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On a Subclass of Analytic Functions defined by Convolution [1]

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Abstract

We introduce and study a comprehensive family of analytic univalent functions which contains various well-known classes of analytic univalent functions as well as many new ones. In this paper, we obtain coefficient bounds, distortion bounds and extreme points. Convolution conditions and convex combination are also determined for functions in this family.

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1 Introduction

Let A denote the class of functions of the form $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions in U.

A function $f \in S$ is said to be starlike of order $\alpha, 0 \leq \alpha < 1$, denoted by $f \in S^*(\alpha)$, if $Re\frac{zf'(z)}{f(z)} > \alpha$ for $z \in U$ and is said to be convex of order $\alpha, 0 \leq \alpha < 1$, denoted by $f \in K(\alpha)$ if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$ for $z \in U$. $S^*(0) = S^*$ and $K(0) = K$ are respectively the classes of starlike and convex functions in S. These classes were studied by Robertson [5] and Silverman [6]. (See also [4]). A function $f \in S$ is said to be close-to-convex of order α, denoted by $f \in C(\alpha), 0 \leq \alpha < 1$ if there exists a function $g \in S^*$ such that

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha, \qquad z \in U.$$

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$C(0) = C$ is the class of close-to-convex functions.

For $1 < \beta \le \min\left\{\dfrac{2(\lambda_2+2)}{4+\lambda_2 t}, \dfrac{2+t}{2t}\right\}$, $S(\phi, t, \beta)$ denote the subclass of S consisting of functions $f \in S$ which satisfy the condition

$$Re\left\{\frac{f(z) * \phi(z)}{f_t(z) \diamond \phi(z)}\right\} < \beta \tag{1}$$

where $f_t(z) = (1-t)z + tf(z)$, $0 \le t \le 1$ and $\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ is analytic in U with the condition $\lambda_n \ge 0$. The operator "$*$" stands for the convolution of two power series and defined for functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n$$

as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n$$

while "$\diamond$" stands for the integral convolution of two power series and defined for $f(z)$ and $F(z)$ as

$$(f \diamond F)(z) = f(z) \diamond F(z) = z + \sum_{n=2}^{\infty} \frac{a_n A_n}{n} z^n.$$

Further, let V be the subclass of S consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

Let $VS(\phi, t, \beta) = S(\phi, t, \beta) \cap V$.

By specializing the parameters we obtain the following known subclasses of S studied earlier by various authors.

1. $S\left(\dfrac{z}{(1-z)^2}, 1, \beta\right) \equiv M(\beta), VS\left(\dfrac{z}{(1-z)^2}, 1, \beta\right) \equiv V(\beta)$,
   $S\left(\dfrac{z+z^2}{(1-z)^3}, 1, \beta\right) \equiv L(\beta), VS\left(\dfrac{z+z^2}{(1-z)^3}, 1, \beta\right) \equiv U(\beta)$, subclasses of S studied by Uralegaddi et al. [7].

2. $VS\left(\frac{z}{(1-z)^2}, 0, \beta\right) \equiv R(\beta)$ the class of close-to-convex functions in U studied by Uralegaddi et al. [8].

3. $VS\left(z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n, 1, \beta\right) \equiv P_\lambda(\beta)$ studied by Dixit and Pathak [2].

4. $VS\left(z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}z^n, 0, \beta\right) \equiv R_\lambda(\beta)$ studied by Dixit and Pathak [3].

5. $VS\left(z+\sum_{n=2}^{\infty}n^{k+1}z^n, 1, \beta\right) \equiv R(K,\beta)$ studied by Dixit and Chandra [1].

In this paper, we extend the above results to the families $S(\phi, t, \beta)$ and $VS(\phi, t, \beta)$.

We obtain coefficient inequalities, extreme points, distortion bounds, convolution conditions and convex combinations for the class $VS(\phi, t, \beta)$. It is worth mentioning that many of our results are either extensions or new approaches to the corresponding previously known results.

2 Main Results

We begin with a sufficient coefficient condition for functions in $S(\phi, t, \beta)$.

Theorem 1 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* S*. If*

$$\sum_{n=2}^{\infty}\frac{\lambda_n(n-\beta t)}{n(\beta-1)}|a_n| \leq 1, \tag{3}$$

where $1 < \beta \leq \min\left\{\frac{2(\lambda_2+2)}{4+\lambda_2 t}, \frac{2+t}{2t}\right\}$, $0 \leq t \leq 1$ *then* $f \in S(\phi, t, \beta)$.

Proof. Let $\sum_{n=2}^{\infty}\frac{\lambda_n(n-\beta t)}{n(\beta-1)}|a_n| \leq 1$. It sufficies to show that

$$\left|\frac{\frac{f(z)*\phi(z)}{f_t(z)\diamond\phi(z)}-1}{\frac{f(z)*\phi(z)}{f_t(z)\diamond\phi(z)}-(2\beta-1)}\right| < 1, \qquad z \in U.$$

We have

$$\left|\frac{\dfrac{f(z)*\phi(z)}{f_t(z)\diamond\phi(z)}-1}{\dfrac{f(z)*\phi(z)}{f_t(z)\diamond\phi(z)}-(2\beta-1)}\right| \le \frac{\sum_{n=2}^{\infty}\dfrac{\lambda_n(n-t)}{n}|a_n|\ |z|^{n-1}}{2(\beta-1)-\sum_{n=2}^{\infty}\lambda_n\dfrac{(n-2\beta t+t)}{n}|a_n|\ |z|^{n-1}}$$

$$\le \frac{\sum_{n=2}^{\infty}\dfrac{\lambda_n(n-t)}{n}|a_n|}{2(\beta-1)-\sum_{n=2}^{\infty}\lambda_n\dfrac{(n-2\beta t+t)}{n}|a_n|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\frac{\lambda_n(n-t)}{n}|a_n| \le 2(\beta-1)-\sum_{n=2}^{\infty}\frac{\lambda_n(n-2\beta t+t)}{n}|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{\infty}\lambda_n\frac{(n-\beta t)}{n} \le \beta-1. \tag{4}$$

But (4) is true by hypothesis and the theorem is proved. In the following theorem, it is proved that the condition (3) is also necessary for functions f of the form (2).

Theorem 2 *Let the functions f be given by* (2)*. Then $f \in VS(\phi,t,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\lambda_n\frac{(n-\beta t)}{n}|a_n| \le \beta-1, \tag{5}$$

where $1<\beta\le\min\left\{\dfrac{2(\lambda_2+2)}{4+\lambda_2 t},\dfrac{2+t}{2t}\right\},\ 0\le t\le 1.$

Proof. In view of Theorem 1 it suffices to show the only if part.

Suppose

$$Re\left\{\frac{f(z)*\phi(z)}{f_t(z)\diamond\phi(z)}\right\}<\beta, \qquad z\in U,$$

is equivalent to

$$Re\left\{\frac{z+\sum_{n=2}^{\infty}\lambda_n|a_n|z^n}{z+\sum_{n=2}^{\infty}\dfrac{\lambda_n|a_n|t}{n}}\right\}<\beta. \tag{6}$$

Choose values of z on the positive real axis so that $\dfrac{f(z) * \phi(z)}{f_t(z) \diamond \phi(z)}$ is real. Then letting $z \to 1$ through real values we obtain

$$1 + \sum_{n=2}^{\infty} \lambda_n |a_n| \leq \beta \left(1 + \sum_{n=2}^{\infty} \frac{\lambda_n |a_n| t}{n}\right),$$

i.e.,

$$\sum_{n=2}^{\infty} \lambda_n \left(\frac{n - \beta t)}{n}\right) |a_n| \leq \beta - 1$$

and the proof is complete.

Remark 1 *If we put* $\phi(z) = \dfrac{z}{(1-z)^2}, t = 1$ *in Theorem 2, we obtain the corresponding result for the class* $V(\beta)$ *given earlier by Uralegaddi et al. [7].*

Remark 2 *If we put* $\phi(z) = \dfrac{z + z^2}{(1-z)^3}, t = 1$ *in Theorem 2, we obtain the corresponding result for the class* $U(\beta)$ *given earlier by Uralegaddi et al. [7].*

Remark 3 *If we put* $\phi(z) = \dfrac{z}{(1-z)^2}, t = 0$ *in Theorem 2, we obtain the corresponding result for the class* $R(\beta)$ *given earlier by Uralegaddi et al. [8].*

Remark 4 *If we put* $\phi(z) = z + \sum_{n=2}^{\infty} \dfrac{n\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,\ t = 1$ *in Theorem 2, we obtain the corresponding result for the class* $P_\lambda(\beta)$ *given by Dixit and Pathak [2].*

Remark 5 *If we put* $\phi(z) = z + \sum_{n=2}^{\infty} \dfrac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,\ t = 0$ *in Theorem 2, we obtain the corresponding result for the class* $R_\lambda(\beta)$ *given by Dixit and Pathak [3].*

Remark 6 $\phi(z) = z + \sum_{n=2}^{\infty} n^{k+1} z^n,\ t = 1$ *in Theorem 2, we obtain the corresponding result for the class* $R(k, \beta)$ *given by Dixit and Chandra [1].*

Next we determine the bounds for functions in $VS(\phi, t, \beta)$, which yields a covering result for this family.

Theorem 3 *Let* $f \in VS(\phi, t, \beta)$ *and* $\lambda_k \geq \lambda_2$ *for* $k \geq 2$, *then*

$$r - \frac{2(\beta - 1)}{\lambda_2(2 - \beta t)} r^2 \leq |f(z)| \leq r + \frac{2(\beta - 1)}{\lambda_2(2 - \beta t)} r^2, \qquad (|z| = r) \tag{7}$$

with equality for $f(z) = z + \frac{2(\beta-1)}{\lambda_2(2-\beta t)} z^2$. $(z = \pm r)$

Proof. Since $f(z) \in VS(\phi, t, \beta)$, in view of Theorem 2, we have

$$\lambda_2 \frac{(2-\beta t)}{2} \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{\lambda_n (n-\beta t)}{n} |a_n| \leq (\beta - 1),$$

which evidently gives

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{2(\beta-1)}{\lambda_2(2-\beta t)}. \tag{8}$$

Consequently, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n| \; |z|^n \\ &\leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq r + r^2 \frac{2(\beta-1)}{\lambda_2(2-\beta t)}, \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| \; |z|^n \\ &\geq r - r^2 \sum_{n=2}^{\infty} |a_n| \\ &\geq r - r^2 \frac{2(\beta-1)}{\lambda_2(2-\beta t)}. \end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.

Corollary 1 *Let $f \in VS(\phi, t, \beta)$ and $\lambda_k \geq \lambda_2$ for $k \geq 2$. Then*

$$\left\{ w : |w| < \frac{2(\lambda_2+1) - \beta(\lambda_2 t + 2)}{\lambda_2(2-\beta t)} \subset f(U) \right\}.$$

Next, we determine the extreme points of $VS(\phi, t, \beta)$.

Theorem 4 *Let $f_1(z) = z$ and*

$$f_n(z) = z + \frac{n(\beta-1)}{\lambda_n(n-\beta t)} z^n, \quad (n = 2, 3, ...).$$

Then $f \in VS(\phi, t, \beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \gamma_n f_n(z)$, where $\gamma_n \geq 0$ and $\sum_{n=1}^{\infty} \gamma_n = 1$.

Proof. The proof of Theorem 4 is similar to that of Theorem 9 in [6]. Therefore we omit the details involved.

For our next theorem, we need to define the convolution of two analytic functions.

For functions of the form $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ and $F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n$ we define the convolution of two functions f and F as

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n A_n| z^n. \tag{9}$$

Using this definition we show that the class $VS(\phi, t, \beta)$ is closed under convolution.

Theorem 5 *If $f \in VS(\phi, t, \beta)$ and $F \in VS(\phi, t, \beta)$ then $f * F \in VS(\phi, t, \beta)$.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ be in $VS(\phi, t, \beta)$ and $F(z) = z + \sum_{n=2}^{\infty} |A_n| z^n$ be in $VS(\phi, t, \beta)$.

Then the convolution $f * F$ is given by (9).

We wish to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 2.

For $F \in VS(\phi, t, \beta)$ we note that $|A_k| \leq 1$. Now, for the convolution function $f * F$ we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\lambda_n (n - \beta t)}{n(\beta - 1)} |a_n A_n| \\
\leq &\sum_{n=2}^{\infty} \frac{\lambda_n (n - \beta t)}{n(\beta - 1)} |a_n| \\
\leq &1, \qquad \text{(since } f \in VS(\phi, t, \beta)).
\end{aligned}$$

Therefore $f * F \in VS(\phi, t, \beta)$. Next we show that $VS(\phi, t, \beta)$ is closed under convex combination of its members.

Theorem 6 *The class $VS(\phi, t, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3...$ let $f_i(z) \in VS(\phi, t, \beta)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{n=2}^{\infty} |a_{n_i}| z^n.$$

Then by Theorem 2

$$\sum_{n=2}^{\infty} \frac{\lambda_n (n - \beta t)}{n(\beta - 1)} |a_{n_i}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) z^n.$$

Now

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\lambda_n(n-\beta t)}{n(\beta-1)} \left(\sum_{i=1}^{\infty} t_i |a_{n_i}| \right) \\ =& \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\lambda_n(n-\beta t)}{n(\beta-1)} |a_{n_i}| \right) \\ \le& \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Therefore $\sum_{i=1}^{\infty} t_i f_i(z) \in VS(\phi, t, \beta)$.

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