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Regularity and Normality on L-Topological Spaces: (I) [1]

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Abstract

In the present paper, we define regularity and normality like strong S_1 regularity, S_1 regularity, weak S_1 regularity, strong S_1 normality, S_1 normality as well as weak S_1 normality on L-topological spaces. Also we investigate some of their properties and the relations between them.

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1 Introduction

The concept of fuzzy topology was first defined in 1968 by chang [2] and redefined by Hutton and Reilly [4] and others. A new definition of fuzzy topology introduced by Badard [1] under the name of "smooth topology". The smooth topological space was rediscovered by Ramadan [5].

In the present work, it has been studied the concepts of separation axioms like strong S_1 regularity, S_1 regularity, weak S_1 regularity, strong S_1 normality, S_1 normality aswell as weak S_1 normality on L-topological spaces. Also it has been investigated some of their properties and the relations between them.

2 Preliminaries

Throughout this paper, L, L' represent two completely distributive lattice with the smallest element 0 (or $\perp$) and the greatest element 1 (or $\top$), where $0 \neq 1$. We define M(L) to be the set of all non-zero $\vee$-irreducible (or coprime) elements in L such

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that $a \in M(L)$ iff $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. Let X be a non-empty usual set, and L^X be the set of all L-fuzzy sets on X.

For each $a \in L$, let $\underline{a}$ denote the constant-valued L-fuzzy set with a as its value. Let $\underline{0}$ and $\underline{1}$ be the smallest element and the greatest element in L^X, respectively. For the empty set $\emptyset \subset L$, we define $\wedge\emptyset = 1$ and $\vee\emptyset = 0$.

For every L-fuzzy subset $A \in L^X$, define its support set by $\{x \in X \quad : A(x) > 0\}$, denoted by $supp(A)$.

Definition 1 *A L-fuzzy topology on X is a map $\tau : L^X \to L$ satisfying the following three axioms:*
1) $\tau(\underline{\top}) = \top$;
2) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ for every $A, B \in L^X$;
3) $\tau(\vee_{i\in\triangle} A_i) \geq \bigvee_{i\in\triangle} \tau(A_i)$ for every family $\{A_i | i \in \triangle\} \subseteq L^X$.

The pair (X, τ) is called an L-fuzzy topological space. For every $A \in L^X, \tau(A)$ is called the degree of openness of the fuzzy subset A. For $a \in L$ and a map $\tau : L^X \to L$, we define

$$\tau_{[a]} = \{A \in L^X \mid \tau(A) \geq a\}.$$

Definition 2 *A smooth topological space (sts) [3] is an ordered pair (X, τ), where X is a non-empty set and $\tau : L^X \to L'$ is a mapping satisfying the following properties :*
(O1) $\tau(\underline{0}) = \tau(\underline{1}) = 1_L$,
(O2) $\forall A_1, A_2 \in L^X, \tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$,
(O3) $\forall I, \tau(\bigcup_{i\in I} A_i) \geq \bigwedge_{i\in I} \tau(A_i)$.

Definition 3 *A smooth cotopology is defined as a mapping $\Im : L^X \to L'$ which satisfies*
(C1) $\Im(\underline{0}) = \Im(\underline{1}) = 1_{L'}$,
(C2) $\forall B_1, B_2 \in L^X, \Im(B_1 \cup B_2) \geq \Im(B_1) \wedge \Im(B_2)$,
(C3) $\forall I, \Im(\bigcap_{i\in I} B_i) \geq \bigwedge_{i\in I} \Im(B_i)$.

In this paper we suppose $L' = L$.
The mapping $\Im_t : L^X \to L'$, defined by $\Im_t(A) = \tau(A^c)$ where τ is a smooth topology on X, is smooth cotopology on X. Also $\tau_\Im : L^X \to L'$, defined by $\tau_\Im(A) = \Im(A^c)$ where $\Im$ is a smooth cotopology on X, is a smooth topology on X where A^c denotes the complement of A [3].

Definition 4 *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping ; then [3], f is smooth continuous iff $\Im_{\tau_2}(A) \leq \Im_{\tau_1}(f^{-1}(A)), \forall A \in L^Y$.*

Definition 5 *A map $f : X \to Y$ is called smooth open (resp. closed) with respect to the smooth topologies τ_1 an τ_2 (resp. cotopologies $\Im_1$ and $\Im_2$), respectively, iff for each $A \in L^X$ we have $\tau_1(A) \le \tau_2(f(A))$ (resp. $\Im_1(A) \le \Im_2(f(A))$), where*

$$f(C)(y) = sup\left\{C(x) : x \in f^{-1}(\{y\})\right\}, \textit{ if } \quad f^{-1}(\{y\}) \neq \emptyset,$$
$$\textit{and } f(C)(y) = 0 \quad \textit{otherwise.}$$

Let $\tau : L^X \to L$ be a sts, and $A \in L^X$, then the τ-smooth closure of A, denoted by $\overline{A}$, is defined by

$$\overline{A} = A, \quad \text{if } \Im_\tau(A) = 1_L,$$
$$\text{and } \overline{A} = \bigcap \left\{F : F \in L^X, F \supseteq A, \Im_\tau(F) > \Im_\tau(A)\right\}, \quad \text{if } \Im_\tau(A) \neq 1_L.$$

Definition 6 *A map $f : X \to Y$ is called a smooth homeomorphism with respect to the smooth topologies τ_1 and τ_2 iff f is bijective and f and f^{-1} are smooth continuous.*

Let (X, τ_1) and (Y, τ_2) be two smooth topological spaces and $f : X \to Y$ a bijective map. The following statements are equivalent [3]:
1. f is a smooth homeomorphism,
2. f is a smooth open and smooth continuous,
3. f is a smooth closed and smooth continuous.
A property which is preserved under smooth homeomorphism is said to be a smooth topological property.

A map $f : X \to Y$is called L-preserving (resp. strictly L-preserving) with respect to the L-topologies $\tau_{1_{[a]}}$ and $\tau_{2_{[a]}}$, for each $a \in M(L)$ respectively, iff for every $A, B \in L^Y$ with $\tau_2(A), \tau_2(B) \ge a$, we have

$$\tau_2(A) \ge \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) \ge \tau_1(f^{-1}(B))$$
$$(\text{resp. } \tau_2(A) > \tau_2(B) \Rightarrow \tau_1(f^{-1}(A)) > \tau_1(f^{-1}(B)) \text{ [3].}$$

Let $f : X \to Y$ be a strictly L-preserving and continuous map with respect to the L-topologies $\tau_{1_{[a]}}$ and $\tau_{2_{[a]}}$, respectively, then for every $A \in L^Y$ with $\tau_2(A) \ge a, f^{-1}(\overline{A}) \supseteq \overline{f^{-1}(A)}$.

3 Main results

Definition 7 *A L-topology space $(X, \tau_{[a]})$ for each $a \in M(L)$ is called*
(a) strong s_1 regular (resp. strong S_2 regular) space iff for each $C \in L^X$, satisfying $\Im_\tau(C) > 0$, and each $x \in X$ satisfying $x \notin suppC$, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \ge a$ such that $x \in suppA$ (resp. $x \in supp(A \setminus \overline{B}), \tau(A) \ge A(x), C \subseteq B, \tau(B) \ge \Im_\tau(C)$ and $\overline{A} \cap \overline{B} = \underline{0}$ (resp. $\overline{A} \subseteq (\overline{B})^C$),
(b) s_1 regular (resp. S_2 regular) space iff for each $C \in L^X$, satisfying $\Im_\tau(C) > 0$,

and each $x \in X$ *satisfying* $x \notin suppC$*, there exist* $A, B \in L^X$ *with* $\tau(A), \tau(B) \geq a$ *such that* $x \in suppA$ *(resp.* $x \in supp(A \setminus B), \tau(A) \geq A(x), C \subseteq B, \tau(B) \geq \Im_\tau(C)$ *and* $A \cap B = \underline{0}$ *(resp.* $A \subseteq B^c$*),*
(c) weak s_1 *regular (resp. weak* S_2 *regular) space iff for each* $C \in L^X$*, satisfying* $\Im_\tau(C) > 0$*, and each* $x \in X$ *satisfying* $x \notin suppC$*, there exist* $A, B \in L^X$ *with* $\tau(A), \tau(B) \geq a$ *such that* $x \in suppA \setminus suppB^\circ$ *(resp.* $x \in supp(A \setminus B^\circ), \tau(A) \geq A(x), C \subseteq B, \tau(B) \geq \Im_\tau(C)$ *and* $A^\circ \cap B^\circ = \underline{0}$ *(resp.* $A^\circ \subseteq (B^\circ)^c$*).*

Definition 8 *An L-topology space* $(X, \tau_{[a]})$ *for each* $a \in M(L)$ *is called*
(a) strong S_1 *normal (resp. strong* S_2 *normal) space iff for each* $C, D \in L^X$ *such that* $C \subseteq (D^c)$ *(resp.* $C \cap D = \underline{0}$*),* $\Im_\tau(C) > 0$ *and* $\Im_\tau(D) > 0$*, there exist* $A, B \in L^X$ *with* $\tau(A), \tau(B) \geq a$ *such that* $C \subseteq A, \tau(A) \geq \Im_\tau(C), D \subseteq B, \tau(B) \geq \Im_\tau(D)$ *and* $\overline{A} \cap \overline{B} = \underline{0}$ *(resp.* $\overline{A} \subseteq (\overline{B})^c$*),*
(b) S_1 *normal (resp.* S_2 *normal) space iff for each* $C, D \in L^X$ *such that* $C \subseteq (D^c)$ *(resp.* $C \cap D = \underline{0}$*),* $\Im_\tau(C) > 0$ *and* $\Im_\tau(D) > 0$*, there exist* $A, B \in L^X$ *with* $\tau(A), \tau(B) \geq a$ *such that* $C \subseteq A, \tau(A) \geq \Im_\tau(C), D \subseteq B, \tau(B) \geq \Im_\tau(D)$ *and* $A \cap B = \underline{0}$ *(resp.* $A \subseteq (B)^c$*),*
(c) weak S_1 *normal (resp. weak* S_2 *normal) space iff for each* $C, D \in L^X$ *such that* $C \subseteq (D^c)$ *(resp.* $C \cap D = \underline{0}$*),* $\Im_\tau(C) > 0$ *and* $\Im_\tau(D) > 0$*, there exist* $A, B \in L^X$ *with* $\tau(A), \tau(B) \geq a$ *such that* $C \subseteq A, \tau(A) \geq \Im_\tau(C), D \subseteq B, \tau(B) \geq \Im_\tau(D)$ *and* $A^\circ \cap B^\circ = \underline{0}$ *(resp.* $A^\circ \subseteq (B^\circ)^c$*).*

Lemma 1 *Let* $(X, \tau_{[a]})$ *be an L-topology space for each* $a \in M(L), A, B \in L^X$ *and* $\tau(A), \tau(B) \geq a$*. Then the following properties hold:*
(i) $suppA \setminus suppB \subseteq supp(A \setminus B)$*,*
(ii) $suppA \setminus supp\overline{B} \subseteq suppA \setminus suppB \subseteq suppA \setminus suppB^\circ$*,*
(iii) $A \setminus \overline{B} \subseteq A \setminus B \subseteq A \setminus B^\circ$*,*
(iv) $A \cap B = \underline{0} \Rightarrow A \subseteq B^c$*.*

Proof. (i) Consider $x \in suppA \setminus suppB$. Then we obtain $A(x) > 0$ and $B(x) = 0$. Hence, $min(A(x), 1 - B(x)) = A(x) > 0$, i.e., $x \in supp(A \setminus B)$. The reverse inclusion in (i) is not true as can be seen from the following counterexample. Let $X = \{x_1, x_2\}, A(x_1) = 0.5, B(x_1) = 0.3$. Then we have $x_1 \in supp(A \setminus B)$ and $x_1 \notin suppA \setminus suppB$.
(ii) and (iii) easily follow from $B^\circ \subseteq B \subseteq \overline{B}$.
(iv) See [4]. □

Proposition 1 *Let* $(X, \tau_{[a]})$ *be an L-topology space for each* $a \in M(L)$*. Then the relationships as shown in Fig. 2. hold.*

Proof. All the implications in Fig.2 are straightforward consequences of Lemma 1 As an example we prove that strong S_1 normal implies strong S_2 normal. Suppose that $(X, \tau_{[a]})$ is a strong S_1 normal space and let $C, D \in L^X$ such that $C \cap D = \underline{0}, \Im_\tau(C) > 0$ and $\Im_\tau(D) > 0$.

From Lemma 1 (iv) it follows that $C \subseteq D^c$. Since $(X, \tau_{[a]})$ is strong S_1 normal there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $C \subseteq A, \tau(A) \geq \Im_\tau(C), D \subseteq B, \tau(B) \geq \Im_\tau(D)$ and $\overline{A} \cap \overline{B} = \underline{0}$. From Lemma 1 it follows that $\overline{A} \subset (\overline{B})^c$ and hence $(X, \tau_{[a]})$ is strong S_2 normal. □

strong S_1 regular $\Rightarrow$ S_1 regular $\Rightarrow$ weak S_1 regular

$\Downarrow$ $\Downarrow$ $\Downarrow$

strong ST_2 regular $\Rightarrow$ ST_2 regular $\Rightarrow$ weak ST_2 regular

strong S_1 normal $\Rightarrow$ S_1 normal $\Rightarrow$ weak S_1 normal

$\Downarrow$ $\Downarrow$ $\Downarrow$

strong S_2 normal $\Rightarrow$ S_2 normal $\Rightarrow$ weak S_2 normal

Fig. 2. Relationship between the different regularity and normality notions.

Proposition 2 *The S_i (i = 1, 2) regularity (resp. normality) property is a topological property. when $f : (X, \tau_1) \to (Y, \tau_2)$ be an smooth homeomorphism or $f : (X, \tau_{1_{[a]}}) \to (Y, \tau_{2_{[a]}})$ be an homeomorphism for each $a \in M(L)$.*

Proof. As an example we give the proof for S_2 normality when $f : X \to Y$ be a homeomorphism from S_2 normal space $(X, \tau_{1_{[a]}})$ onto a space $(Y, \tau_{2_{[a]}})$ for each $a \in M(L)$. Let $C, D \in L^Y$ such that $C \cap D = \underline{0}, \Im_{\tau_2}(C) > 0$ and $\Im_{\tau_2}(D) > 0$. Since f is bijective and continuous, from $C' \in \tau_{2_{[a]}}$ we have $f^{-1}(C') \in \tau_{1_{[a]}}$. From here, $\tau_2(C') \geq a$ then $\tau_1(f^{-1}(C')) \geq a$. It follows that $\tau_1(f^{-1}(C')) \geq \tau_2(C')$, hence $\tau_1((f^{-1}(C))') \geq \tau_2(C') > 0$.
Now we obtain that $\Im_{\tau_1}(f^{-1}(C)) \geq \Im_{\tau_2}(C) > 0$. Similarly, $\Im_{\tau_1}(f^{-1}(D)) \geq \Im_{\tau_2}(D) > 0$. we know that $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D) = f^{-1}(\underline{0}) = \underline{0}$. Since $(X, \tau_{1_{[a]}})$ is S_2 normal, there exist $A, B \in L^X$ with $\tau(A), \tau(B) \geq a$ such that $f^{-1}(C) \subseteq A, \tau_1(A) \geq \Im_{\tau_1}(f^{-1}(C)), f^{-1}(D) \subseteq B, \tau_1(B) \geq \Im_{\tau_1}(f^{-1}(D))$ and $A \subseteq B^c$. Since f is L-open and L-closed, it follows that$\tau_2(f(A)) \geq \tau_1(A), \tau_2(f(B)) \geq \tau_1(B), \Im_{\tau_2}(C) \geq \Im_{\tau_1}(f^{-1}(C))$ and $\Im_{\tau_2}(D) \geq \Im_{\tau_1}(f^{-1}(D))$, and hence, $\tau_2(f(A)) \geq \Im_{\tau_1}(f^{-1}(C)) = \Im_{\tau_2}(C)$,
$\tau_2(f(B)) \geq \Im_{\tau_1}(f^{-1}(D)) = \Im_{\tau_2}(D), C \subseteq f(A), D \subseteq f(B)$ and $f(A) \subseteq f(B^c) = (f(B))^c$. So $(Y, \tau_{2_{[a]}})$ is S_2 normal. □

Proposition 3 *Let $f : X \to Y$ be an injective, L-closed, L-continuous map with respect to the L-topologies $\tau_{1_{[a]}}$ and $\tau_{2_{[a]}}$ respectively for each $a \in M(L)$. If $(Y, \tau_{2_{[a]}})$ is S_i (i = 1, 2) regular (resp. normality); then so is $(X, \tau_{1_{[a]}})$.*

Proof. As an example we give the proof for S_1 regularity. Let $C \in L^X$, satisfy $\Im_{\tau_1}(C) > 0$ and let $x \in X$ be such that $x \notin suppC$. Since f is injective and L-closed we have $f(x) \notin suppf(C)$ and $\Im_{\tau_2}(f(C)) \geq \Im_{\tau_1}(C) > 0$. Since $(Y, \tau_{2_{[a]}})$ is S_1 regular, there exist $A, B \in L^Y$ with $\tau(A), \tau(B) \geq a$ such that $f(x) \in suppA, \tau_2(A) \geq A(f(x)), f(C) \subseteq B, \tau_2(B) \geq \Im_{\tau_2}(f(C))$ and $A \cap B = \underline{0}$. Since f is injective and L-continuous, if $A \in \tau_{2_{[a]}}$ then $f^{-1}(A) \in \tau_{1_{[a]}}$. Hence when $\tau_2(A) \geq a$ then $\tau_1(f^{-1}(A)) \geq a$. Thus $\tau_1(f^{-1}(A)) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x)$. Similarly, $\tau_1(f^{-1}(B)) \geq \tau_2(B) \geq \Im_{\tau_1}(C)$. we know that $C \subseteq (f^{-1}(B)), f^{-1}(A)(x) = A(f(x)) > 0$, i.e., $x \in suppf^{-1}(A)$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\underline{0}) = \underline{0}$ and hence $(X, \tau_{1_{[a]}})$ is S_1 regular. $\square$

Proposition 4 *Let $f : X \to Y$ be a strictly L-preserving, injective, L-closed and L-continuous map with respect to the L-topologies $\tau_{1_{[a]}}$ and $\tau_{2_{[a]}}$ respectively for each $a \in M(L)$. If $(Y, \tau_{2_{[a]}})$ is strong S_i (i = 1, 2) regular (resp. normal) ; then so is $(X, \tau_{1_{[a]}})$.*

Proof. As an example we proof the strong S_2 regularity. Let $C \in L^X$, satisfying $\Im_{\tau_1}(C) > 0$ and let $x \in X$ such that $x \notin suppC$. Since f is injective and L-closed we have $f(x) \notin suppf(C)$ and $\Im_{\tau_2}(f(C)) \geq \Im_{\tau_1}(C) > 0$. Since $(Y, \tau_{2_{[a]}})$ is S_2 regular, there exist $A, B \in L^Y$ with $\tau(A), \tau(B) \geq a$ such that $f(x) \in supp(A \setminus \overline{B}), \tau_2(A) \geq A(f(x)), f(C) \subseteq B, \tau_2(B) \geq \Im_{\tau_2}(f(C))$ and $\overline{A} \subseteq (\overline{B})^c$. As f is injective, L-continuous and strictly L-preserving it follows that $\tau_1(f^{-1}(A)) \geq \tau_2(A) \geq A(f(x)) = f^{-1}(A)(x), \tau_1(f^{-1}(B)) \geq \Im_{\tau_1}(C), C \subseteq (f^{-1}(B)), [\overline{f^{-1}(A) \setminus f^{-1}(B)}](x) = [f^{-1}(A) \cap (\overline{f^{-1}(B)})^c](x) \geq (A) \cap f^{-1}(\overline{B})^c](x) = f^{-1}(A \cap (\overline{B})^c)(x) = f^{-1}(A \setminus \overline{B})(x) = (A \setminus \overline{B})f(x) > 0$, i.e., $x \in supp(f^{-1}(A) \setminus \overline{f^{-1}(B)})$ and $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A}) \subseteq f^{-1}(\overline{B})^c \subseteq (\overline{f^{-1}(B)})^c$, and hence $(X, \tau_{1_{[a]}})$ is strong S_2 regular. $\square$

References

[1] R. Badard, *Smooth axiomatics*, Ist IFSA Congress, Palma de Mallorea, July 1986.

[2] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., 24, 1968, 182-193.

[3] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal.Appl., 18, 1967, 145-174.

[4] A. A. Ramadan, *Smooth topological space*, Fuzzy Sets and Systems 48, 1992, 371-375.

[5] F. G. Shi, *Theory of molecular nested sets and its applications*, J. Yantai Teachers Univ., 1, 1996, 33-36.

[6] G. J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems, 47, 1992, 351-376.

[7] L. A. Zadeh, *Fuzzy sets*, Inform. and Control., 8, 1965, 338-353.

[8] J. Zhang, F. G. Shi, C. Y. Zheng, *On L-fuzzy topological spaces*, Fuzzy Sets and Systems, 149, 2005, 473-484.

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