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An Investigation on Minimal Surfaces of Multivalent Harmonic Functions [1]

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Abstract

The projection on the base plane of a regular minimal surface S in $\mathbb{R}^3$ with isothermal parameters defines a complex-valued univalent harmonic function $f = h(z) + \overline{g(z)}$. The aim of this paper is to obtain the distortion inequalities for the Weierstrass-Enneper parameters of the minimal surface for the harmonic multivalent functions for which analytic part is an m-valent convex function.

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1 Preliminaries

Minimal surfaces are most commonly known as those which have the minimum area amongst all other surfaces spanning a given closed curve in $\mathbb{R}^3$. Geometrically, the definition of a minimal surface is that the mean curvature H is zero at every point of the surface. If locally one can write the minimal surface in $\mathbb{R}^3$ as $(x, y, \Phi(x, y))$, then the minimal surface equation $H = 0$ is equivalent to

$$(1) \qquad (1 + \Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (1 + \Phi_x^2)\Phi_{yy} = 0\ .$$

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There exists a choice of isothermal parameters $(u, v) \in \Omega \subset \mathbb{R}^2$ so that the surface $X(u, v) = (x(u, v), y(u, v), \Phi(u, v)) \in \mathbb{R}^3$ satisfying the minimal surface equation is given by

$$E = |X_u|^2 = |X_v|^2 = G > 0, \quad F =< X_u, X_v >= 0, \quad \triangle_{(u,v)} X = 0 \ , \tag{2}$$

where Δ denotes the Laplacian operator. The general solution of such an equation is called the local Weierstrass-Enneper representation [2].

A complex-valued function f which is harmonic in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(z_0) = 0$ for some prescribed point $z_0 \in \mathbb{D}$. According to a theorem of H. Lewy [1], f is locally univalent if and only if its Jacobian $(|f_z|^2 - |f_{\overline{z}}|^2 = |h^{'}(z)|^2 - |g^{'}(z)|^2)$ does not vanish. The function f is said to be sense-preserving if its Jacobian is positive. In this case then $h^{'}(z) \neq 0$ and the analytic function $w(z) = \dfrac{g'(z)}{h'(z)}$, called the second dilatation of f, has the property $|w(z)| < 1$ for all $z \in \mathbb{D}$. Throughout this paper we will assume that f is locally univalent and sense -preserving, and we will call f a harmonic mapping.

A harmonic mapping $f = h + \overline{g}$ can be lifted locally to a regular minimal surface given by conformal (or isothermal) parameters if and only if its dilatation is the square of an analytic function $w(z) = q^2(z)$ for some analytic function q with $|q(z)| < 1$. Equivalently, the requirement is that any zero of w be of even order, unless $w \equiv 0$ on its domain, so that there is no loss of generality in supposing that z ranges over the unit disc $\mathbb{D}$, because any other isothermal representation can be precomposed with a conformal map from the unit disc $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem. For such a harmonic mapping $f = u + iv$, the minimal surface has the Weierstrass-Enneper representation with parameters (u, v, t) given by

$$\begin{array}{c} u = Re\{f(z)\} = Re\{\int_0^z \varphi_1(\zeta) d\zeta\}, \\ u = Re\{f(z)\} = Re\{\int_0^z \varphi_1(\zeta) d\zeta\}, \\ v = Im\{f(z)\} = Re\{\int_0^z \varphi_2(\zeta) d\zeta\}, \\ t = Re\{\int_0^z \varphi_3(\zeta) d\zeta\} \end{array} \tag{3}$$

for $z \in \mathbb{D}$ with

$$\begin{array}{c} \varphi_1 = h' + g' = p(1 + q^2) = \dfrac{\partial u}{\partial z}, \\ \varphi_2 = -i(h' - g') = -ip(1 - q^2) = \dfrac{\partial v}{\partial z}, \\ \varphi_3 = -2ipq = \dfrac{\partial t}{\partial z}, \quad \varphi_3^2 = -4w(h')^2 \quad \text{and} \quad h' = p. \end{array} \tag{4}$$

(see [1] and [4, p. 176]).

The metric of the surface has the form $ds = \lambda|dz|$, where $\lambda = \lambda(z) > 0$. Here, the function λ takes the form

$$\lambda = |h'| + |g'| = |h'|(1 + |w|) = |p|(1 + |q|^2) \ . \tag{5}$$

A classical theorem of differential geometry says that if a regular surface is represented by conformal parameters (or isothermal parameters) so that its metric has the form $ds = \lambda|dz|$ for some positive function λ, then the Gauss curvature of the surface is $K = -\lambda^{-2}\Delta(log\lambda)$. The quantity K is also known as the curvature of the metric. In our special case of a minimal surface associated with a harmonic mapping $f = h + \overline{g}$, the formula for curvature reduces to

$$K = -\frac{4|q'|^2}{|p|^2(1 + |q|^2)^4} \ , \tag{6}$$

since the underlying harmonic mapping f has dilatation $w = \dfrac{g'}{h'} = q^2$ and $h' = p$. An equivalent expression is the following one:

$$K = -\frac{|w'|^2}{|h'g'|(1 + |w|)^4} \ . \tag{7}$$

Now we define the following class of harmonic functions [2], which is used throughout this paper.

Let $\mathcal{H}$ be the family of functions $f = h(z) + \overline{g(z)}$ which are harmonic and sense-preserving in the open disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a fixed $m \in \mathbb{Z}^+$, let $\mathcal{H}(m)$ be the set of all harmonic multivalent and sense-preserving functions in $\mathbb{D}$ of the form $f = h(z) + \overline{g(z)}$, where

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1}z^{n+m-1}, g(z) = \sum_{n=1}^{\infty} b_{n+m-1}z^{n+m-1}, |b_m| < 1. \tag{8}$$

are analytic in $\mathbb{D}$, and called analytic and co-analytic parts of f respectively (see [7], [8], [10], [11] and [12]).

Let Ω be the family of functions $\phi(z)$ which are regular and satisfying the conditions $\phi(0) = 0$, and $|\phi(z)| < 1$ for every $z \in \mathbb{D}$; and let $\Omega(a)$, where $0 < a < 1$, be the class of functions $w(z)$ which are regular in $\mathbb{D}$ and satisfy $w(0) = a$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. We let $\Omega_{\cup}$ be the union of all classes $\Omega(a)$ where a ranges over $(0, 1)$. Denote by $\mathcal{P}(m)$ (with m a positive integer) the class of functions $p(z) = m + p_1z + \cdots$ which are analytic in $\mathbb{D}$, and satisfying conditions $p(0) = m$, $Re(p(z)) > 0$ for all $z \in \mathbb{D}$ and such that $p(z) \in \mathcal{P}(m)$ if and only if

$$p(z) = m \cdot \frac{1 + \phi(z)}{1 - \phi(z)} \tag{9}$$

for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$.

Let $F(z) = z + d_2 z^2 + \cdots$ and $G(z) = z + e_2 z^2 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$, then we say that $F(z)$ is subortinate to $G(z)$ and we write $F(z) \prec G(z)$.

Finally, let $\mathcal{A}_m (m \geq 1)$ denote the class of functions $s(z) = z^m + \alpha_{m+1} z^{m+1} + \alpha_{m+2} z^{m+2} + \cdots$ which are analytic in $\mathbb{D}$, for $s(z) \in \mathcal{A}_m (m \geq 1)$ we say that $s(z)$ belongs to the class $C(m)$ (the class of m-valent convex functions) if

$$Re\{1 + z\frac{s''(z)}{s'(z)}\} > 0, \quad z \in \mathbb{D} \ . \tag{10}$$

We denote by $\mathcal{H}C(m)$ the subclass of $\mathcal{H}(m)$ consisting of all harmonic multivalent and sense-preserving functions for which analytic part is an m-valent convex function.

2 Main Results

Lemma 1 *Let $w(z)$ be an element of $\Omega_{\cup}$. Then*

$$\frac{|a - r|}{1 - ar} \leq |w(z)| \leq \frac{a + r}{1 + ar} \tag{11}$$

for all $z \in \mathbb{D}$.

Proof. The inequality (11) is clear for $z = 0$, whence $r = |z| = 0$. Now, let $z \in \mathbb{D} \setminus \{0\}$, and define

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)} w(z)}, \quad z \in \mathbb{D},$$

where $w(0) = a \in (0, 1)$. This function satisfies the conditions of Schwarz's lemma. The estimation of Schwarz's lemma, $|\phi(z)| \leq |z| = r$, gives

$$|\phi(z)| = \left| \frac{w(z) - a}{1 - aw(z)} \right| \leq r \Rightarrow |w(z) - a| \leq r|1 - aw(z)| \ . \tag{12}$$

The inequality (12) is equivalent to

$$\left| w(z) - \frac{a(1-r^2)}{1-a^2r^2} \right| \leq \frac{r(1-a^2)}{1-a^2r^2} \ . \tag{13}$$

The equality holds in the inequality (13) only for the function

$$w(z) = \frac{z+a}{1+az}, \quad z \in \mathbb{D}.$$

If we use the triangle inequality in the inequality (13), we get

$$\left| |w(z)| - \left| \frac{a(1-r^2)}{1-a^2r^2} \right| \right| \leq \left| w(z) - \frac{a(1-r^2)}{1-a^2r^2} \right| \leq \frac{r(1-a^2)}{1-a^2r^2}$$

$$\Rightarrow |w(z)| - \left| \frac{a(1-r^2)}{1-a^2r^2} \right| \leq \frac{r(1-a^2)}{1-a^2r^2}$$

$$\Rightarrow -\frac{r(1-a^2)}{1-a^2r^2} \leq |w(z)| - \left| \frac{a(1-r^2)}{1-a^2r^2} \right| \leq \frac{r(1-a^2)}{1-a^2r^2}$$

$$\Rightarrow -\frac{r(1-a^2)}{1-a^2r^2} + \left| \frac{a(1-r^2)}{1-a^2r^2} \right| \leq |w(z)| \leq \frac{r(1-a^2)}{1-a^2r^2} + \left| \frac{a(1-r^2)}{1-a^2r^2} \right|$$

$$\Rightarrow \frac{a-r}{1-ar} \leq |w(z)| \leq \frac{a+r}{1+ar} \ . \tag{14}$$

Similarly, if we replace a with r in the inequality (12), we get

$$\Rightarrow \frac{r-a}{1-ar} \leq |w(z)| \leq \frac{a+r}{1+ar} \ . \tag{15}$$

From the inequalities (14) and (15), we obtain (12). □

Corollary 1 *If* $w(z) \in \Omega_{\cup}$, *then*

$$\frac{(1-a)(1-r)}{1+ar} \leq (1-|w(z)|) \leq \frac{1-ar-|a-r|}{1-ar} \ , \tag{16}$$

$$\frac{1-ar+|a-r|}{1-ar} \leq 1+|w(z)| \leq \frac{(1+a)(1+r)}{1+ar} \ , \tag{17}$$

$$\frac{(1+a)(1-r)}{1-ar} \leq |1+w(z)| \leq \frac{(1+a)(1+r)}{1+ar} \tag{18}$$

and

$$\frac{(1-a)(1-r)}{1+ar} \leq |1-w(z)| \leq \frac{(1-a)(1+r)}{1-ar} \ . \tag{19}$$

Proof. These inequalities are simple consequences of Lemma 1 and the inequality (13). □

Lemma 2 *Let $s(z)$ be an element of $C(m)$. Then*

$$\frac{r^{-(1-m)}}{(1+r)^{2m}} \leq |s'(z)| \leq \frac{r^{-(m-1)}}{(1-r)^{2m}} \ . \tag{20}$$

This inequality is sharp because the extremal function function is

$$s'(z) = \frac{z^{m-1}}{(1-z)^{2m}} \ . \tag{21}$$

Proof. Using the definition of the class $C(m)$ and the definition of subordination, we can write

$$1 + z\frac{s''(z)}{s'(z)} = p(z) \prec m(\frac{1+z}{1-z}) \ . \tag{22}$$

The relation (22) shows that

$$\left|1 + z\frac{s''(z)}{s'(z)} - m.\frac{1+r^2}{1-r^2}\right| \leq \frac{2mr}{1-r^2} \ . \tag{23}$$

After simple calculations from the inequality (23) we get

$$-\frac{(1+m)r+(1-m)}{1+r} \leq Re(z\frac{s''(z)}{s'(z)}) \leq \frac{(1+m)r-(1-m)}{1-r} \ . \tag{24}$$

On the other hand, we have

$$Re(z\frac{s''(z)}{s'(z)}) = r\frac{\partial}{\partial r}\log|s'(z)|.$$

Therefore the inequality (24) can be written in the following form:

$$-\frac{(1+m)r+(1-m)}{1+r} \leq r\frac{\partial}{\partial r}\log|s'(z)| \leq \frac{(1+m)r-(1-m)}{1-r} \ . \tag{25}$$

Then, integrating both sides of the inequality (25) from zero to r, we obtain (20). □

Example 1 *An example of a minimal surface that satisfies the about properties is*

$$f(z) = z^m + |b_m|(\overline{z})^m + \frac{m(1-|b_m|)}{m+1}(\overline{z})^{m+1}, \quad m \in \mathbb{Z}^+, z \in \mathbb{D} \quad \text{and} \quad |b_m| < 1.$$

Indeed, f is harmonic, since $\Delta f = 4\dfrac{\partial^2 f}{\partial z \partial \overline{z}} = 0$ and it is clear that f is multivalent.

The functions $h(z) = z^m$ and $g(z) = |b_m|z^m + \dfrac{m(1-|b_m|)}{m+1}z^{m+1}$, the analytic and co-analytic parts of f, are analytic in $\mathbb{D}$. Hence the second dilatation $w(z) = |b_m| + (1-|b_m|)z$ of f satisfies $|w(z)| < 1$ and is the square of the analytic function $q(z) = \sqrt{|b_m| + (1-|b_m|)z}$ in $\mathbb{D}$. Thus the harmonic multivalent mapping f can be lifted locally to a regular minimal surface. Furthermore, the analytic part h of f is an m-valent convex function, since $Re\{1 + z\dfrac{h''(z)}{h'(z)}\} = m > 0$ for every $z \in \mathbb{D}$.

Corollary 2 *Let $f = h(z) + \overline{g(z)}$ be element of $\mathcal{H}C(m)$. Then*

(26) $$\frac{|a-r|r^{-(1-m)}}{(1-ar)(1+r)^{2m)}} \leq |g'(z)| \leq \frac{(a+r)r^{-(m-1)}}{(1+ar)(1-r)^{2m}} \;.$$

Proof. This corollary is a simple consequence of the definition of second dilatation of f and the lemmas 1 and 2. □

Theorem 1 *Let the functions φ_k, $(k = 1, 2, 3)$ be the Weierstrass-Enneper parameters of a regular minimal surface of $f = (h + \overline{g}) \in \mathcal{H}C(m)$. Then*

(27) $$\frac{(1+a)(1-r)r^{-(1-m)}}{(1-ar)(1+r)^{2m}} \leq |\varphi_1| \leq \frac{(1+a)(1+r)r^{-(m-1)}}{(1+ar)(1-r)^{2m}} \;,$$

(28) $$\frac{(1-a)(1-r)r^{-(1-m)}}{(1+ar)(1+r)^{2m}} \leq |\varphi_2| \leq \frac{(1-a)(1+r)r^{-(m-1)}}{(1-ar)(1-r)^{2m}}$$

and

(29) $$\frac{4|a-r|r^{-2(1-m)}}{(1-ar)(1+r)^{4m}} \leq |\varphi_3|^2 \leq \frac{4(a+r)r^{-2(m-1)}}{(1+ar)(1-r)^{4m}} \;.$$

Proof. Using (4), Lemma 1, the inequalities (18) and (19), and Lemma 2, we get (27), (28) and (29). □

Theorem 2 *Let K be the Gaussian curvature of the regular minimal surface of $f = (h + \overline{g}) \in \mathcal{H}C(m)$. Then*

$$|K| \leq \frac{(1-ar-|a-r|)^2(1-ar)^3(1+a)^2(1+r)^{4m}}{(1-ar+|a-r|)^4(1+ar)^2|a-r|(1-r)^2 r^{-2(1-m)}} \quad . \tag{30}$$

Proof. Using the Lemma 2, Corollary 2 and after simple calculations, we get

$$|K| = \frac{|w'(z)|^2}{|g'(z)h'(z)|(1+|w(z)|)^4} \leq \frac{|w'(z)|^2(1-ar)(1+r)^{4m}}{(1+|w(z)|)^4|a-r|r^{-2(1-m)}} \tag{31}$$

and using the Schwarz-Pick's Lemma for the function

$$\psi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)}w(z)},$$

we obtain

$$|w'(z)|^2 \leq \frac{(1-|w(z)|^2)^2}{(1-r^2)^2} \quad . \tag{32}$$

The inequalities (16), (17) and (32) now yield (30). □

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