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Certain Classes of Multivalent Functions Defined by a Fractional Differential Operator [1]

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Abstract

In this paper, the fractional differential operator $\mathcal{D}^{n,\alpha}_{p,\lambda}$ is introduced and applied to define the classes $\mathcal{S}^{n;\alpha}_{p,\lambda}(\delta,\beta)$ and $\mathcal{T}^{n;\alpha}_{p,\lambda}(\delta,\beta)$ of $\beta-$uniformly convex and starlike $p-$valent functions and $p-$valent functions with negative coefficients, respectively. Several results concerning coefficient estimates and extreme points mainly for the class $\mathcal{T}^{n;\alpha}_{p,\lambda}(\delta,\beta)$ are obtained. Also results for a family of class preserving integral operators are considered.

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1 Introduction

Let $\mathcal{A}(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N}) \tag{1}$$

that are analytic and $p-$valent in the open unit disk $E = \{z : z \in \mathbb{C}, \mid z \mid < 1\}$

Let $\mathcal{T}(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of functions that are analytic and $p-$valent $f(z)$ expressed in the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0, p \in \mathbb{N}) \tag{2}$$

The fractional calculus are defined as follows (e.g.,[9,14])

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Definition 1 *The Riemann-Liouville fractional integral of order α is defined for the function f by*

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t)dt \qquad (\alpha > 0) \tag{3}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 2 *he Riemann-Liouville fractional derivative of order α is defined for the function f by*

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (z-t)^{-\alpha} f(t)dt \qquad (0 \leq \alpha < 1) \tag{4}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 3 *Under the hypothesis of Definition 1.2, the Riemann-Liouville fractional derivative of order $(n+\alpha)$ is defined for the function f by*

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^{\alpha} f(z) \qquad (0 \leq \alpha < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\ ). \tag{5}$$

The Gauss's hypergeometric function ${}_2F_1(a,b;c;z)$ is defined for $z \in E$ by the series expression

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \ c \neq 0, -1, -2, ... \tag{6}$$

where $(\sigma)_k$ is the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\sigma)_k = \frac{\Gamma(\sigma+k)}{\Gamma(\sigma)} = \begin{cases} 1 & ,(k=0, \sigma \in \mathbb{C} \setminus \{0\}) \\ \sigma(\sigma+1)...(\sigma+n-1) & ,(k=n \in \mathbb{N}, \sigma \in \mathbb{C}) \end{cases} \tag{7}$$

Let the function $\varphi_p(a,c;z)$ be defined as

$$\varphi_p(a,c;z) = z_2F_1(1,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p}, \quad c \neq 0, -1, -2, ...; z \in E \tag{8}$$

Now, let us define the operator $\Omega_p^{\alpha} : \mathcal{A}(p) \to \mathcal{A}(p)$ as

$$\begin{aligned} \Omega_p^{\alpha} f(z) &= \tfrac{\Gamma(1+p-\alpha)}{\Gamma(1+p)} z^{\alpha} D_z^{\alpha} f(z) \qquad , \ \alpha \neq p+1, p+2, ... \\ &= z^p + \sum_{k=1}^{\infty} \frac{(p+1)_k}{(p-\alpha+1)_k} a_{k+p} z^{k+p} \end{aligned} \tag{9}$$

$$= \varphi_p(p+1, p-\alpha+1; z) * f(z)$$

where $f * g$ is the well known Hadamard product (or convolution) written as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \tag{10}$$

for any two functions $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$ and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ belonging to $\mathcal{A}(p)$.
Now we introduce the linear fractional differential operator $\mathcal{D}_{p,\lambda}^{n,\alpha} : \mathcal{A}(p) \to \mathcal{A}(p)$ as follows

$$\begin{aligned}
\mathcal{D}_{p,\lambda}^{0,0} f(z) &= f(z) \\
\mathcal{D}_{p,\lambda}^{1,\alpha} f(z) &= (1-\lambda)\Omega_p^{\alpha} f(z) + \frac{\lambda z}{p}(\Omega_p^{\alpha} f(z))' = \mathcal{D}_{p,\lambda}^{\alpha} f(z) \\
\mathcal{D}_{p,\lambda}^{2,\alpha} f(z) &= \mathcal{D}_{p,\lambda}^{\alpha}(\mathcal{D}_{p,\lambda}^{1,\alpha} f(z)) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\mathcal{D}_{p,\lambda}^{n,\alpha} f(z) &= \mathcal{D}_{p,\lambda}^{\alpha}(\mathcal{D}_{p,\lambda}^{n-1,\alpha} f(z))
\end{aligned} \tag{11}$$

for $n \in \mathbb{N}$, $\lambda \geq 0$ and $0 \leq \alpha < 1$.

If f is given by (1), then making use of (9) and (11) we conclude that

$$\mathcal{D}_{p,\lambda}^{n,\alpha} f(z) = z^p + \sum_{k=1}^{\infty} \psi_{k,n}(\alpha, \lambda, p) a_{k+p} z^{k+p}, \quad n \in \mathbb{N}_0 \tag{12}$$

where

$$\psi_{k,n}(\alpha, \lambda, p) = \left[\frac{(p+1)_k}{(p-\alpha+1)_k} (1 + \frac{\lambda k}{p}) \right]^n \tag{13}$$

when $p = 1$, we get the fractional differential operator introduced and studied by Al-Oboudi and Al-Amoudi [3,4], when $p = 1$ and $\alpha = 0$ we get Al-Oboudi differential operator [2],when $\alpha = 0$, $\lambda = 1$ and $p = 1$ the Sălăgean operator is obtained [12], on setting $\lambda = 0$, $n = 1$ and $p = 1$ we obtain the Owa-Srivastava fractional differential operator [10] and when $\alpha = 0$ and $\lambda = 1$, we get the differential operator studied by Eker and Şeker [5].

Note that $\mathcal{D}_{p,\lambda}^{n,\alpha} f(z)$ can easily be written in terms of convolution as

$$\mathcal{D}_{p,\lambda}^{n,\alpha} f(z) = G(z) * f(z) \tag{14}$$

where $G(z) = \underbrace{\{(\varphi_p(p+1,p-\alpha+1;z)*g_{p,\lambda}(z))*...*(\varphi_p(p+1,p-\alpha+1;z)*g_{p,\lambda}(z))\}}_{n-times}$

and $g_{p,\lambda}(z) = \dfrac{z^p - (1-\frac{\lambda}{p})z^{p+1}}{(1-z)^2}$.If $f(z) \in \mathcal{T}(p)$is given by (2), then

$$D_{p,\lambda}^{n,\alpha} f(z) = z^p - \sum_{k=1}^{\infty} \psi_{k,n}(\alpha,\lambda,p) a_{k+p} z^{k+p}, \quad n \in \mathbb{N}_0 \tag{15}$$

where $\psi_{k,n}(\alpha,\lambda,p)$ is as given by (13).
If $(f \circledast g)(z)$ is the modified Hadamard product written as

$$(f \circledast g)(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}, \quad (a_{k+p}, b_{k+p} \geq 0) \tag{16}$$

for the functions $f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$ and $g(z) = z^p - \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ belonging to $\mathcal{T}(p)$. Then $\mathcal{D}_{p,\lambda}^{n,\alpha} f(z)$ can be written as

$$D_{p,\lambda}^{n,\alpha} f(z) = W(z) \circledast f(z) \tag{17}$$

where

$$W(z) = \underbrace{\{(\omega_p(p+1,p-\alpha+1;z) \circledast h_{p,\lambda}(z)) \circledast ... \circledast (\omega_p(p+1,p-\alpha+1;z) \circledast h_{p,\lambda}(z))\}}_{n-times},$$

$\omega_p\ (p+1,p-\alpha+1;z) = 2z^p - \varphi_p(p+1,p-\alpha+1;z)$ and $h_{p,\lambda}(z) = 2z^p - g_{p,\lambda}(z)$.
Consider the subclass $\mathcal{S}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ consisting of functions $f \in \mathcal{A}(p)$ and satisfying

$$\operatorname{Re}\left\{ \frac{z\left(D_{p,\lambda}^{n,\alpha} f(z)\right)'}{D_{p,\lambda}^{n,\alpha} f(z)} - \delta \right\} \geq \beta \left| \frac{z\left(D_{p,\lambda}^{n,\alpha} f(z)\right)'}{D_{p,\lambda}^{n,\alpha} f(z)} - p \right| \tag{18}$$

where

$$z \in E,\ \beta \geq 0,\ -p \leq \delta \leq p,\ 0 \leq \alpha < 1,\ \lambda \geq 0,\ p \in \mathbb{N} \text{ and } n \in N_0. \tag{19}$$

Define the subclass $\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ of functions $f \in \mathcal{T}(p)$ as

$$\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta) = \mathcal{S}_{p,\lambda}^{n,\alpha}(\delta,\beta) \cap \mathcal{T}(p) \tag{20}$$

It is interesting to note that the classes $\mathcal{S}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ and $\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ extends to the classes of starlike , convex, $\beta-$uniformly starlike and $\beta-$uniformly convex for suitable choice of the parameters n, α, p, λ, δ and β. For instant
(i) For $n = 1, \lambda = 0$ and $p = 1$, then $\mathcal{S}_{1,0}^{1,\alpha}(\delta,\beta) \equiv \beta - \mathcal{SB}_\alpha(\delta)$ and $\mathcal{T}_{1,0}^{1,\alpha}(\delta,\beta) \equiv \beta - \mathcal{TSB}_\alpha(\delta)$ which are the classes introduced and studied by Akbarally and

Darus [1].
(ii) For $\alpha = 0$ and $n = 0$, then $\mathcal{S}^{0,0}_{p,\lambda}(\delta,\beta) \equiv \mathcal{UST}(\delta,\beta,p)$ and for $n = 1$, $\lambda = 0$ and $\alpha \to 1$, then $\mathcal{S}^{1,1}_{p,0}(\delta,\beta) \equiv \mathcal{UCV}(\delta,\beta,p)$ where $\mathcal{UST}(\delta,\beta,p)$ and $\mathcal{UCV}(\delta,\beta,p)$ are the subclasses of $\beta-$uniformly starlike and $\beta-$uniformly convex functions studied recently by Khairnar and More [7]. Several other classes studied by several authors can be derived from the classes $\mathcal{S}^{n,\alpha}_{p,\lambda}(\delta,\beta)$ and $\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$, see e.g. [6,11,13].

The purpose of the present investigation is to investigate results concerning coefficient estimates and extreme points for $p-$valent functions $f(z)$ of negative coefficients belonging to the class $\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$. Also, we obtain some results for the integral operators known as Komatu and Jung-Kim-Srivastava integral operators.

2 Coefficient Estimates

Theorem 1 *A function $f(z) \in \mathcal{A}(p)$ defined by (1) is in $\mathcal{S}^{n,\alpha}_{p,\lambda}(\delta,\beta)$ if*

$$\sum_{k=1}^{\infty}[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)\,|a_{k+p}| \leq p-\delta \tag{21}$$

with the limits of parameters given in (19) and $\psi_{k,n}(\alpha,\lambda,p)$ given by (13).

Proof. It suffices to show that

$$\beta\left|\frac{z\left(D^{n,\alpha}_{p,\lambda}f(z)\right)'}{D^{n,\alpha}_{p,\lambda}f(z)}-p\right|-\mathrm{Re}\left\{\frac{z\left(D^{n,\alpha}_{p,\lambda}f(z)\right)'}{D^{n,\alpha}_{p,\lambda}f(z)}-\delta\right\}\leq p-\delta$$

Notice that

$$\beta\left|\frac{z\left(D^{n,\alpha}_{p,\lambda}f(z)\right)'}{D^{n,\alpha}_{p,\lambda}f(z)}-p\right|-\mathrm{Re}\left\{\frac{z\left(D^{n,\alpha}_{p,\lambda}f(z)\right)'}{D^{n,\alpha}_{p,\lambda}f(z)}-\delta\right\}$$

$$\leq(1+\beta)\left|\frac{z\left(D^{n,\alpha}_{p,\lambda}f(z)\right)'}{D^{n,\alpha}_{p,\lambda}f(z)}-p\right|\leq\frac{(1+\beta)\sum\limits_{k=1}^{\infty}k\psi_{k,n}(\alpha,\lambda,p)\,|a_{k+p}|}{1-\sum\limits_{k=1}^{\infty}\psi_{k,n}(\alpha,\lambda,p)\,|a_{k+p}|}$$

This expression is bounded by $(p-\delta)$ if

$$(1+\beta)\sum_{k=1}^{\infty}k\psi_{k,n}(\alpha,\lambda,p)\,|a_{k+p}|\leq(p-\delta)-(p-\delta)\sum_{k=1}^{\infty}\psi_{k,n}(\alpha,\lambda,p)\,|a_{k+p}|$$

which directly yields the inequality (21). □

Now we state and prove the necessary and sufficient condition for $f(z)$ to belong to the class $\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$.

Theorem 2 *A function $f(z) \in \mathcal{T}(p)$ defined by (2) is in $\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)a_{k+p} \leq p-\delta \tag{22}$$

with the limits of parameters given in (19) and $\psi_{k,n}(\alpha,\lambda,p)$ given by (13).

Proof. In view of Theorem 1, we need only to prove the sufficient part. Let $f(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ and z be real. Then by the relations (15) and (20), we get

$$\frac{p-\sum\limits_{k=1}^{\infty}(k+p)\psi_{k,n}(\alpha,\lambda,p)a_{k+p}z^k}{1-\sum\limits_{k=1}^{\infty}\psi_{k,n}(\alpha,\lambda,p)a_{k+p}z^k} - \delta \geq \beta \left| \frac{\sum\limits_{k=1}^{\infty}k\psi_{k,n}(\alpha,\lambda,p)a_{k+p}z^k}{1-\sum\limits_{k=1}^{\infty}\psi_{k,n}(\alpha,\lambda,p)a_{k+p}z^k} \right|$$

Allowing $z \to 1$ along the real axis, we get the desired inequality. The result is sharp for

$$f(z) = z^p - \frac{p-\delta}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}z^{k+p}, \qquad k \in \mathbb{N} \tag{23}$$

and the proof is complete. □

Corollary 1 *Let the function $f(z) \in \mathcal{T}(p)$ defined by (2) be in the class $\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$. Then*

$$a_{k+p} \leq \frac{p-\delta}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}, \qquad k \in \mathbb{N} \tag{24}$$

with equality for the function $f(z)$ given by (23) and the limits of parameters are given in (19) and $\psi_{k,n}(\alpha,\lambda,p)$ is given by (13).

3 Extreme Points for $\mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$

Theorem 3 *Let $f_0(z) = z^p$ and*

$$f_k(z) = z^p - \frac{(p-\delta)}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}z^{k+p}, \quad k \in \mathbb{N} \tag{25}$$

Then $f \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty}\eta_k f_k(z) \tag{26}$$

where $\eta_k \geq 0$ and $\sum\limits_{k=0}^{\infty}\eta_k = 1$, and $\psi_{k,n}(\alpha,\lambda,p)$ is given by (13).

Proof. Suppose that

$$f(z)=\sum_{k=0}^{\infty}\eta_k f_k(z)=z^p-\sum_{k=1}^{\infty}\eta_k\frac{(p-\delta)}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}z^{k+p}.$$

Then

$$\sum_{k=1}^{\infty}\frac{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)}\frac{(p-\delta)}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}\eta_k$$

$$=\sum_{k=1}^{\infty}\eta_k=(1-\eta_0)\leq 1$$

Therefore, $f(z)\in\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$.
Conversely, suppose that $f\in\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$, thus by (24)

$$a_{k+p}\leq\frac{p-\delta}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}$$

Setting $\eta_k=\frac{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)}a_{k+p}$, $k\in\mathbb{N}$ and $\eta_0=1-\sum_{k=1}^{\infty}\eta_k$, then $f(z)$ can be expressed in the form (26), and the proof is complete. □

Corollary 2 *The extreme points of* $\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$ *are the functions* $f_0(z)=z^p$ *and* $f_k(z)=z^p-\dfrac{(p-\delta)}{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}z^{k+p}$, $k\in\mathbb{N}$, *where* $\psi_{k,n}(\alpha,\lambda,p)$ *is given by (13).*

4 Family of Class Preserving Integral Operators

In this section, we discuss some class preserving integral operators. We recall here the Komatu operator [8] defined by

$$\mathcal{H}(z)=\mathcal{P}^d_{c,p}f(z)=\frac{(c+p)^d}{\Gamma(d)z^c}\int_0^z t^{c-1}\left(\log\frac{z}{t}\right)^{d-1}f(t)dt \tag{27}$$

where $d>0,c>-p$ and $z\in E$.
Also we recall the generalized Jung-Kim-Srivastava integral operator [7] defined by

$$\mathcal{I}(z)=\mathcal{Q}^d_{c,p}f(z)=\begin{pmatrix} d+c+p-1\\ c+p-1\end{pmatrix}\frac{d}{z^c}\int_0^z t^{c-1}\left(1-\frac{t}{z}\right)^{d-1}f(t)dt \tag{28}$$

Theorem 4 *If* $f(z)\in\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$, *then* $\mathcal{H}(z)\in\mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$.

Proof. Let the function $f(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ be defined by (2), then it can be easily verified that

$$\mathcal{H}(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{c+p}{c+k+p}\right)^d a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0, p \in \mathbb{N}) \tag{29}$$

Now $\mathcal{H}(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ if

$$\sum_{k=1}^{\infty} \frac{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)} \left(\frac{c+p}{c+k+p}\right)^d a_{k+p} \quad \leq 1 \tag{30}$$

Now, by Theorem 2, $f(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)} a_{k+p} \leq 1$$

Since $\dfrac{c+p}{c+k+p} \leq 1$ for $k \in \mathbb{N}$, then (30) holds true. Therefore $\mathcal{H}(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$.

Theorem 5 *Let $d > 0, c > -p$ and $f(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$. Then $\mathcal{H}(z)$ defined by (27) is $p-$valent in the disk $|z| < R_1$, where*

$$R_1 = \inf_k \left\{\frac{p[k(1+\beta)+p-\delta](c+k+p)^d \psi_{k,n}(\alpha,\lambda,p)}{(k+p)(c+p)^d(p-\delta)}\right\}^{\frac{1}{k}} \tag{31}$$

Proof. In order to prove the assertion , it is enough to show that

$$\left|\frac{\mathcal{H}'(z)}{z^{p-1}} - p\right| \leq p \tag{32}$$

Now, in view of (29), we get

$$\left|\frac{\mathcal{H}'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=1}^{\infty}(k+p)\left(\frac{c+p}{c+k+p}\right)^d a_{k+p} \; z^k \right|$$
$$\leq \sum_{k=1}^{\infty}(k+p)\left(\frac{c+p}{c+k+p}\right)^d a_{k+p} \; |z|^k$$

This expression is bounded by p if

$$\sum_{k=1}^{\infty} \frac{(k+p)}{p}\left(\frac{c+p}{c+k+p}\right)^d a_{k+p} \; |z|^k \leq 1 \tag{33}$$

Given that $f(z) \in \mathcal{T}_{p,\lambda}^{n,\alpha}(\delta,\beta)$, so in view of Theorem 2, we have

$$\sum_{k=1}^{\infty} \frac{[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)} a_{k+p} \leq 1$$

Thus, (33) holds if

$$(k+p)\left(\frac{c+p}{c+k+p}\right)^{d} a_{k+p} \ |z|^{k} \leq \frac{p[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(p-\delta)}, \qquad k \in \mathbb{N}$$

that is

$$|z| \leq \left\{\frac{p[k(1+\beta)+p-\delta](c+k+p)^{d}\psi_{k,n}(\alpha,\lambda,p)}{(k+p)(c+p)^{d}(p-\delta)}\right\}^{\frac{1}{k}}$$

The result follows by setting $|z| = R_1$. □

Following similar steps as in the proofs of Theorem 4.1 and Theorem 4.2, we can state the following two theorems concerning the generalized Jung-Kim-Srivastava integral operator $\mathcal{I}(z)$.

Theorem 6 *If* $f(z) \in \mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$, *then* $\mathcal{I}(z) \in \mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$.

Theorem 7 *Let* $d > 0, c > -p$ *and* $f(z) \in \mathcal{T}^{n,\alpha}_{p,\lambda}(\delta,\beta)$. *Then* $\mathcal{I}(z)$ *defined by (28) is* $p-$*valent in the disk* $|z| < R_2$, *where*

$$R_2 = \inf_{k}\left\{\frac{p[k(1+\beta)+p-\delta]\psi_{k,n}(\alpha,\lambda,p)}{(k+p)((p-\delta)}\frac{(p+c+d)_k}{(p+c)_k}\right\}^{\frac{1}{k}}. \tag{34}$$

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