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Buffon's problem with a cluster of needles and a lattice of rectangles [1]

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Abstract

In this paper we consider a cluster of n needles ($1 \leq n < \infty$) dropped at random onto a plane lattice of rectangles with side lengths A and B. Each needle of the cluster has length ℓ with $0 < \ell \leq \frac{1}{2}\min(A,B)$. All needles are connected with one joint. There are at most $2n$ intersections between the cluster and the lattice. The probabilities of the number of intersections are calculated, distributions are shown, limit distributions are stated and results of computer based experiments are presented.

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1 Introduction

We consider a cluster $\mathcal{Z}_{n,\ell}$ of n needles thrown at random onto a plane lattice $\mathcal{R}_{A,B}$ of rectangles (see figure 1). The fundamental cell of $\mathcal{R}_{A,B}$ is a rectangle with side lengths A and B. $\mathcal{R}_{A,B}$ is the union of the lattice $\mathcal{R}_A$ of vertical lines and the lattice $\mathcal{R}_B$ of horizontal lines. All needles of $\mathcal{Z}_{n,\ell}$ are connected with one rotating joint. Each needle has equal length ℓ.

Clusters $\mathcal{Z}_{n,\ell}$ and lattices of parallel lines are considered in [3, pp. 81-95]. A cluster $\mathcal{Z}_{2,\ell}$ ("needle with a joint") and a lattice of regular triangles as well as a lattice of regular hexagons are considered in [4] and [5] respectively.

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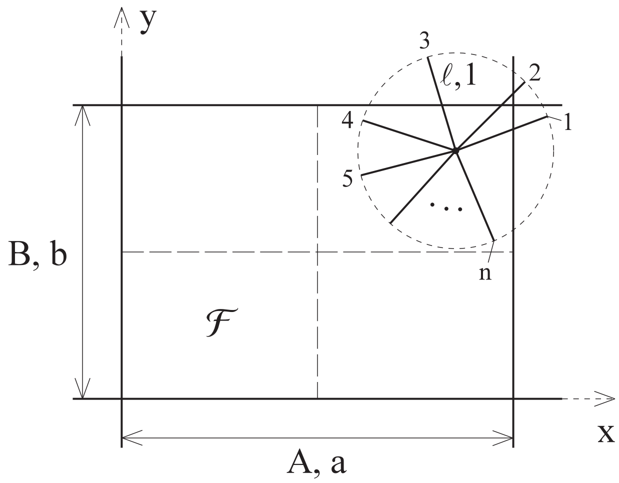

Figure 1: Lattice $\mathcal{R}_{A,\,B}$ ($\mathcal{R}_{a,\,b}$) and cluster $\mathcal{Z}_{n,\,\ell}$ ($\mathcal{Z}_n$)

We assume $\min(A, B) \geq 2\ell$ so that the cluster $\mathcal{Z}_{n,\,\ell}$ can intersect at most one line of $\mathcal{R}_A$ and (at the same time) one line of $\mathcal{R}_B$ (except sets with measure zero). A *random throw of* $\mathcal{Z}_{n,\,\ell}$ *onto* $\mathcal{R}_{A,\,B}$ means the following: After throwing $\mathcal{Z}_{n,\,\ell}$ onto $\mathcal{R}_{A,\,B}$ the coordinates x and y of the joint point are random variables uniformly distributed in $[0, A]$ and $[0, B]$ resp.; the angle ϕ_i between the x-axis and the needle i is for $i \in \{1, \ldots, n\}$ a random variable uniformly distributed in $[0, 2\pi]$. All $n + 2$ random variables are stochastically independent.

In order to simplify notation we define the cluster $\mathcal{Z}_n := \mathcal{Z}_{n,\,1}$ and the lattice $\mathcal{R}_{a,\,b}$, where $a := A/\ell$ and $b := B/\ell$. It has no influence on the probabilities to be calculated to use $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b}$ instead of $\mathcal{Z}_{n,\,\ell}$ and $\mathcal{R}_{A,\,B}$. If required we consider $\mathcal{R}_{a,\,b}$ as the union of the lattice $\mathcal{R}_a$ of vertical lines with distance a apart and the lattice of horizontal lines with distance b apart. Furthermore we will use the abbreviations $\lambda := 1/a = \ell/A$ and $\mu := 1/b = \ell/B$.

2 Intersection probabilities

The aim of this paragraph is the calculation of the probabibilities $p(i)$, $i \in \{0, \ldots, 2n\}$, of exactly i intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,\,b}$. Due existing symmetries it is sufficient to consider only the subset $\mathcal{F}$ of the fundamental cell (figure 1). For the calculations it is necessary to consider $\mathcal{F}$ as union of five subsets $F_1, \ldots, F_5$ (see figure 2):

$$\begin{aligned} F_1 &= \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le x \le a/2,\ 1 \le y \le b/2\}\,, \\ F_2 &= \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1,\ 1 \le y \le b/2\}\,, \\ F_3 &= \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le x \le a/2,\ 0 \le y \le 1\}\,, \\ F_4 &= \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1,\ \sqrt{1-x^2} \le y \le 1\}\,, \\ F_5 &= \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\,\}\,. \end{aligned}$$

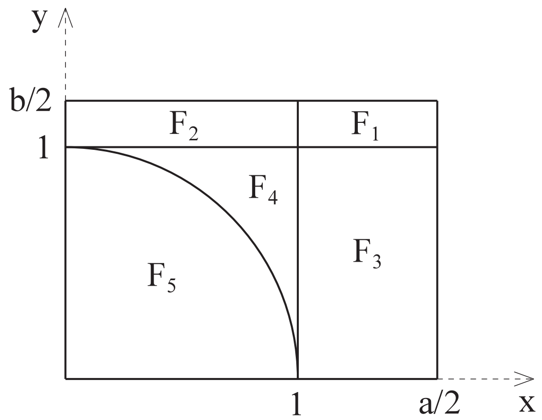

Figure 2: $\mathcal{F} = F_1 \cup \ldots \cup F_5$

With $p_k(i \,|\, (x,y))$ we denote the conditional probability, that $\mathcal{Z}_n$ with joint point $(x,y) \in F_k$ has exactly i intersections with $\mathcal{R}_{a,b}$. Considering the case distinctions for the subsets F_k the probabilities are calculated with

$$\begin{aligned} p(i) &= \frac{\sum_{k=1}^{5} \iint_{F_k} p_k(i \,|\, (x,y))\, \mathrm{d}x\, \mathrm{d}y}{\iint_{\mathcal{F}} \mathrm{d}x\, \mathrm{d}y} \\ &= \frac{4}{ab} \sum_{k=1}^{5} \iint_{F_k} p_k(i \,|\, (x,y))\, \mathrm{d}x\, \mathrm{d}y \\ &= 4\lambda\mu \sum_{k=1}^{5} \iint_{F_k} p_k(i \,|\, (x,y))\, \mathrm{d}x\, \mathrm{d}y\,. \qquad (1) \end{aligned}$$

If the joint point (x,y) of $\mathcal{Z}_n$ is in F_k and one needle of $\mathcal{Z}_n$ intersects $\mathcal{R}_{a,b}$ in exactly j points, this needle is inside an angle or in a disjoint union of angles. $\alpha_k^j(x,y)$ denotes the value of this angle or the sum of the values of this disjoint union. We set $\alpha_k^j(x,y) = 0$, if such an angle or such a union does not exist. Now we shall determine the functions $\alpha_k^j : F_k \to [0, 2\pi]$. With these functions we can calculate the conditional probabilities $p_k(i \,|\;) : F_k \to [0, 1]$ and with formula (1) the probabilites $p(i)$.

Subset F_1: If the joint point (x, y) of $\mathcal{Z}_n$ is inside F_1, we have $\alpha_1^0(x, y) = 2\pi$, $\alpha_1^1(x, y) = 0$ and $\alpha_1^2(x, y) = 0$, since the needles of $\mathcal{Z}_n$ cannot intersect any line of $\mathcal{R}_{a,b}$. Hence $p_1(0 \,|\, (x, y)) = 1$, $p_1(1 \,|\, (x, y)) = 0$ and $p_1(2 \,|\, (x, y)) = 0$.

Subset F_2: If the joint point (x, y) is inside F_2, the needles can intersect only the line $x = 0$ of the fundamental cell. Hence

$$\alpha_2^1(x, y) = 2 \arccos x \,, \quad \alpha_2^2(x, y) = 0 \,, \quad \alpha_2^0(x, y) = 2\pi - \alpha_2^1(x, y) \,.$$

For the conditional probabilities we get

$$p_2(i \,|\, (x, y)) = \frac{1}{(2\pi)^n} \binom{n}{i} \alpha_2^1(x, y)^i \, \alpha_2^0(x, y)^{n-i} \,, \quad i \in \{0, 1, \ldots, n\} \,.$$

Subset F_3: If the joint point (x, y) is inside F_3, the needles can intersect only the line $y = 0$ of the fundamental cell. Hence

$$\alpha_3^1(x, y) = 2 \arccos y \,, \quad \alpha_3^2(x, y) = 0 \,, \quad \alpha_3^0(x, y) = 2\pi - \alpha_3^1(x, y)$$

and

$$p_3(i \,|\, (x, y)) = \frac{1}{(2\pi)^n} \binom{n}{i} \alpha_3^1(x, y)^i \, \alpha_3^0(x, y)^{n-i} \,, \quad i \in \{0, 1, \ldots, n\} \,.$$

Subset F_4: If the joint point (x, y) is inside F_4, a single needle can intersect $\mathcal{R}_{a,b}$ in at most one point. If this needle intersects $\mathcal{R}_{a,b}$, one of the alternative events occurs: a) the needle intersects the line $x = 0$ or b) the needle intersects the line $y = 0$. Consequently we have

$$\alpha_4^1(x, y) = 2(\arccos x + \arccos y) \,, \quad \alpha_4^2(x, y) = 0 \,, \quad \alpha_4^0(x, y) = 2\pi - \alpha_4^1(x, y)$$

and

$$p_4(i \,|\, (x, y)) = \frac{1}{(2\pi)^n} \binom{n}{i} \alpha_4^1(x, y)^i \, \alpha_4^0(x, y)^{n-i} \,, \quad i \in \{0, 1, \ldots, n\} \,.$$

Subset F_5: If the joint point (x, y) is inside F_5, a single needle can intersect $\mathcal{R}_{a,b}$ in 0, 1 or 2 points. The needle can intersect a) the line $x = 0$ or b) the line $y = 0$. These are not alternative events. We have

$$\begin{aligned} \alpha_5^1(x, y) &= \pi \,, \quad \alpha_5^2(x, y) = \arccos y - \arcsin x \ (\geq 0 \text{ in } F_5) \,, \\ \alpha_5^0(x, y) &= 2\pi - \alpha_5^1(x, y) - \alpha_5^2(x, y) \,. \end{aligned}$$

We denote by $i_0, i_1, i_2 \in \mathbb{N} \cup \{0\}$ the numbers of needles, that have 0, 1 and 2 resp. intersections with $\mathcal{R}_{a,b}$. There are exactly i intersecstions between $\mathcal{Z}_n$ and $\mathcal{R}_{a,b}$ for all i_1, i_2 with $i_1 + i_2 = i$ and $i_0 = n - i_1 - i_2$. According to rules of combinatorics the number of permutations of

$$\underbrace{2, \dots, 2}_{i_2\text{-times}}, \underbrace{1, \dots, 1}_{i_1\text{-times}}, \underbrace{0, \dots, 0}_{i_0\text{-times}} \quad \text{with} \quad i_0 + i_1 + i_2 = n$$

is given by

$$\frac{n!}{i_0!\, i_1!\, i_2!}\,.$$

Hence

$$p_5(i \,|\, (x,y)) = \frac{1}{(2\pi)^n} \sum_{j=0}^{[i/2]} \frac{n!}{j!\,(i-2j)!\,(n-i+j)!} \times \\ \times \alpha_5^2(x,y)^j\, \alpha_5^1(x,y)^{i-2j}\, \alpha_5^0(x,y)^{n-i+j}\,, \quad i \in \{0, 1, \dots, 2n\}\,,$$

where $[i/2]$ denotes the integer part of $i/2$. (Note that all summands with $n - i + j < 0$ vanish.)

So we have provided all formulas for the calculation of the intersection probabilities with formula (1).

3 Results

For $n = 1, 2, 3, 4$ the following intersection probabilities have been calculated with Mathematica.

<u>$n = 1$:</u>

$$p(0) = 1 - \frac{2}{\pi}(\lambda + \mu) + \frac{1}{\pi}\lambda\,\mu\,, \quad p(1) = \frac{2}{\pi}(\lambda + \mu) - \frac{2}{\pi}\lambda\,\mu\,, \quad p(2) = \frac{1}{\pi}\lambda\,\mu\,.$$

These are the intersection probabilities of the Laplace problem [6]. For $\mu = 0$ one gets

$$p(0) = 1 - \frac{2\lambda}{\pi}\,, \quad p(1) = \frac{2\lambda}{\pi} \quad \text{and} \quad p(2) = 0\,.$$

This is the solution of the classical Buffon problem.

<u>$n = 2$:</u>

$$p(0) = 1 - \left(\tfrac{4}{\pi^2} + \tfrac{2}{\pi}\right)(\lambda + \mu) + \left(\tfrac{2}{\pi^2} + \tfrac{5}{2\pi}\right)\lambda\,\mu\,, \quad p(1) = \tfrac{8}{\pi^2}(\lambda + \mu) - \tfrac{5}{\pi}\lambda\,\mu\,,$$

$$p(2) = \left(-\tfrac{4}{\pi^2} + \tfrac{2}{\pi}\right)(\lambda + \mu) + \left(-\tfrac{4}{\pi^2} + \tfrac{2}{\pi}\right)\lambda\,\mu\,, \quad p(3) = \tfrac{1}{\pi}\lambda\,\mu\,,$$

$$p(4) = \left(\tfrac{2}{\pi^2} - \tfrac{1}{2\pi}\right)\lambda\,\mu\,.$$

From $p(0)$ we see that

$$\left(\frac{4}{\pi^2}+\frac{2}{\pi}\right)(\lambda+\mu)-\left(\frac{2}{\pi^2}+\frac{5}{2\pi}\right)\lambda\,\mu$$

is the probability of at least one intersection. After slight manipulations this result is the same as in [1].

<u>$n=3$:</u>

$$\begin{aligned}
p(0) &= 1+\left(\tfrac{12}{\pi^3}-\tfrac{12}{\pi^2}-\tfrac{3}{2\pi}\right)(\lambda+\mu)+\left(\tfrac{33}{2\pi^3}+\tfrac{3}{2\pi^2}+\tfrac{3}{\pi}\right)\lambda\,\mu\,,\\
p(1) &= \left(-\tfrac{36}{\pi^3}+\tfrac{24}{\pi^2}-\tfrac{3}{2\pi}\right)(\lambda+\mu)+\left(-\tfrac{36}{\pi^3}+\tfrac{3}{\pi^2}-\tfrac{15}{4\pi}\right)\lambda\,\mu\,,\\
p(2) &= \left(\tfrac{36}{\pi^3}-\tfrac{12}{\pi^2}+\tfrac{3}{2\pi}\right)(\lambda+\mu)+\left(\tfrac{45}{2\pi^3}-\tfrac{3}{2\pi^2}-\tfrac{3}{\pi}\right)\lambda\,\mu\,,\\
p(3) &= \left(-\tfrac{12}{\pi^3}+\tfrac{3}{2\pi}\right)(\lambda+\mu)+\left(-\tfrac{12}{\pi^3}-\tfrac{6}{\pi^2}+\tfrac{9}{2\pi}\right)\lambda\,\mu\,,\\
p(4) &= \left(\tfrac{27}{2\pi^3}-\tfrac{3}{2\pi^2}\right)\lambda\,\mu\,,\quad p(5)=\left(\tfrac{3}{\pi^2}-\tfrac{3}{4\pi}\right)\lambda\,\mu\,,\\
p(6) &= \left(-\tfrac{9}{2\pi^3}-\tfrac{3}{2\pi^2}\right)\lambda\,\mu\,.
\end{aligned}$$

<u>$n=4$:</u>

$$\begin{aligned}
p(0) &= 1+\left(\tfrac{48}{\pi^4}+\tfrac{24}{\pi^3}-\tfrac{24}{\pi^2}-\tfrac{1}{\pi}\right)(\lambda+\mu)+\left(\tfrac{84}{\pi^4}+\tfrac{21}{2\pi^3}+\tfrac{3}{4\pi^2}+\tfrac{11}{4\pi}\right)\lambda\,\mu\,,\\
p(1) &= \left(-\tfrac{192}{\pi^4}-\tfrac{48}{\pi^3}+\tfrac{48}{\pi^2}-\tfrac{2}{\pi}\right)(\lambda+\mu)+\left(-\tfrac{384}{\pi^4}+\tfrac{57}{\pi^3}+\tfrac{3}{\pi^2}-\tfrac{1}{\pi}\right)\lambda\,\mu\,,\\
p(2) &= \left(\tfrac{288}{\pi^4}-\tfrac{24}{\pi^2}\right)(\lambda+\mu)+\left(\tfrac{624}{\pi^4}-\tfrac{177}{\pi^3}+\tfrac{3}{\pi^2}-\tfrac{25}{4\pi}\right)\lambda\,\mu\,,\\
p(3) &= \left(-\tfrac{192}{\pi^4}+\tfrac{48}{\pi^3}+\tfrac{2}{\pi}\right)(\lambda+\mu)+\left(-\tfrac{384}{\pi^4}+\tfrac{117}{\pi^3}-\tfrac{3}{\pi^2}+\tfrac{2}{\pi}\right)\lambda\,\mu\,,\\
p(4) &= \left(\tfrac{48}{\pi^4}-\tfrac{24}{\pi^3}+\tfrac{1}{\pi}\right)(\lambda+\mu)+\left(\tfrac{24}{\pi^4}-\tfrac{12}{\pi^3}-\tfrac{15}{2\pi^2}+\tfrac{17}{4\pi}\right)\lambda\,\mu\,,\\
p(5) &= \left(\tfrac{27}{\pi^3}-\tfrac{3}{\pi^2}-\tfrac{1}{\pi}\right)\lambda\,\mu\,,\quad p(6)=\left(\tfrac{48}{\pi^4}-\tfrac{15}{\pi^3}+\tfrac{3}{\pi^2}-\tfrac{3}{4\pi}\right)\lambda\,\mu\,,\\
p(7) &= \left(-\tfrac{9}{\pi^3}+\tfrac{3}{\pi^2}\right)\lambda\,\mu\,,\quad p(8)=\left(-\tfrac{12}{\pi^4}+\tfrac{3}{2\pi^3}+\tfrac{3}{4\pi^2}\right)\lambda\,\mu\,,
\end{aligned}$$

It is easy to verify that

$$\sum_{i=0}^{2n} p(i)=1 \quad\text{and}\quad \sum_{i=1}^{2n} i\,p(i)=\frac{2n(\lambda+\mu)}{\pi}$$

hold in all cases.

Additionally we give the following numerical approximations:

<u>$n=1$:</u>

$$\begin{aligned}
p(0) &\approx 1-0,63662\,(\lambda+\mu)+0,31831\,\lambda\,\mu\,,\\
p(1) &\approx 0,63662\,(\lambda+\mu)-0,63662\,\lambda\,\mu\,,\\
p(2) &\approx 0,31831\,\lambda\,\mu\,.
\end{aligned}$$

$\underline{n = 2:}$

$$\begin{aligned} p(0) &\approx 1 - 1,0419\,(\lambda+\mu) + 0,998417\,\lambda\,\mu\,, \\ p(1) &\approx 0,810569\,(\lambda+\mu) - 1,59155\,\lambda\,\mu\,, \\ p(2) &\approx 0,231335\,(\lambda+\mu) + 0,231335\,\lambda\,\mu\,, \\ p(3) &\approx 0,31831\,\lambda\,\mu\,, \\ p(4) &\approx 0,0434874\,\lambda\,\mu\,. \end{aligned}$$

$\underline{n = 3:}$

$$\begin{aligned} p(0) &\approx 1 - 1,3063\,(\lambda+\mu) + 1,63906\,\lambda\,\mu\,, \\ p(1) &\approx 0,793188\,(\lambda+\mu) - 2,05075\,\lambda\,\mu\,, \\ p(2) &\approx 0,422666\,(\lambda+\mu) - 0,381252\,\lambda\,\mu\,, \\ p(3) &\approx 0,0904464\,(\lambda+\mu) + 0,437449\,\lambda\,\mu\,, \\ p(4) &\approx 0,283414\,\lambda\,\mu\,, \\ p(5) &\approx 0,0652311\,\lambda\,\mu\,, \\ p(6) &\approx 0,00684987\,\lambda\,\mu\,. \end{aligned}$$

$\underline{n = 4:}$

$$\begin{aligned} p(0) &\approx 1 - 1,48321\,(\lambda+\mu) + 2,15233\,\lambda\,\mu\,, \\ p(1) &\approx 0,707655\,(\lambda+\mu) - 2,11815\,\lambda\,\mu\,, \\ p(2) &\approx 0,524894\,(\lambda+\mu) - 0,988022\,\lambda\,\mu\,, \\ p(3) &\approx 0,213625\,(\lambda+\mu) + 0,163949\,\lambda\,\mu\,, \\ p(4) &\approx 0,0370402\,(\lambda+\mu) + 0,452273\,\lambda\,\mu\,, \\ p(5) &\approx 0,248518\,\lambda\,\mu\,, \\ p(6) &\approx 0,0742253\,\lambda\,\mu\,, \\ p(7) &\approx 0,0136997\,\lambda\,\mu\,, \\ p(8) &\approx 0,001176\,\lambda\,\mu\,. \end{aligned}$$

4 Distributions

In this paragraph $p(i, n, \lambda, \mu)$ denotes the probability of exactly i intersections between $\mathcal{Z}_n$ and $\mathcal{R}_{a,b}$. We define the random variable

$$Z_{n,\lambda,\mu} := \frac{\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_{a,b}}{n}$$

and denote by $F_{Z_{n,\lambda,\mu}}$ the distribution of $Z_{n,\lambda,\mu}$

$$F_{Z_{n,\lambda,\mu}}(x) = P(Z_{n,\lambda,\mu} \le x) = \sum_{i:\,\frac{i}{n} \le x} p(i,n,\lambda,\mu)\,.$$

$X_{n,\lambda}$ and $Y_{n,\mu}$ denote the random variables

$$X_{n,\lambda} := \frac{\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_a}{n}$$

and

$$Y_{n,\mu} := \frac{\text{number of intersections between } \mathcal{Z}_n \text{ and } \mathcal{R}_b}{n}$$

resp. with distributions $F_{X_{n,\lambda}}(x) = P(X_{n,\lambda} \le x)$ and $F_{Y_{n,\mu}}(x) = P(Y_{n,\mu} \le x)$ respectively. We shall consider the convergence behaviour of $F_{Z_{n,\lambda,\mu}}$ as $n \to \infty$. It can be proved (see [3, pp. 90-93]), that

$$F_{X_\lambda}(x) := \lim_{n\to\infty} F_{X_{n,\lambda}}(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0\,, \\ 1 - 2\lambda\cos\pi x & \text{for} \quad 0 \le x < \frac{1}{2}\,, \\ 1 & \text{for} \quad \frac{1}{2} \le x < \infty \end{cases}$$

and analogously

$$F_{Y_\mu}(x) := \lim_{n\to\infty} F_{Y_{n,\mu}}(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0\,, \\ 1 - 2\mu\cos\pi x & \text{for} \quad 0 \le x < \frac{1}{2}\,, \\ 1 & \text{for} \quad \frac{1}{2} \le x < \infty\,. \end{cases}$$

Provided that the random variables X_λ and Y_μ are stochastically independent, the distribution $F_{Z_{\lambda,\mu}}$ of $Z_{\lambda,\mu} := X_\lambda + Y_\mu$ is given by

$$F_{Z_{\lambda,\mu}}(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0\,, \\ 1 - 2(\lambda+\mu)\cos\pi x \\ \qquad + 2(2\cos\pi x - \pi x\sin\pi x)\lambda\mu & \text{for} \quad 0 \le x < \frac{1}{2}\,, \\ 1 + 2\pi(x-1)\lambda\mu\sin\pi x & \text{for} \quad \frac{1}{2} \le x < 1\,, \\ 1 & \text{for} \quad 1 \le x < \infty\,. \end{cases}$$

A proof can be found in [2]. Although the random variables $X_{n,\lambda}$ and $Y_{n,\mu}$ are not stochastically independent for finite n, we suppose that the random variables $X_\lambda = \lim_{n\to\infty} X_{n,\lambda}$ and $Y_\mu = \lim_{n\to\infty} Y_{n,\mu}$ are stochastically independent as the diagrams in the figures 3, …, 6 suggest. Every diagram shows the limit distribution $F_{Z_{1/4,\,1/3}}$ compared to $F_{Z_{2,\,1/4,\,1/3}}$ in figure 3, to $F_{Z_{10,\,1/4,\,1/3}}$ in figure 4, $F_{Z_{20,\,1/4,\,1/3}}$ in figure 5 and to $F_{Z_{100,\,1/4,\,1/3}}$ in figure 6. (Remark: The distributions $F_{Z_{n,\,1/4,\,1/3}}$ with $n \in \{10, 20, 100\}$ are calculated numerically with Mathematica.)

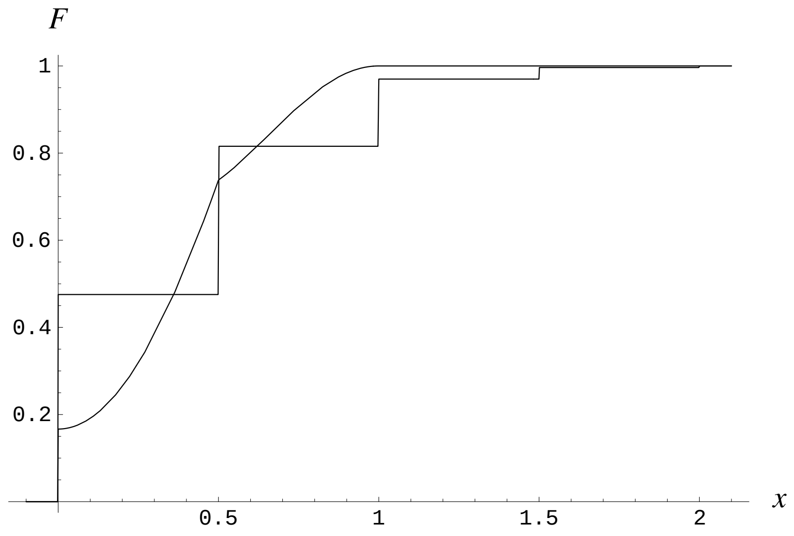

Figure 3: Distributions $F_{Z_{2,\,1/4,\,1/3}}$ and $F_{Z_{1/4,\,1/3}}$

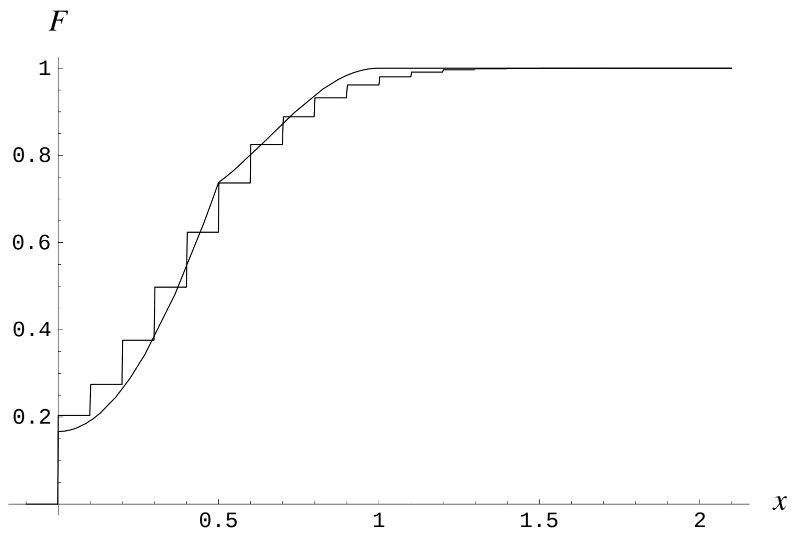

Figure 4: Distributions $F_{Z_{10,\,1/4,\,1/3}}$ and $F_{Z_{1/4,\,1/3}}$

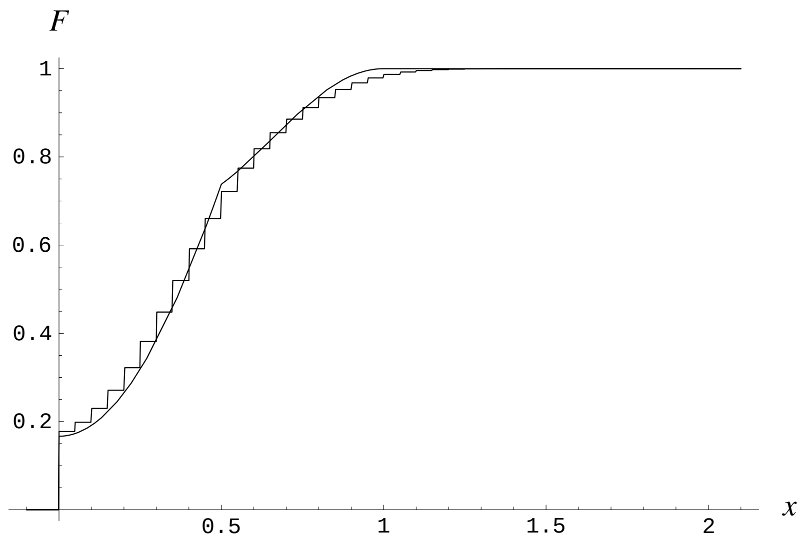

Figure 5: Distributions $F_{Z_{20,\,1/4,\,1/3}}$ and $F_{Z_{1/4,\,1/3}}$

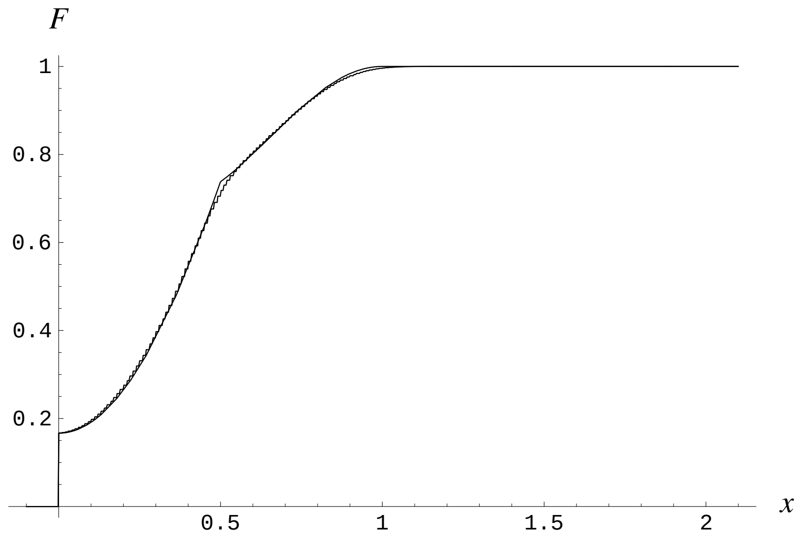

Figure 6: Distributions $F_{Z_{100,\,1/4,\,1/3}}$ and $F_{Z_{1/4,\,1/3}}$

5 Simulation

The theoretical results have been verified by computer based random experiments with a program written in Turbo-Pascal. With this program relative frequencies $h(i, n, \lambda, \mu)$ for the number of intersections are determined for several combinations of the parameters. In the following table one example is given:

Cluster $\mathcal{Z}_5$ and lattice $\mathcal{R}_{4,3}$ ($n = 5$, $\lambda = 1/4$, $\mu = 1/3$), 10^6 trials:

i	probability $p(i, 5, 1/4, 1/3)$	frequency $h(i, 5, 1/4, 1/3)$
0	0,275820	0,277043
1	0,189043	0,189301
2	0,208469	0,207930
3	0,162813	0,162663
4	0,0937358	0,093157
5	0,0436462	0,043567
6	0,0182233	0,018071
7	0,00640372	0,006487
8	0,00158286	0,001513
9	0,000245084	0,000247
10	0,0000178561	0,000021

References

[1] U. Bäsel, *Geometrische Wahrscheinlichkeiten für variable Testelemente auf Rechteckgittern*, FernUniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik, 78, 2007, 1-6.

[2] U. Bäsel, *Buffon's Problem with a Star of Needles and a Lattice of Rectangles*, FernUniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik, 79, 2008, 1-11.

[3] U. Bäsel, *Geometrische Wahrscheinlichkeiten für nichtkonvexe Testelemente*, Dissertation, FernUniversität Hagen, Hagen 2008.

[4] U. Bäsel and A. Duma, *Schnittwahrscheinlichkeiten für eine gleichschenklige Gelenknadel und ein reguläres Dreiecksgitter*, FernUniversität Hagen:

Seminarberichte aus der Fakultät für Mathematik und Informatik, 80, 2008, 121-138.

[5] U. Bäsel and A. Duma, *Intersection probabilities for a needle with a joint and a lattice of hexagons*, Annales de l'I.S.U.P., Université Pierre et Marie Curie, Paris, to appear.

[6] A. Duma, *Das Problem von Buffon (Tage der Forschung 1998, Hagener Universitätsreden 33.3)*, FernUniversität-Gesamthochschule Hagen, Hagen 1999.

[7] A. Duma and M. I. Stoka, *Geometrische Wahrscheinlichkeiten für Hindernisse und nicht-konvexe Testkörper in der Ebene*, FernUniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik. 78, 2007, 37-51.

[8] Santaló, L. A.: *Integral Geometry and Geometric Probability.* Addison-Wesley, London, 1976.

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