

Common fixed point results in b-cone metric spaces over topological vector spaces ¹

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Abstract

In this paper we introduce the notion of b-cone metric space over topological vector space (for short *b-TVS cone metric space*) and prove some common fixed point theorems in complete b-cone metric spaces over topological vector spaces. These results generalize, extend and unify several well-known recent related results in literature.

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1 Introduction

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity. In 1976, Jungck [9], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. This theorem has many applications but suffers from one drawback - the results require the continuity of one of the two maps involved. Sessa [20] introduced the notion of weakly commuting maps. Jungck [10] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts.

Fixed point theory in K-metric and K-normed spaces was developed by A. I. Perov and his consortiums (see [13], [15], [16]). The main idea consists to use an ordered Banach space instead of the set of real numbers, as the codomain for a

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metric. For more details on fixed point theory in K -metric and K -normed spaces, we refer the reader to [21]. Without mentioning these previous works, Huang and Zhang [8] reintroduced such spaces under the name of cone metric spaces but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. After that, fixed point results in cone metric spaces have been studied by many other authors. References [1], [2], [17], [18], [19] are some works in this line of research. However, very recently W.-S. Du in [7] uses the scalarization function and investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that many of the fixed point results in ordered K -metric spaces for maps satisfying contractive conditions of a linear type in K -metric spaces can be considered as the corollaries of corresponding theorems in metric spaces.

The concept of b -metric space appeared in some works, such as I. A. Bakhtin [3], S. Czerwik [6]. Several papers deal with the fixed point theory for singlevalued and multivalued operators in b -metric spaces (see [4], [14]).

The aim of present paper is to introduce the notion of *b-cone metric space over topological vector space* (for short *b-TVS cone metric space*) and to extend the results of Abbas and Jungck [1] at such space. To set up our results in the next sections we recall some definitions and facts (see [5], [7]).

Let E be a topological vector space (for short *TVS*) with its zero vector θ_E .

Definition 1 *A subset K of E is called a cone if:*

- (i) K is closed, nonempty and $K \neq \{\theta_E\}$;
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in K$ imply $ax + by \in K$;
- (iii) $K \cap -K = \{\theta_E\}$.

For a given cone $K \subset E$, we can define a partial ordering \leq_K with respect to K by

$$(1) \quad x \leq_K y \text{ if and only if } y - x \in K.$$

We shall write $x <_K y$ to indicate that $x \leq_K y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}K$ (interior of K).

In the following, unless otherwise specified, we always suppose that Y is a locally convex Hausdorff TVS with its zero vector θ , K a cone in Y with $\text{int}K \neq \emptyset$, $e \in \text{int}K$ and \leq_K a partial ordering with respect to K . The nonlinear scalarization function $\xi_e : Y \rightarrow \mathbb{R}$ is defined as follows

$$\xi_e(y) = \inf\{r \in \mathbb{R} \mid y \in r \cdot e - K\}.$$

Lemma 1 (see [5]) *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

- (i) $\xi_e(y) \leq r$ if and only if $y \in r \cdot e - K$;
- (ii) $\xi_e(y) > r$ if and only if y is not in $r \cdot e - K$;
- (iii) $\xi_e(y) \geq r$ if and only if y is not in $r \cdot e - \text{int}K$;
- (iv) $\xi_e(y) < r$ if and only if $y \in r \cdot e - \text{int}K$;
- (v) $\xi_e(\cdot)$ is positively homogeneous and continuous on Y ;
- (vi) if $y_1 \in y_2 + K$ then $\xi_e(y_2) \leq \xi_e(y_1)$;
- (vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, for all $y_1, y_2 \in Y$.

Now, we introduce the concepts of *b-TVS cone metric* and *b-TVS cone metric space* by

Definition 2 Let X be a nonempty set. Suppose that a mapping $d : X \times X \rightarrow Y$ satisfies:

- (i) $\theta \leq_K d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (iii) there exists $a \geq 1$ such that

$$d(x, y) \leq_K a \cdot (d(x, z) + d(z, y)),$$

for all $x, y, z \in X$.

Then d is called a *b-TVS cone metric* on X and (X, d) is called a *b-TVS cone metric space*.

Example 1 Any TVS cone metric space is a b-TVS cone metric space.

Example 2 Let L_p ($0 < p < 1$) be the space of all real functions $x(t)$, $t \in [0, 1]$, such that

$$\int_0^1 |x(t)|^p dt < \infty,$$

$P = \{(x, y) \in E \mid x, y \geq 0\} \subset \mathbb{R}^2$ and $\alpha \geq 0$. Then

$$d : L_p \times L_p \rightarrow \mathbb{R}^2,$$

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}, \alpha \left\{ \int_0^1 |x(t) - y(t)|^p dt \right\}^{\frac{1}{p}}$$

is a *b-TVS cone metric* on L_p , with $a = 2^p$.

Definition 3 Let $\{x_n\}$ be a sequence in X and $x \in X$.

- (i) x_n **b-TVS cone converges** to x whenever for every $0 \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$, for all $n > N$. We denote this by $x_n \xrightarrow{b\text{-cone}} x$;
- (ii) x_n is a **b-TVS cone Cauchy sequence** whenever for every $c \in E$, $0 \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $m, n > N$;
- (iii) (X, d) is **b-TVS complete** if every b-TVS cone Cauchy sequence is b-TVS cone convergent.

Definition 4 [1] Let f and g be self maps of a set X . If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g .

Jungck [11] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points.

Proposition 1 [1] Let f and g be weakly compatible self maps of a set X . If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .

2 Main results

In this section we obtain several coincidence and common fixed point theorems for mappings defined on a b-TVS cone metric space. Follows the idea of W.-S. Du we can prove the following

Theorem 1 Let (X, d) be a b-TVS cone metric space. Then $D : X \times X \rightarrow \mathbb{R}$ defined by $D = \xi_e \circ d$ is a b-metric.

Proof. Since $\theta \leq_K d(x, y)$, we have that $d(x, y) \notin -intK$ for all $x, y \in X$. By Lemma 1 (iii) it follows that $D(x, y) \geq 0$ for all $x, y \in X$. If $D(x, y) = 0$ then from Lemma 1 (i) we have that $d(x, y) \in -K \cap K = \{\theta\}$ i.e. $x = y$. Conversely, if $x = y$ then $d(x, y) = \theta$. Hence, $D(x, y) = \xi_e(\theta) = 0$. Also, it is obvious that $D(x, y) = D(y, x)$. Since

$$d(x, y) \leq_K a \cdot (d(x, z) + d(z, y))$$

we have that

$$a \cdot (d(x, z) + d(z, y)) \in d(x, y) + K,$$

for all $x, y \in X$. Then, via Lemma 1 (v), (vi) and (vii), we obtain that

$$\xi_e(d(x, y)) \leq \xi_e(a \cdot (d(x, z) + d(z, y))) \leq a \cdot (\xi_e(d(x, z)) + \xi_e(d(z, y))),$$

for all $x, y \in X$. Thus, $D(x, y) \leq D(x, z) + D(y, z)$ for all $x, y \in X$.

Theorem 2 Let (X, d) be a b-TVS cone metric space and $x \in X$ and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X . Then the following statements hold:

- (i) if $\{x_n\}$ b-TVS cone converges to x , then $D(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) if $\{x_n\}$ is a b-TVS cone Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence with respect to b-metric D ;
- (iii) if (X, d) is a b-TVS cone complete, then (X, d) is a complete b-metric space.

Proof.

(i) Let be $\varepsilon > 0$. Since $\{x_n\}$ b-TVS cone converges to x , we have that there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll \varepsilon \cdot e$ for all $n \geq n_0$. Therefore, $\varepsilon \cdot e - d(x_n, x) \in \text{Int}K$. Thus, $d(x_n, x) \in \varepsilon \cdot e - \text{int}K$. Hence, via Lemma 1 (iv), we obtain that $D(x_n, x) = \xi_e(d(x_n, x)) < \varepsilon$, for all $n \geq n_0$.

(ii) Let be $\varepsilon > 0$. Since $\{x_n\}$ is a b-TVS cone Cauchy sequence, we have that there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll \varepsilon \cdot e$, for all $n, m \geq n_0$. Thus, $d(x_n, x_m) \in \varepsilon \cdot e - \text{int}K$. Hence, via Lemma 1 (iv), we obtain that $D(x_n, x_m) = \xi_e(d(x_n, x_m)) < \varepsilon$, for all $n, m \geq n_0$.

(iii) The conclusion (iii) is immediate from conclusions (i) and (ii).

Theorem 3 Let (X, d) be a b-TVS cone metric space. Suppose mappings $f, g : X \rightarrow X$ satisfy:

- (i) the range of g contains the range of f and $g(X)$ is b-TVS cone complete subspace of X ;
- (ii) there exists $k \in [0, \frac{1}{\alpha})$ such that $d(fx, fy) \leq_K k \cdot d(gx, gy)$ for all $x, y \in X$.

Then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X . Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$.

Since

$$d(fx, fy) \leq_K k \cdot d(gx, gy),$$

we have that $k \cdot d(gx, gy) \in d(fx, fy) + K$ for all $x, y \in X$. It follows that

$$\xi_e(d(fx, fy)) \leq \xi_e(k \cdot d(gx, gy)) \leq k \xi_e(d(gx, gy)),$$

for all $x, y \in X$.

Thus,

$$D(fx, fy) \leq kD(gx, gy),$$

for all $x, y \in X$.

Then

$$D(gx_{n+1}, gx_n) = D(fx_n, fx_{n-1}) \leq kD(gx_n, gx_{n-1}) \leq$$

$$\leq k^2 D(gx_{n-1}, gx_{n-2}) \leq \cdots \leq k^n D(gx_1, gx_0).$$

Then, for all $p \geq 1$ we have that

$$\begin{aligned} D(gx_n, gx_{n+p}) &\leq aD(gx_n, gx_{n+1}) + a^2D(gx_{n+1}, gx_{n+2}) + \cdots + a^pD(gx_{n+p-1}, gx_{n+p}) \leq \\ &\leq (ak^n + a^2k^{n+1} + \cdots + a^pk^{n+p-1})D(gx_1, gx_0) \leq \frac{ak^n}{1-ak}D(gx_1, gx_0). \end{aligned}$$

It follows that $\{gx_n\}$ is a Cauchy sequence in b -metric space $(g(X), D)$. Since $(g(X), D)$ is complete, there exists $q \in g(X)$ such that $gx_n \xrightarrow{D} q$ as $n \rightarrow \infty$. Consequently, we can find $p \in X$ such that $g(p) = q$. Further, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$D(gx_n, fp) = D(fx_{n-1}, fp) \leq kD(gx_{n-1}, gp) < \varepsilon.$$

It follows that $gx_n \xrightarrow{D} fp$ as $n \rightarrow \infty$. The uniqueness of a limit implies that $fp = gp = q$. Now we show that f and g have a unique point of coincidence. For this, we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. It follows that

$$D(gp_1, gp) = D(fp_1, fp) \leq k \cdot D(gp_1, gp).$$

From the above we obtain that $D(gp_1, gp) = 0$ i.e. $gp_1 = gp$. From Proposition 1 f and g have a unique common fixed point.

Corollary 1 *Let (X, d) be a complete b -TVS cone metric space. Suppose the mapping $f : X \rightarrow X$ satisfies:*

$$(i) \text{ there exists } k \in [0, \frac{1}{a}) \text{ such that } d(fx, fy) \leq_K k \cdot d(x, y) \text{ for all } x, y \in X.$$

Then f has a unique fixed point in X .

Proof. The proof uses Theorem 3 by replacing g with identity mapping.

Remark 1 *For $a = 1$ we obtain Theorem 2.3 of W.-S. Du [7].*

Theorem 4 *Let (X, d) be a b -TVS cone metric space. Suppose mappings $f, g : X \rightarrow X$ satisfy:*

$$(i) \text{ the range of } g \text{ contains the range of } f \text{ and } g(X) \text{ is a } b\text{-TVS cone complete subspace of } X;$$

$$(ii) \text{ there exists } k \in [0, \frac{1}{1+a}) \text{ such that } d(fx, fy) \leq_K k \cdot (d(fx, gx) + d(fy, gy)) \text{ for all } x, y \in X.$$

Then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X . Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$.

Since

$$d(fx, fy) \leq_K k \cdot (d(fx, gx) + d(fy, gy)),$$

we have that $k \cdot (d(fx, gx) + d(fy, gy)) \in d(fx, fy) + K$ for all $x, y \in X$. It follows that

$$\begin{aligned} \xi_e(d(fx, fy)) &\leq \xi_e(k \cdot (d(fx, gx) + d(fy, gy))) \leq k\xi_e(d(fx, gx) + d(fy, gy)) \leq \\ &\leq k \cdot (\xi_e(d(fx, gx)) + \xi_e(d(fy, gy))), \end{aligned}$$

for all $x, y \in X$.

Thus,

$$D(fx, fy) \leq k \cdot (D(fx, gx) + D(fy, gy)),$$

for all $x, y \in X$.

Then

$$\begin{aligned} D(gx_{n+1}, gx_n) &= D(fx_n, fx_{n-1}) \leq k(D(fx_n, gx_n) + D(gx_n, gx_{n-1})) = \\ &= k(D(gx_{n+1}, gx_n) + D(gx_n, gx_{n-1})). \end{aligned}$$

So, $D(gx_{n+1}, gx_n) \leq_K h \cdot D(gx_n, gx_{n-1})$, where $h = \frac{k}{1-k} \in [0, \frac{1}{a})$. Then, for all $p \geq 1$ we get that

$$\begin{aligned} D(gx_n, gx_{n+p}) &\leq aD(gx_n, gx_{n+1}) + a^2D(gx_{n+1}, gx_{n+2}) + \cdots + a^pD(gx_{n+p-1}, gx_{n+p}) \leq \\ &\leq (ah^n + a^2h^{n+1} + \cdots + a^ph^{n+p-1})D(gx_1, gx_0) \leq \frac{ah^n}{1-ah}D(gx_1, gx_0). \end{aligned}$$

We obtain that $\{gx_n\}$ is a Cauchy sequence in b -metric space $(g(X), D)$. Since $g(X)$ is complete, there exists $q \in g(X)$ such that $gx_n \xrightarrow{D} q$ as $n \rightarrow \infty$. Consequently, we can find $p \in X$ such that $g(p) = q$. Further, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\begin{aligned} D(gp, fp) &\leq a \cdot D(gp, gx_n) + a \cdot D(gx_n, fp) = a \cdot D(gp, gx_n) + a \cdot D(fx_{n-1}, fp) \leq \\ &\leq a \cdot D(gp, gx_n) + a \cdot k(D(fx_{n-1}, gx_{n-1}) + D(fp, gp)) = \\ &= a \cdot D(gp, gx_n) + a \cdot kD(gx_n, gx_{n-1}) + a \cdot kD(fp, gp). \end{aligned}$$

Hence,

$$\begin{aligned} D(gp, fp) &\leq \frac{a}{1-ak}D(gp, gx_n) + \frac{ak}{1-ak}D(gx_n, gx_{n-1}) \leq \\ &\leq \frac{a}{1-ak}D(gp, gx_n) + \frac{akh^{n-1}}{1-ak}D(gx_1, gx_0) < \varepsilon. \end{aligned}$$

From the above we obtain that $d(gp, fp) = 0$ i.e. $fp = gp = q$.

Now we show that f and g have a unique point of coincidence. For this, we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. It follows that

$$D(gp_1, gp) = D(fp_1, fp) \leq k \cdot (D(fp_1, gp_1) + D(fp, gp)) = 0,$$

which gives $D(gp_1, gp) = 0$ i.e. $gp_1 = gp$. From Proposition 1 f and g have a unique common fixed point.

Corollary 2 *Let (X, d) be a complete b -TVS cone metric space. Suppose the mapping $f : X \rightarrow X$ satisfies:*

- (i) *there exists $k \in [0, \frac{1}{1+a})$ such that $d(fx, fy) \leq_K k \cdot (d(fx, gx) + d(fy, gy))$ for all $x, y \in X$.*

Then f has a unique fixed point in X .

Proof. The proof uses Theorem 4 by replacing g with identity mapping.

Theorem 5 *Let (X, d) be a b -TVS cone metric space. Suppose mappings $f, g : X \rightarrow X$ satisfy:*

- (i) *the range of g contains the range of f and $g(X)$ is a b -TVS cone complete subspace of X ;*
- (ii) *there exists $k \in [0, \frac{1}{a^2+a})$ such that $d(fx, fy) \leq_K k \cdot (d(fx, gy) + d(fy, gx))$ for all $x, y \in X$.*

Then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X . Choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$.

Since

$$d(fx, fy) \leq_K k \cdot (d(fx, gy) + d(fy, gx)),$$

we have that $k \cdot (d(fx, gy) + d(fy, gx)) \in d(fx, fy) + K$ for all $x, y \in X$. It follows that

$$\begin{aligned} \xi_e(d(fx, fy)) &\leq \xi_e(k \cdot (d(fx, gy) + d(fy, gx))) \leq k\xi_e(d(fx, gy) + d(fy, gx)) \leq \\ &\leq k \cdot (\xi_e(d(fx, gy)) + \xi_e(d(fy, gx))), \end{aligned}$$

for all $x, y \in X$.

Thus,

$$D(fx, fy) \leq k \cdot (D(fx, gy) + D(fy, gx)),$$

for all $x, y \in X$.

Then

$$D(gx_{n+1}, gx_n) = D(fx_n, fx_{n-1}) \leq k(D(fx_n, gx_{n-1}) + D(fx_{n-1}, gx_n)) =$$

$$= kD(gx_{n+1}, gx_{n-1}) \leq a \cdot kD(gx_{n+1}, gx_n) + a \cdot kD(gx_n, gx_{n-1}).$$

So, $D(gx_{n+1}, gx_n) \leq h \cdot D(gx_n, gx_{n-1})$, where $h = \frac{ak}{1-ak} \in [0, \frac{1}{a})$. Then, for all $p \geq 1$ we infer that

$$\begin{aligned} D(gx_n, gx_{n+p}) &\leq aD(gx_n, gx_{n+1}) + a^2D(gx_{n+1}, gx_{n+2}) + \dots + a^pD(gx_{n+p-1}, gx_{n+p}) \leq \\ &\leq (ah^n + a^2h^{n+1} + \dots + a^ph^{n+p-1})D(gx_1, gx_0) \leq \frac{ah^n}{1-ah}D(gx_1, gx_0). \end{aligned}$$

We obtain that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists $q \in g(X)$ such that $gx_n \rightarrow q$ as $n \rightarrow \infty$. Consequently, we can find $p \in X$ such that $g(p) = q$. Further, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\begin{aligned} D(gp, fp) &\leq a \cdot D(gp, gx_n) + a \cdot D(gx_n, fp) = a \cdot D(gp, gx_n) + a \cdot D(fx_{n-1}, fp) \leq \\ &\leq a \cdot D(gp, gx_n) + a \cdot k(D(fx_{n-1}, gp) + D(fp, gx_{n-1})) = \\ &= a \cdot D(gp, gx_n) + a \cdot kD(gx_n, gp) + a \cdot kD(fp, gx_{n-1}) \leq \\ &\leq a \cdot D(gp, gx_n) + a \cdot kD(gx_n, gp) + a^2 \cdot kD(fp, gp) + a^2 \cdot kD(gp, gx_{n-1}). \end{aligned}$$

Hence,

$$D(gp, fp) \leq \frac{a+ak}{1-a^2k}D(gp, gx_n) + \frac{a^2k}{1-a^2k}D(gp, gx_{n-1}) \ll c.$$

It follows that $D(gp, fp) = 0$ i.e. $fp = gp = q$.

Now we show that f and g have a unique point of coincidence. For this, we assume that there exists another point $p_1 \in X$ such that $fp_1 = gp_1$. It follows that

$$D(gp_1, gp) = D(fp_1, fp) \leq k \cdot (D(fp_1, gp) + D(fp, gp_1)) = 2kD(gp, gp_1),$$

i.e. $D(gp_1, gp) = 0$. From Proposition 1 f and g have a unique common fixed point.

Corollary 3 *Let (X, d) be a complete b-TVS cone metric space. Suppose the mapping $f : X \rightarrow X$ satisfies:*

- (i) *there exists $k \in [0, \frac{1}{a^2+a})$ such that $d(fx, fy) \leq_K k \cdot (d(fx, gy) + d(fy, gx))$ for all $x, y \in X$.*

Then f has a unique fixed point in X .

Proof. The proof uses Theorem 5 by replacing g with identity mapping.

Remark 2 *For $a = 1$ and Y be a Banach space we obtain Theorem 2.1, Theorem 2.2 and Theorem 2.3 of [1].*

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