mathbb{S}^7$.*

Proof. Again, we will only prove one since the proofs of others are similar. From the Corollary 1-(b) we have

$$K(P_X V) = \frac{<(\nabla_X T)_V V, X> + <A_X V, A_X V> - <T_V X, T_V X>}{<X, X><V, V>}$$

Since the fiber $\mathbb{S}^3$ is totally geodesic, T is identically zero. On the other hand, we know that $K(P_X V) = 1$, so we have

$$1 = \frac{<A_X V, A_X V>}{<X, X>}$$

Hence we obtain $<A_X V, A_X V> = <X, X>$, that is, $\|j(X)\|^2 = \|X\|^2 \Rightarrow \|j(X)\| = \|X\|$. Which means that j is an isometry.

Lemma 5 *Let X be any horizontal vector field on $\mathcal{HX}(\mathbb{S}^7)$. Then the set $\{X, i(X), j(X), k(X)\}$ is an orthogonal basis on $\mathcal{HX}(\mathbb{S}^7)$.*

Proof. $<X, i(X)> = <X, A_X U> = -<U, A_X X> = 0$, since $A_X X = 0$. Similarly,

$$<X, j(X)> = <X, k(X)> = 0 \ . \tag{5}$$

$<j(X), k(X)> = <A_X V, A_X W>$. From the Theorem 1-(b) we have
$<\mathcal{R}_{XV} Y, W> = <(\nabla_V A)_X Y, W> + <A_X V, A_Y W>$, since $T \equiv 0$.
Put $Y = X$ in this equation,
$\mathcal{R}_{XV} X = 1.\{<X, X> V - <X, V> X\} = \|X\|^2 V$, since
$<X, V> = 0$, thus
$<\mathcal{R}_{XV} X, W> = \|X\|^2 <V, W> = 0$, since $<V, W> = 0$.
On the other hand, we have from Corollary 3, $<(\nabla_V A)_X X, W> = 0$. Thus we get $<A_X V, A_X W> = 0$. Similarly,

$$<i(X), j(X)> = <i(X), k(X)> = 0 \ . \tag{6}$$

From the (5) and (6) the result follows.

Lemma 6 $i^2 = -I, \quad j^2 = -I, \quad k^2 = -I,$ *where I denotes the identity tensor of type $(1,1)$ in $\mathbb{S}^7$.*

Proof. We will only prove one since the proofs of others are similar. We must show that $i^2(X) = -X$, for any horizontal vector field X.

Let Y be an arbitrary horizontal vector field, since i is skew-symmetric we write $<i^2(X), Y> = <i(i(X)), Y> = -<i(X), i(Y)>$. On the other hand we know that i is an isometry, thus we have
$-<i(X), i(Y)> = -<X, Y>$. Finally, $<i^2(X), Y> = -<X, Y>$. From this, we deduce that $i^2(X) = -X$.

Lemma 7 $ij = -ji, \quad ik = -ki, \quad jk = -kj.$

Proof. We will only prove one since the proofs of others are similar. For example, $ij = -ji$. We must show that $ij(X) = -ji(X)$ for any horizontal vector field X.

Let Y be an arbitrary horizontal vector field, since i is skew-symmetric we can write

$< ij(X), Y >=< i(A_XV), Y >= - < A_XV, i(Y) >= - < A_XV, A_YU >$. Here we use Theorem 1-(b) and $T \equiv 0$ we get,

$< \mathcal{R}_{XV}Y, U >=< (\nabla_V A)_X Y, U > + < A_XV, A_YU >$. But

$\mathcal{R}_{XV}Y = 1.\{< X, Y > V - < V, Y > X\} =< X, Y > V$,

hence $< \mathcal{R}_{XV}Y, U >=< X, Y >< V, U >= 0$, since $< V, U >= 0$.

Thus we have, $- < A_XV, A_YU >=< (\nabla_V A)_X Y, U >$.

On the other hand we have from Corollary 3

$< (\nabla_V A)_X Y, U >= - < (\nabla_U A)_X Y, V >$. Therefore we obtain

$< ij(X), Y >= - < (\nabla_U A)_X Y, V >$, this is equal to $< A_XU, A_YV >$, using properties of A $(1')$ and $(3')$, we get

$$< A_XU, A_YV >= - < A_YA_XU, V >=< A_{A_XU}Y, V >$$
$$= - < A_{A_XU}V, Y >= - < ji(X), Y > .$$

From this, we deduce that $ij(X) = -ji(X)$.

Lemma 8 $ij = k$ *or* $ij = -k.$

Proof. We must show that $ij(X) = k(X)$ or $ij(X) = -k(X)$ for any $X \in \mathcal{HX}(\mathbb{S}^7)$. If $X = 0$, then it is clear. Therefore we can assume $X \neq 0$. Since $ij(X) \in \mathcal{HX}(\mathbb{S}^7)$ and $\{X, i(X), j(X), k(X)\}$ is an orthogonal basis on $\mathcal{HX}(\mathbb{S}^7)$, we can write

$ij(X) = \lambda_1 X + \lambda_2 i(X) + \lambda_3 j(X) + \lambda_4 k(X)$, where $\lambda_a (1 \leq a \leq 4)$ are real-valued functions on fiber $\mathbb{S}^3$, where,

$\lambda_1 = \frac{<ij(X),X>}{<X,X>}, \quad \lambda_2 = \frac{<ij(X),i(X)>}{<i(X),i(X)>}, \quad \lambda_3 = \frac{<ij(X),j(X)>}{<j(X),j(X)>}, \quad \lambda_4 = \frac{<ij(X),k(X)>}{<k(X),k(X)>}$

$$< ij(X), X >=< A_{A_XV}U, X >= - < A_XU, A_XV >$$
$$= - < i(X), j(X) >= 0, \text{ so}$$

$$< ij(X), X >= 0 \quad . \tag{7}$$

$< ij(X), i(X) >=< A_{A_XV}U, A_XU >= A_{A_XV} < U, A_XU > - < U, A_{A_XV}A_XU >$ $=< U, A_{A_XV}A_XU >$, since $< U, A_XU >= 0$. From the Theorem 1-(b) we can write

$< U, A_{A_XV}A_XU >=< \mathcal{R}_{(A_XV)U}X, U > - < (\nabla_U A)_{A_XV}X, U >$. Where, since

$\mathcal{R}_{(A_XV)U}X = 1.\{< X, A_XV > U - < X, U > A_XV\} = 0$, we obtain

$< ij(X), i(X) >= - < (\nabla_U A)_{A_XV}X, U >$. But from the Corollary 3 we deduce that

$- < (\nabla_U A)_{A_XV}X, U >= 0$, so

$$< ij(X), i(X) >= 0 \quad . \tag{8}$$

Similarly,

$$< ij(X), j(X) >= 0 \ . \tag{9}$$

Now, we compute $< ij(X), k(X) >$.

$< ij(X), k(X) >=< A_{A_X V}U, A_X W >= - < A_{A_X V}A_X W, U >$, since $A_{A_X V}A_X W$ is vertical, we can write $A_{A_X V}A_X W = \alpha U + \beta V + \gamma W$, where α, β and γ are real-valued functions on $\mathbb{S}^3$.

$< A_{A_X V}A_X W, V >= - < A_{A_X V}V, A_X W >= - < j^2(X), k(X) >$
$=< X, k(X) >= 0.$

Similarly,

$< A_{A_X V}A_X W, W >=< X, j(X) >= 0$. Thus we have

$A_{A_X V}A_X W = \alpha U$. Let put $A_X V = Y$ and $A_X W = Z$. Then from the Corollary 2-(c) we get

$\frac{3<A_Y Z, A_Y Z>}{<Y,Y><Z,Z>-<Y,Z>^2} = K_*(P_{Y_*}Z_*) - K(P_Y Z) = 4 - 1 = 3$

$\Rightarrow < A_Y Z, A_Y Z >= \|Y\|^2\|Z\|^2$,

since $< Y, Z >=< A_X V, A_X W >=< j(X), k(X) >= 0$.

On the other hand $\|Y\|^2 =< A_X V, A_X V >=< X, X >= \|X\|^2$ and

$\|Z\|^2 =< A_X W, A_X W >=< X, X >= \|X\|^2$, so

$< A_Y Z, A_Y Z >= \|X\|^4$.

$\|X\|^4 =< A_Y Z, A_Y Z >=< \alpha U, \alpha U >= \alpha^2 < U, U >= \alpha^2.1 = \alpha^2$, so $\alpha = \pm\|X\|^2$. Hence,

$< ij(X), k(X) >= - < A_{A_X V}A_X W, U >$
$= - < \pm\|X\|^2 U, U >= \mp\|X\|^2$. Finally

$$\lambda_4 = \frac{<ij(X),k(X)>}{<k(X),k(X)>} = \frac{\mp\|X\|^2}{\|X\|^2} = \mp 1 \ . \tag{10}$$

From the $(7), (8)$ and (9), we get $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and from the (10), $\lambda_4 = \mp 1$. Thus we have the required result.

Remark 1 *In case* $ij = k$, *we get* $ijk = kk = k^2 = -I \Rightarrow ijk = -I$.

If $ij = -k$, *then* $jik = -ijk = -(-k)k = k^2 = -I \Rightarrow jik = -I$.

In each case, we order i, j *and* k. *such that their triple multiplication is equal to* $-I$. *Therefore, there will be no matter if we admit* $ijk = -I$, *so we have*
$ij = -ji = k, \quad ki = -ki = j, \quad jk = -kj = i.$

Let us denote by N the subspace of $\mathcal{HX}(\mathbb{S}^7)$ spanned by $\{i(X), j(X), k(X)\}$. By using Lemma 6, Lemma 7, Lemma 8 and Remark 1 we have the following.

Theorem 2 *N is an almost quaternion structure in $\mathcal{HX}(\mathbb{S}^7)$.*

Corollary 4 *Given almost quaternion structure in Theorem above N is integrable.*

Proof. We must verify (2) in the Definition 3. For any $X \in \mathcal{HX}(\mathbb{S}^7)$, using Lemma 5 and Remark 1, we can write

$$\begin{aligned} i(X) &= 0.X + 1.i(X) + 0.j(X) + 0.k(X) \\ i(i(X)) &= -1.X + 0.i(X) + 0.j(X) + 0.k(X) \\ i(j(X)) &= 0.X + 0.i(X) + 0.j(X) + 1.k(X) \\ i(k(X)) &= 0.X + 0.i(X) - 1.j(X) + 0.k(X) \end{aligned}$$

Thus the matrix of i,

$$[i]_{4\times 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Similarly, the matrix of j,

$$[j]_{4\times 4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and the matrix of k,

$$[k]_{4\times 4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which means that N is integrable.

References

[1] R. H. Escobales, Jr., *Riemannian submersion with totally geodesic fibers*, J. Differential Geometry, 10, 1975, 253-276.

[2] R. H. Escobales, Jr., *The integrability tensor for bundle-like foliations*, Transactions of the American Mathematical Society, 270, 1982, 333-339.

[3] A. Gray, *Pseudo-Riemannian almost product manifols and submersion*, J. Math. Mech., 16, 1967, 715-737.

[4] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Intersience, New York, 1963.

[5] K. Yano, M. Kon, *Structures on manifolds*, World Scientific Publishing Co. Rte. Ltd., 1984.

[6] A.M. Magid, *Submersions from anti-de sitter space with totally geodesic fibers*, J. Differential Geometry, 16, 1981, 323-331.

[7] B. O'Neill, *Semi-Riemannian Geometry with applications to Relativity*, Academic Press, 1983.

[8] B. O'Neill, *The Fundamental equations of a submersion*, Michigan Math. J., 13, 1966, 459-469.

[9] A. Ranjan, *Riemannian submersions of spheres with totally geodesic fibers*, Osaka J. Math., 22, 1985, 243-260.

Hakan Mete Taştan
Istanbul University
Department of Mathematics
Vezneciler 34134, Istanbul, Turkey
e-mail: hakmete@istanbul.edu.tr