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Certain classes of p-valent functions defined by convolution [1]

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Abstract

Making use of a differential operator and a given p-valent analytic function g the authors introduce and study two new classes $TS_g^*(p,q,n,\alpha)$ and $TC_g(p,q,n,\alpha)$ of p-valent analytic functions with negative coefficients. They also indicate relevant connections of these classes with sevesral other families of multivalent functions which were studied by earlier workers as well as new.

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1 Introduction

Let $T(p,n)$ denote the class of functions of the form:

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k \quad (a_k\geqslant 0; k\geqslant n+p; p,n\in N=\{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U=\{z:|z|<1\}$.

A function $f(z)$ in the class $T(p,n)$ is said to be p-valently starlike of order α $(0\leq\alpha<p)$ if it satisfies the condition :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \ \ (z\in U; 0\leq\alpha<p; p\in N), \tag{1.2}$$

and is said to be p-valently convex of order α $(0\leq\alpha<p)$ if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \ \ (z\in U; 0\leq\alpha<p; p\in N). \tag{1.3}$$

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Denote by $T_n^*(p,\alpha)$ and $C_n(p,\alpha)$ the classes of p-valently starlike and p-valently convex functions of order α, respectively, which were introduced and studied by Owa [5]. It is known that (see [3] and [5])

$$f(z) \in C_n(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in T_n^*(p,\alpha). \tag{1.4}$$

The classes $T_1^*(p,\alpha) = T^*(p,\alpha)$ and $C_1(p,\alpha) = C(p,\alpha)$ were studied by Owa [4].

For a function $f(z)$ in the class $T(p,n)$, we define

$$D_p^0 f(z) = f(z),$$

$$D_p^1 f(z) = D_p f(z) = \frac{z}{p} f'(z),$$

$$D_p^2 f(z) = D(D_p f(z)),$$

$$D_p^\sigma f(z) = D(D_p^{\sigma-1} f(z)) = z^p - \sum_{k=n+p}^{\infty} (\frac{k}{p})^\sigma a_k z^k \ \ (\sigma \in N). \tag{1.5}$$

For $p = n = 1$, the differential operator D^σ was introduced by Sālāgean [7].

Let $(f * g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geqslant 0), \tag{1.6}$$

then

$$(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k. \tag{1.7}$$

For a function $f(z)$ of the form (1.1) we define the classes $TS_g^*(p,q,n,\alpha)$ and $TC_g(p,q,n,\alpha)$ as follows:

Definition 1. *Let $g(z)$ be a fixed function defined by* (1.6). *The function $f(z)$ of the form* (1.1) *is said to be in the class $TS_g^*(p,q,n,\alpha)$ if and only if*

$$Re\left\{\frac{z((f*g)(z))^{(q+1)}}{((f*g)(z))^{(q)}}\right\} > \alpha \ (0 \le \alpha < p-q, p \in N, q \in N_0 = N \cup \{0\}\,; p > q), \tag{1.8}$$

and is in the class $TC_g(p,q,n,\alpha)$ if and only if

$$Re\left\{1 + \frac{z((f*g)(z))^{(q+2)}}{((f*g)(z))^{(q+1)}}\right\} > \alpha \ (0 \le \alpha < p-q, p \in N, q \in N_0 = N \cup \{0\}\,; p > q), \tag{1.9}$$

where

$$(1.10) \qquad f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q},$$

$$(1.11) \qquad \delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j=0) \\ i(i-1)...(i-j+1) & (j \neq 0). \end{cases}$$

It follows from (1.8) and (1.9) that

$$(1.12) \qquad f^{(q)}(z) \in TC_g(p,q,n,\alpha) \Leftrightarrow \frac{zf^{(q+1)}(z)}{p-q} \in TS_g^*(p,q,n,\alpha).$$

Several well-known subclasses of functions are special cases of the classes $TS_g^*(p,q,n,\alpha)$ and $TC_g(p,q,n,\alpha)$ for suitable choices of $g(z)$. For example :

(i) for $q=0$ and replacing $n+p$ by m, we have, $TS_g^*(p,0,n,\alpha) = TS_g^*(p,m,\alpha)$ (Ali et al.[1]);

(ii) for $q=0$ and $b_k = 1$ $(k \geqslant n+p)$, we have

$$TS_g^*(p,0,n,\alpha) = \begin{cases} T_n^*(p,\alpha) & \text{(Owa [5])} \\ T_\alpha(p,n) & \text{(Yamakawa [9])} \end{cases}$$

and

$$TC_g(p,0,n,\alpha) = \begin{cases} C_n(p,\alpha) & \text{(Owa [5]} \\ C_\alpha(p,n) & \text{(Yamakawa [9].} \end{cases}$$

We also have the following new classes:

(iii) for $b_k = (\frac{k}{p})^\sigma$, we get the class:

$$(1.13) \qquad TS^*(p,q,n,\alpha,\sigma) = \left\{ f(z) \in T(p,n) : Re\left(\frac{z(D_p^\sigma f(z))^{(q+1)}}{(D_p^\sigma f(z))^{(q)}} \right) > \alpha \right\},$$

where $0 \leq \alpha < p-q, p \in N,\ q;\sigma \in N_0 = N \cup \{0\}\,; p > q$ and D_p^σ is defined by (1.5);

(iv) for $b_k = \begin{pmatrix} \lambda+k-1 \\ k-p \end{pmatrix}$ $(\lambda > -p)$ we get the class:

$$(1.14) \qquad TS^*(p,q,n,\alpha,\lambda) = \left\{ f(z) \in T(p,n) : Re\left(\frac{z(\Omega^{p,\lambda} f(z))^{(q+1)}}{(\Omega^{p,\lambda} f(z))^{(q)}} \right) > \alpha \right\},$$

where $0 \leq \alpha < p-q, p \in N,\ q \in N_0; \lambda > -p; p > q$ and $\Omega^{p,\lambda}$ is the extended Ruscheweyh derivative of order λ which was investigated by Raina and Srivastava [6];

(v) for $b_k = \frac{(\alpha_1)_{k-p}(\alpha_2)_{k-p}...(\alpha_r)_{k-p}}{(\beta_1)_{k-p}(\beta_2)_{k-p}...(\beta_s)_{k-p}(k-p)!} \geqslant 0$ $(\alpha_j \in C, j = 1, 2, ...r; \beta_i \in C \backslash \{0, -1, -2, ...\}; j = 1, ...s)$, we get the class:

$$TS^*(p, q, n, \alpha, \lambda) = \left\{ f(z) \in T(p, n) : Re \left(\frac{z(H_s^r[\alpha_1]f)^{(q+1)}(z)}{(H_s^r[\alpha_1]f)^{(q)}(z)} \right) > \alpha \right\}$$

$$(0 \leq \alpha < p - q; z \in U; r \leq s + 1; q, r, s \in N_0; p \in N; p > q).$$

The operator

$$H_s^r[\alpha_1]f)(z) = H_s^r(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s)f(z),$$

is the Dziok-Srivastava linear operator (see for details [2]).

2 Coefficient Estimate

Theorem 1. *Let the function* $f(z)$ *be in the form* (1.1). *Then* $f(z)$ *is in the class* $TS_g^*(p, q, n, \alpha)$ *if and only if*

$$\sum_{k=n+p}^{\infty} (k - q - \alpha)\delta(k, q)b_k a_k \leq (p - q - \alpha)\delta(p, q), \tag{2.1}$$

where $\delta(i, j)$ *is given by* (1.11).

Proof. Let the function $f(z)$ be in the class $TS_g^*(p, q, n, \alpha)$, then we have

$$Re\left\{ \frac{z((f * g)(z))^{(q+1)}}{((f * g)(z))^{(q)}} \right\} = Re \left\{ \frac{\delta(p, q + 1) - \sum\limits_{k=n+p}^{\infty} \delta(k, q + 1)b_k a_k z^{k-p}}{\delta(p, q) - \sum\limits_{k=n+p}^{\infty} \delta(k, q)b_k a_k z^{k-p}} \right\} > \alpha.$$

Letting $z \to 1^-$ through real values, we can see that

$$\delta(p, q + 1) - \sum_{k=n+p}^{\infty} \delta(k, q + 1)b_k a_k \geqslant \alpha \left[\delta(p, q) - \sum_{k=n+p}^{\infty} \delta(k, q)b_k a_k \right].$$

Thus we have the required inequality (2.1).

For the converse, let (2.1) holds true, then

$$\left| \frac{z((f * g)(z))^{q+1}}{((f * g)(z))^{q}} - (p - q) \right|$$

$$= \left| \frac{\sum\limits_{k=n+p}^{\infty} (k - p)\delta(k, q)b_k a_k z^k}{\delta(p, q) - \sum\limits_{k=n+p}^{\infty} \delta(k, q)b_k a_k z^k} \right|$$

$$\leq \frac{\sum\limits_{k=n+p}^{\infty}(k-p)\delta(k,q)b_k a_k}{\delta(p,q)-\sum\limits_{k=n+p}^{\infty}\delta(k,q)b_k a_k} \leq p-q-\alpha.$$

This shows that the values of $\frac{z((f*g)(z))^{q+1}}{((f*g)(z))^q}$ lie in a circle centered at $w = p-q$ and whose radius is $p-q-\alpha$. Hence $f(z) \in TS^*_g(p,q,n,\alpha)$ which completes the proof of Theorem 1.

Theorem 2. *Let the function* $f(z)$ *be in the form* (1.1). *Then* $f(z)$ *is in the class* $TC_g(p,q,n,\alpha)$ *if and only if*

$$\sum_{k=n+p}^{\infty}\left(\frac{k-q}{p-q}\right)(k-q-\alpha)\delta(k,q)b_k a_k \leq (p-q-\alpha)\delta(p,q). \tag{2.2}$$

Proof. Since $f^{(q)}(z) \in TC_g(p,q,n,\alpha) \Leftrightarrow \frac{zf^{(q+1)}(z)}{p-q} \in TS^*_g(p,q,n,\alpha)$, we may replace a_k by $\frac{(k-q)a_k}{p-q}$ in Theorem 1. Thus we have the theorem.

Corollary 1. *If the function* $f(z)$ *of the form* (1.1) *is in the class* $TS^*_g(p,q,n,\alpha)$, *then*

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k} \quad (k \geqslant n+p; p,n \in N). \tag{2.3}$$

Equality is attained for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k}z^k \quad (k \geqslant n+p; p,n \in N). \tag{2.4}$$

If the function $f(z)$ *of the form* (1.1) *is in the class* $TC_g(p,q,n,\alpha)$, *then*

$$a_k \leq \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(k-q)(k-q-\alpha)\delta(k,q)b_k} \quad (k \geqslant n+p; p,n \in N). \tag{2.5}$$

Equality is attained for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(k-q)(k-q-\alpha)\delta(k,q)b_k}z^k \quad (k \geqslant n+p; p,n \in N). \tag{2.6}$$

3 Distortion Theorems

Theorem 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $TS^*_g(p,q,n,\alpha)$. *Then for* $|z| = r < 1$ *and* $b_k \geqslant b_{n+p}$, $k \geqslant n+p$, *we have*

$$\left|f^{(m)}(z)\right| \leq \left[\delta(p,m) + \frac{(p-q-\alpha)\delta(p,q)\delta(n+p,m)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right]r^{p-m}, \tag{3.1}$$

and

$$\left|f^{(m)}(z)\right| \geqslant \left[\delta(p,m) - \frac{(p-q-\alpha)\delta(p,q)\delta(n+p,m)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^{p-m}. \tag{3.2}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}z^{n+p} \quad (z \in U). \tag{3.3}$$

Proof. Under the hypothesis of Theorem 3, we find from the assertion (2.1) of Theorem 1 that

$$\sum_{k=n+p}^{\infty} k!a_k \leq \frac{(p-q-\alpha)(n+p-q)!\delta(p,q)}{(n+p-q-\alpha)b_{n+p}} \; (n,p \in N; q \in N_0; p > q). \tag{3.4}$$

Now, the inequalities (3.1) and (3.2) would follow readily when we make use of (3.4) in conjunction with the series expansion for $f^{(m)}(z)$ $(m \in N_0)$ given by (1.10).

Putting $m = 0$ in Theorem 3, we have

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $TS_g^*(p,q,n,\alpha)$. Then for $|z| = r < 1$ and $b_k \geqslant b_{n+p}$ $(k \geqslant n+p)$, we have*

$$|f(z)| \leq \left[1 + \frac{(p-q-\alpha)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^p, \tag{3.5}$$

and

$$|f(z)| \geqslant \left[1 - \frac{(p-q-\alpha)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^p. \tag{3.6}$$

The result is sharp for the function $f(z)$ given by (3.3).

Putting $m = 1$ in Theorem 3, we have

Corollary 3. *Let the function $f(z)$ defined by (1.1) be in the class $TS_g^*(p,q,n,\alpha)$. Then for $|z| = r < 1$ and $b_k \geqslant b_{n+p}$ $(k \geqslant n+p)$, we have*

$$\left|f'(z)\right| \leq \left[p + \frac{(p-q-\alpha)(n+p)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^{p-1}, \tag{3.7}$$

and

$$\left|f'(z)\right| \geqslant \left[p - \frac{(p-q-\alpha)(n+p)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^{p-1}. \tag{3.8}$$

The result is sharp for the function $f(z)$ given by (3.3).

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $TC_g(p,q,n,\alpha)$. Then for $|z| = r < 1$ and $b_k \geqslant b_{n+p}$ $(k \geqslant n+p)$, we have*

$$\left|f^{(m)}(z)\right| \leq \left[\delta(p,m) + \frac{(p-q)(p-q-\alpha)\delta(p,q)\delta(n+p,m)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}}r^n\right] r^{p-m}, \tag{3.9}$$

and

$$\left|f^{(m)}(z)\right| \geqslant \left[\delta(p,m) - \frac{(p-q)(p-q-\alpha)\delta(p,q)\delta(n+p,m)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} r^n\right] r^{p-m}. \tag{3.10}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} z^{n+p} \quad (z \in U). \tag{3.11}$$

Putting (i) $m = 0$ and (ii) $m = 1$ in Theorem 4, we obtain the following results.
Corollary 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $TC_g(p,q,n,\alpha)$. *Then for* $|z| = r < 1$ *and* $b_k \geqslant b_{n+p}$ $(k \geqslant n+p)$, *we have*

$$|f(z)| \leq \left[1 + \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} r^n\right] r^p, \tag{3.12}$$

and

$$|f(z)| \geqslant \left[1 - \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} r^n\right] r^p. \tag{3.13}$$

The result is sharp for the function $f(z)$ *given by* (3.11).
Corollary 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $TC_g(p,q,n,\alpha)$. *Then for* $|z| = r < 1$ *and* $b_k \geqslant b_{n+p}$ $(k \geqslant n+p)$, *we have*

$$\left|f'(z)\right| \leq \left[p + \frac{(p-q)(p-q-\alpha)(n+p)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} r^n\right] r^{p-1}, \tag{3.14}$$

and

$$\left|f'(z)\right| \geqslant \left[p - \frac{(p-q)(p-q-\alpha)(n+p)\delta(p,q)}{(n+p-q)(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} r^n\right] r^{p-1}. \tag{3.15}$$

The result is sharp for the function $f(z)$ *given by* (3.11).

4 Closure Theorems

Let the functions $f_\nu(z)$ $(\nu = 1, 2, ...l)$ be defined by

$$f_\nu(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geqslant 0). \tag{4.1}$$

We shall prove the following results for the closure functions in the class $TS_g^*(p,q,n,\alpha)$.
Theorem 5. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2, ...l)$ *defined by* (4.1) *be in the class* $TS_g^*(p,q,n,\alpha)$. *Then the function* $h(z)$ *defined by*

$$h(z) = \sum_{\nu=1}^{l} c_\nu f_\nu(z) \quad (c_\nu \geqslant 0), \tag{4.2}$$

is also in the class $TS_g^*(p,q,n,\alpha)$, *where*

$$\sum_{\nu=1}^{l} c_\nu = 1.$$

Proof. According to the definition of $h(z)$, it can be written as

$$\begin{aligned} h(z) &= \sum_{\nu=1}^{l} c_\nu [z^p - \sum_{k=n+p}^{\infty} a_{k,\nu} z^k] \\ &= \sum_{\nu=1}^{l} c_\nu z^p - \sum_{\nu=1}^{l} \sum_{k=n+p}^{\infty} c_\nu a_{k,\nu} z^k \end{aligned}$$

$$= z^p - \sum_{k=n+p}^{\infty} \sum_{\nu=1}^{l} c_\nu a_{k,\nu} z^k. \tag{4.3}$$

Furthermore, since the functions $f_\nu(z)$ $(\nu = 1, 2, ...l)$ are in the class $TS_g^*(p,q,n,\alpha)$, then

$$\sum_{k=n+p}^{\infty} (k-q-\alpha)\delta(k,q) b_k a_k \leq (p-q-\alpha)\delta(p,q).$$

Hence

$$\begin{aligned} &\sum_{k=n+p}^{\infty} (k-q-\alpha)\delta(k,q) b_k (\sum_{\nu=1}^{l} c_\nu a_{k,\nu}) \\ = &\sum_{\nu=1}^{l} c_\nu \left\{ \sum_{k=n+p}^{\infty} (k-q-\alpha)\delta(k,q) b_k a_{k,\nu} \right\} \leq (p-q-\alpha)\delta(p,q), \end{aligned}$$

which implies that $h(z)$ be in the class $TS_g^*(p,q,n,\alpha)$.

Corollary 6. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by* (4.1) *be in the class* $TS_g^*(p,q,j,\alpha)$. *Then the function* $h(z)$ *defined by*

$$h(z) = (1-\eta) f_1(z) + \eta f_2(z) \ \ (0 \leq \eta \leq 1), \tag{4.4}$$

is also in the class $TS_g^*(p,q,n,\alpha)$.

As a consequence of Corollary 7 there exists extreme points of the class $TS_g^*(p,q,n,\alpha)$.

Theorem 6. *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q) b_k} z^k \ \ (k \geqslant n+p; p, n \in N). \tag{4.5}$$

Then the function $f(z)$ is in the class $TS_g^*(p,q,n,\alpha)$ if and only if it can be expressed in the form:

$$f(z) = \lambda_p\, z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z), \tag{4.6}$$

where $(\lambda_p \geqslant 0, \lambda_k \geqslant 0, k \geqslant n+p)$ *and* $\lambda_p + \sum\limits_{k=n+p}^{\infty} \lambda_n = 1.$

Proof. Suppose that $f(z)$ is expressed in the form (4.6).Then

$$\begin{aligned} f(z) &= \lambda_p\, z^p + \sum_{k=n+p}^{\infty} \lambda_k \left[z^p - \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k} z^k \right] \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k} \lambda_k z^k. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} . \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k} \lambda_k \\ =& \sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_p \leq 1. \end{aligned}$$

Then, $f(z) \in TS_g^*(p,q,n,\alpha)$.

Conversely, suppose that $f(z) \in TS_g^*(p,q,n,\alpha)$. We may set

$$\lambda_k = \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} a_k,$$

where a_k is given by (2.3).Then

$$\begin{aligned} f(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k} \lambda_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty} [z^p - f_k(z)]\lambda_k \\ &= (1 - \sum_{k=n+p}^{\infty} \lambda_k) z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z) \\ &= \lambda_p z^p + \sum_{k=n+p}^{\infty} \lambda_k f_k(z). \end{aligned}$$

This completes the proof of Theorem 6.

Theorem 7. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2, ...l)$ *defined by* (4.1) *be in the class* $TC_g(p, q, n, \alpha)$. *Then the function* $h(z)$ *defined by*

$$R(z) = \sum_{\nu=1}^{l} c_\nu f_\nu(z) \quad (c_\nu \geqslant 0),$$

is also in the class $TC_g(p, q, n, \alpha)$, *where*

$$\sum_{\nu=1}^{l} c_\nu = 1.$$

Corollary 7. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by* (4.1) *be in the class* $TC_g(p, q, j, \alpha)$. *Then the function* $h(z)$ *defined by*

$$R(z) = (1 - \eta) f_1(z) + \eta f_2(z) \;\; (0 \le \eta \le 1), \tag{4.7}$$

is also in the class $TC_g(p, q, n, \alpha)$.

Corollary 8. *The extreme points of the class* $TC_g(p, q, n, \alpha)$ *are* $f(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{(p-q)(p-q-\alpha)\delta(p,q)}{(k-q)(k-q-\alpha)\delta(k,q)b_k} z^k \;\; (k \geqslant n+p). \tag{4.8}$$

5 Modified Hadamard Products

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by (4.1). The modified Hadamard product of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1} a_{k,2} z^k. \tag{5.1}$$

Theorem 8. *Let the functions* $f_\nu(z)$ $(\nu = 1, 2)$ *defined by* (4.1) *be in the* $TS_g^*(p, q, n, \alpha)$ *and* $b_{n+p} \geqslant b_k$ $(k \geqslant n+p)$. *Then we have* $(f_1 * f_2)(z) \in TS_g^*(p, q, n, \beta)$, *where*

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)b_{n+p} - (p-q-\alpha)^2\delta(p,q)} \;\; (p, n \in N). \tag{5.2}$$

The result is sharp for the functions $f_\nu(z)$ *given by*

$$f_\nu(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(n+p-q-\alpha)\delta(n+p,q)b_{n+p}} z^{n+p} \;\; (\nu = 1, 2). \tag{5.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [8], we need to find the largest $\beta = \beta(p, q, n, \alpha)$ such that

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\beta)\delta(k,q)b_k}{(p-q-\beta)\delta(p,q)} a_{k,1} a_{k,2} \le 1. \tag{5.4}$$

Since the functions $f_\nu(z)$ $(\nu = 1,2)$ belong to the class $TS_g^*(p,q,n,\alpha)$, then from Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} a_{k,\nu} \leq 1. \tag{5.5}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{5.6}$$

Thus, it is sufficient to show that

$$\frac{(k-q-\beta)}{(p-q-\beta)} \sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)},$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-q-\alpha)(p-q-\beta)}{(p-q-\alpha)(k-q-\beta)}. \tag{5.7}$$

But from (5.5) we have

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-q-\alpha)\delta(p,q)}{(k-q-\alpha)\delta(k,q)b_k}. \tag{5.8}$$

Consequently, we need only to prove that

$$\frac{(p-q-\alpha)(k-q-\beta)}{(k-q-\alpha)(p-q-\beta)} \leq \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)},$$

or, equivalently, that

$$\beta \leq (p-q) - \frac{(k-p)(p-q-\alpha)^2\delta(p,q)}{(k-q-\alpha)^2\delta(k,q)b_k - (p-q-\alpha)^2\delta(p,q)}. \tag{5.9}$$

Since the right hand side of (5.9) is an increasing function of k $(k \geqslant n+p)$. Hence, we have

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)b_{n+p} - (p-q-\alpha)^2\delta(p,q)}.$$

This completes the proof of Theorem 8.

Theorem 9. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (4.1) be in the $TC_g(p,q,n,\alpha)$. Then we have $(f_1 * f_2)(z) \in TC_g(p,q,n,\beta)$, where*

$$\beta = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q+1)}{(n+p-q-\alpha)^2\delta(n+p,q+1)b_{n+p} - (p-q-\alpha)^2\delta(p,q+1)}. \tag{5.10}$$

The result is sharp for the functions $f_\nu(z)$ *given by*

$$f_\nu(z) = z^p - \frac{(p-q-\alpha)\delta(p,q+1)}{(n+p-q-\alpha)\delta(n+p,q+1)b_{n+p}} z^{n+p} \ (\nu = 1,2). \tag{5.11}$$

Theorem 10. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (4.1) *be in the* $TS_g^*(p,q,n,\alpha)$. *Then the function*

$$h(z) = z^p - \sum_{k=n+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k, \tag{5.12}$$

is in the class $TS_g^*(p,q,n,\beta)$, *where*

$$\beta = (p-q) - \frac{2n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)b_{n+p} - 2(p-q-\alpha)^2\delta(p,q)}. \tag{5.13}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ *given by* (5.3).
Proof. From Theorem 1, we have

$$\sum_{k=n+p}^{\infty} \left\{ \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} \right\}^2 a_{k,\nu}^2$$

$$\leq \left\{ \sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} a_{k,\nu} \right\}^2 \leq 1 \ (\nu = 1,2). \tag{5.14}$$

It follows that

$$\sum_{k=n+p}^{\infty} \frac{1}{2} \left\{ \frac{(k-q-\alpha)\delta(k,q)b_k}{(p-q-\alpha)\delta(p,q)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{5.15}$$

Therefore, we need to find the largest β such that

$$\frac{(k-q-\beta)}{(p-q-\beta)} \leq \frac{1}{2} \frac{(k-q-\alpha)^2\delta(k,q)b_k}{(p-q-\alpha)^2\delta(p,q)} \ \ (k \geqslant n+p),$$

that is, that

$$\beta \leq (p-q) - \frac{2(k-p)(p-q-\alpha)^2\delta(p,q)}{(k-q-\alpha)^2\delta(k,q)b_k - 2(p-q-\alpha)^2\delta(p,q)}. \tag{5.16}$$

Since the right hand side of (5.16) is an increasing function of k and $b_{k+1} \geqslant b_k$ $(k \geqslant n+p)$, then, setting $k = n+p$ in (5.16), we have (5.13). This completes the proof of Theorem 10.
Theorem 11. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (4.1) *be in the* $TC_g(p,q,n,\alpha)$. *Then the function* $h(z)$ *given by* (5.12) *is in the class* $TC_g(p,q,n,\beta)$, *where*

$$\beta = (p-q) - \frac{2n(p-q-\alpha)^2\delta(p,q+1)}{(n+p-q-\alpha)^2\delta(n+p,q+1)b_{n+p} - 2(p-q-\alpha)^2\delta(p,q+1)}. \tag{5.17}$$

The result is sharp for the functions $f(z)$ $(\nu = 1,2)$ *given by* (5.11).

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