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The continuity of inversion in topological generalized group [1]

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Abstract

In this paper, continuity of inversion in paratopological generalized group are studied. Let G be a topologically periodic paratopological generalized group. It is shown that the inversion $i : x \to x^{-1}$ is continuous provideded G is regular, locally compact(countably compact), and every subset of G_e is nowhere dense in G_e(with subspace topology), if it is nowhere dense in G.

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1 Introduction and preliminaries

A paratopological generalized group G is a semigroup which satisfies the following conditions:
1. For each $x \in G$ there is a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$;
2. For each $x \in G$ there exists $x^{-1} \in G$ such that $x(x^{-1}) = (x^{-1})x = e(x)$;
3. G is a Hausdorff topological space;
4. The mapping

$$\begin{aligned} m_2 : G \times G &\longmapsto G \\ (g, h) &\longmapsto gh \end{aligned}$$

is continuous mapping. A paratopological generalized group with continuous inversion is called a topological generalized group[11].
Let G be a paratopological generalized group. $G_e = \{x \in G; ex = xe = x\}$ is defined for each idempotent e of G.

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The question of finding topological properties on a paratopological generalized group which imply that it is a topological generalized group. We know that: Every regular countably compact paratopological group is a topological group [10]. Every regular locally compact paratopological is a topological group [5]. Every quasi-regular pseudo-compact paratopological group is a topological group [10]. Every topological periodic countably compact paratopological group is a topological group [3]. Every $\tilde{C}$ech-complete paratopological group is a topological group [4]. Every strongly Baire semi-topological group is a topological group [7]. Every compact inverse Clifford topological semigroup is a topological inverse Clifford semigroup [8],[9]. Banakh and Gutik proved that every sequential countably compact inverse topological semigroup is a topological inverse semigroup [2]. They proved if G is a topologically periodic Hausdorff inverse Clifford topological semigroup, The inverse operation $i : G \to G$ is continuous provided G is inversely regular and countably compact or G is regular and $G \times G$ is countably compact [2].
In this paper, we will prove that if G is a regular, locally compact(countably compact), topologically periodic paratopological generalized group and Suppose every subset of G_e is nowhere dense in G_e(with subspace topology), if it is nowhere dense in G, then G is a topological generalized group.
A topological semigroup G is called topologically periodic if for any point $x \in G$ and any neighborhood $U \subset G$ of x there is $n \geq 2$ such that $x^n \in U$.

Definition 1 *A topological semigroup G is called topologically periodic if for any point $x \in G$ and any neighborhood $U \subset G$ of x there is $n \geq 2$ such that $x^n \in U$.*

2 Main result

Now we need some results of [2],[6]:

Theorem 1 *[2] The inversion in a topological periodic paratopological generalized group G is continuous if and only if it is continuous at each idempotent of G.*

Theorem 2 *A topological space is a Hausdorff space if and only if every net in X has at most one limit.*

Theorem 3 *[6] Let G be an paratopological generalized group, then every G_a is closed in G.*

Proof. We only need to prove that $\overline{G_a} = G_a$. we suppose $x \in \overline{G_a}$ therefore there exist a net $\{x_\beta\}_{\beta \in Y}$ of G that $\{x_\beta\}$ converges to x. By Definition G_a, it sufficient to verify that $xe_a = e_a x = x$, since semigroup operation is continuous and by proposition1.6.6 of [6] and lemma 1.2, $xe_a = (lim x_\beta)e_a = lim(x_\beta e_a) = x$. A similar argument shows that $e_a x = x$.

Lemma 1 *Let G is an paratopological generalized group and U is any open neighborhood of one of idempotent element in G; then $\overline{M \cap G_a} \subseteq (M \cap G_a)U^{-1}$ for each subset M of G.*

Proof. Let $x \in \overline{M \cap G_a}$, then there exist a net $\{x_\beta\}_{\beta \in Y}$ where $x_\beta \in M \cap G_a$ converges to x; since semigroup operation is continuous, hence $\{x^{-1}x_\beta\}_{\beta \in Y}$ converges to $x^{-1}x$.
By lemma 1.3, $\{x^{-1}x_\beta\}_{\beta \in Y}$ converges to e_a and because U is an open neighborhood of e_a, therefore there exist $\lambda \in Y$ such that $x^{-1}x_\beta \in U$ for any $\lambda \leq \beta$; then ${x_\beta}^{-1}x = (x^{-1}x_\beta)^{-1} \in U^{-1}$, so $x \in x_\beta U^{-1} \subset (M \cap G_a)U^{-1}$.

Lemma 2 *Suppose that G is an paratopological generalized group and topologically periodic and not topological generalized group. Then there exist an open neighborhood U of at least one of idempotent element of G such that $U \cap U^{-1} \cap G_a$ is nowhere dense in G.*

Proof. The inverse operation in G is discontinuous. By lemma1.1, it is discontinuous at least one of idempotent element e_a, therefore we can choose an open neighborhood W of e_a such that $e_a \notin int(W^{-1})$.
Since the multiplication in G is continuous, we can find an open neighborhood U of e_a such that $U^3 \subset W$. We claim $U \cap U^{-1} \cap G_a$ is nowhere dense in G. Assume the contrary. Then there exists a nonempty open set V such that $V \subset \overline{U \cap U^{-1} \cap G_a}$. By Lemma 1.3 it follows that $V \subset \overline{U \cap U^{-1} \cap G_a} \subset (U \cap U^{-1} \cap G_a)U^{-1} \subset U^{-2}$. Then $VU^{-1} \subset U^{-3} \subset W^{-1}$ Clearly, $V \cap U \cap G_a \neq \varnothing$, and the set VU^{-1} is open in G. Therefore, $e_a \in VU^{-1} \subset int(W^{-1})$, this is a contradiction.

Theorem 4 *Suppose that G is a regular, locally compact, topologically periodic paratopological generalized group and every subset of G_a is nowhere dense in G_a (with subspace topology), if it is nowhere dense in G, then G is a topological generalized group.*

Proof. Step 1: we will prove G_a (with subspace topology) is a topological group.
It is evident that G_a is group with neutral element e_a, since group operation is continuous, therefore $*|_{G_a \times G_a} : G_a \times G_a \to G_a$ is continuous [6]. On the other hand from lemma1.3, G_a is closed subset in G, then G_a is locally compact and regular [7], hence G_a is a topological group [5].
Step 2: suppose that G is not topological generalized group that is, inversion $i : G \to G$ isn't continuous. By lemma1.1, there exist an idempotent element e_a that i isn't continuous in e_a. By lemma2.2, there exist an open neighborhood such as U of e_a such that $U \cap U^{-1} \cap G_a$ is nowhere dense in G. By hypothesis, $U \cap U^{-1} \cap G_a$ is nowhere dense in G_a.
On the other hand ,by step 1, $i|_{G_a} : G_a \to G_a$ is continuous, hence $(U \cap G_a)^{-1} = U^{-1} \cap G_a$ is open in G_a and $U \cap G_a$ is open in G_a, so $U^{-1} \cap U \cap G_a = (U^{-1} \cap G_a) \cap (U \cap G_a)$ is open in G_a . It is a contradiction.

Corollary 1 *Suppose that G is a regular, Countably compact, topologically periodic paratopological generalized group and every subset of G_a is nowhere dense in G_a (with subspace topology), if it is nowhere dense in G, then G is a topological generalized group.*

Proof. We know that closed subspace of countably compact space is countably compact (with subspace topology) [6]. similarly proof of theorem 2.3 is complete [3].

Example 1 *Suppose that G is a regular, locally compact (countably compact) topologically periodic paratopological generalized group and G_a is open in G, then G is a topological generalized group .*

Example 2 *Suppose that G is a regular, locally compact (countably compact) topologically periodic paratopological generalized group and $Card(e(G)) < \infty$, then G is a topological generalized group.*

Proof. $G_a = G - (\bigcup_{b \neq a} G_b)$, since G_a is closed and $(card(e(G)) < \infty$, therefore G_a is open. By example 2.5, G is a topological generalized group.

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