General Mathematics Vol. 19, No. 3 (2011), 85–92

Existence Results for some Integral Equation with Modified Argument [1]

Monica Lauran

Abstract

One of the most important tool in establishing existence or uniqueness theorems for integral equations in the classical Banach's contraction mapping principle or some of its generalizations.
In this paper we shall illustrate how one can establish existence results and approximate the solutions of certain integral equations in the case when the contraction mapping principle does not apply, but we are able to use instead the technique oimations f nonexpansive mappings.

2000 Mathematics Subject Classification: 45B05, 45D05, 47H10.
Key words and phrases: Integral equation, Nonexpansiv mapping, Fixed point theorem.

1 Introduction

Most existence or existence and uniqueness theorems for integral equations are usually obtained by means of fixed point technique, e.g. by Schauder's fixed point theorem or by the contractions mapping principle. One of the most important tool in establishing existence or uniqueness theorems for integral equations in the classical Banach's contraction mapping principle or some of its generalizations.
In this paper we shall illustrate how one can approximate the solutions of certain integral equations in the case when the contraction mapping principle does not apply, but we are able to use instead the technique of nonexpansive mappings.

[1] *Received 09 March, 2010*
Accepted for publication (in revised form) 30 June, 2011

2 Preliminaries

The content of this section is taken from [2] and [5].
Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an *α-contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \le \alpha d(x, y), \ \ \forall x, y \in X.$$

The strict contraction condition ensure the existence and uniquness of fixed point for a α-contraction in a complete metric space and also the convergence of Picard iteration, defined by $x_0 \in X$ and

$$x_{n+1} = Tx_n, \ \ n = 0, 1, 2, ...$$

to that fixed point. In the case when $\alpha = 1$, T is said to be *nonexpansive*.
As a complete study of nonexpansive mappings with respect to their fixed points could be better done in a normed space setting, we shall present some concepts and results that will be used later in the paper. Let K be a nonempty subset of a real normed linear space E and let $T : K \to K$ be a map. A point $x \in K$ is called a *fixed point* of T if $Tx = x$. In this settings, T is *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \ \ \forall x, y \in K.$$

We can now formulate one of the most important fixed point theorems for nonexpansive mappings, due to Browder, Ghode and Kirk, see for example [1]

Theorem 2.1 *Let K be a nonempty closed convex and bounded subset of a uniformly banach space E. Then any nonexpansive mapping $T : K \to K$ has at least a fixed point.*

Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a self mapping. For $x_0 \in K$ and $\lambda \in [0, 1]$ the sequence x_n defined by

$$x_{n+1} = (1 - \lambda) \cdot x_n + \lambda \cdot Tx_n, n = 0, 1, 2...$$

is usually called *Krasnoselskij iteration.*
For $x_0 \in K$ the sequence x_n defined by

$$x_{n+1} = (1 - \lambda_n) \cdot x_n + \lambda_n \cdot Tx_n, n = 0, 1, 2...$$

where $\lambda_n \subset [0, 1]$ is a sequence of real number satisfying some appropiate condition, is called *Mann iteration.*

Theorem 2.2 *([4]) Let K be a subset of a Banach space E and let $T : K \to K$ be a nonexpansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process x_n under the following assumptions:*

(a) $x_n \in K$ for all positive integers;

(b) $0 \le \lambda_n \le b < 1$ *for all positive integers;*

(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$. *If* x_n *is bounded, then* $x_n - Tx_n \to 0$ *as* $n \to \infty$.

The following corolaries will be particulary important for application part of our paper.

Corollary 2.1 *([5]) Let* K *be a convex and compact subset of a Banach space* E *and let* $T : K \to K$ *be a nonexpansive mapping. If the Mann iteration process* x_n *satisfies assumptions (a)-(c) in Theorem 2.2 ,then* x_n *converges strongly to a fixed point of* T.

Corollary 2.2 *([5]) Let* K *be a closed bounded convex subset of a real normed space* E *and* $T : K \to K$ *be a nonexpansive mapping. If* $I - T$ *maps closed bounded subset of* E *into closed subset of* E *and* x_n *is the Mann iteration, with* λ_n *satisfying assuptions (a)-(c) in Theorem 2.2, then* x_n *converges strongly to a fixed point of* T *in* K.

3 Existence and uniqueness theorems for some integral equations

Consider the following integral equation:

$$y(x) = f(x) + \int_{x_0}^{t} K(x, y(y(x)))dx, \tag{1}$$

where $x_0, t \in [a, b]$, $y, f \in C[a, b]$, $K \in C([a, b] \times [a, b])$.
Denote

$$c_x = \max\{x - a, b - x\}, \ \ \forall x \in [a, b],$$

and

$$(*) C_L = \{y \in C([a, b] \times [a, b] : |y(t_1) - y(t_2)| \le L \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a, b]\}; L > 0$$

Theorem 3.1 *Assume that the following condition are satisfied:*

(i) $K \in C([a, b] \times [a, b])$;

(ii) $\exists L_1 > 0, L_2 > 0$ *such that*

$$|K(s, u) - K(s, v)| \le L_1 \cdot |u - v|, \ \ \forall s, u, v \in [a, b],$$

and

$$|f(t_1) - f(t_2)| \le L_2 \cdot |t_1 - t_2|, \ \ \forall t_1, t_2 \in [a, b].$$

(iii) If L *is the Lipschitz constant involved in (*), then*

$$M = \max\{|K(s, u)| : s, u \in [a, b]\}, \ \ \text{and } M + L_2 \le L$$

(iv) If $\exists x_0 \in [a,b]$ *such that* $|f(x)| \le f(x_0)$ *then, either:*

(a) $M \cdot c_{x_0} \le c_{y_0}, \quad y_0 = f(x_0)$*; or*

(b) $x_0 = a,\ M \cdot (b-a) \le b - y_0,\ K(s,u) \ge 0,\ \forall s,u \in [a,b]$*; or*

(c) $x_0 = b,\ M \cdot (b-a) \le y_0 - a,\ K(s,u) \ge 0,\ \forall s,u \in [a,b]$*;*

(v) $L_1 \cdot (L+1) \cdot c_{x_0} \le 1$

Then there exists at least solutions of (1) *in* C_L *wich can be approximate by the Krasnoselskij iteration*

$$y_{n+1} = (1-\lambda) \cdot y_n + \lambda f(x) + \lambda \int\limits_{x_0}^{t} K(x, y_n(y_n(x)))dx, t \in [a,b], n \ge 1$$

where $\lambda \in (0,1)$ *and* $y_1 \in C_L$*.*

Proof. The technique of proof is basically the one introduce in [2].
Consider the integral operator $A : C_L \to C[a,b]$

$$(Ay)(t) = f(t) + \int\limits_{x_0}^{t} K(s, y(y(s)))ds, \ \ x_0, t \in [a,b].$$

It is clear that $y \in C_L$ is a solution of (1) if only if y is a fixed point of A, that is $y = Ay$. We prove that C_L is an invariant set with respect to A, i.e.,we have $A(C_L) \subset C_L$.
If first condition (a) holds, then for any $y \in C_L$ and $t \in [a,b]$ we have:

$$|(Ay)(t)| = \left| f(t) + \int\limits_{x_0}^{t} K(s, y(y(s)))ds \right| \le |f(t)| + \int\limits_{x_0}^{t} |K(s, y(y(s)))| \, ds$$

$$\le f(x_0) + M \cdot |t - x_0| = f(x_0) + M \cdot c_{x_0} \le f(x_0) + c_{y_0} \le b$$

$$|(Ay)(t)| \ge |f(t)| - \left| \int\limits_{x_0}^{t} K(s, y(y(s)))ds \right| \ge -f(x_0) - M \cdot |t - x_0|$$

$$\ge -f(x_0) - c_{y_0} \ge a$$

So, $\forall y \in C_L$ we obtaine $(Ay)(t) \in C[a,b]$. We show that $Ay \in C_L$, $\forall y \in C_L$.
Now, for $\forall t_1, t_2 \in [a,b]$ we have

$$|(Ay)(t_1) - (Ay)(t_2)| \le |f(t_1) - f(t_2)| + \left| \int\limits_{x_0}^{t_1} K(s, y(y(s)))ds - \int\limits_{x_0}^{t_2} K(s, y(y(s)))ds \right|$$

$$\le |f(t_1) - f(t_2)| + \left| \int\limits_{t_1}^{t_2} K(s, y(y(s)))ds \right| \le L_2 \cdot |t_1 - t_2| + \int\limits_{t_1}^{t_2} |K(s, y(y(s)))| \, ds$$

$$\leq L_2 \cdot |t_1 - t_2| + M \cdot |t_1 - t_2| = (L_2 + M) \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2|$$

So, $(Ay) \in C_L$, $\forall y \in C_L$. In a similar way we treat the cases (b) and (c).
Let $y, z \in C_L$ and $t \in [a, b]$. Then

$$|(Ay)(t) - (Ay)(z)| = \left| \int_{x_0}^{t} K(s, y(y(s)))ds - \int_{x_0}^{t} K(s, z(z(s)))ds \right| \leq$$

$$\leq \int_{x_0}^{t} |K(s, y(y(s))) - K(s, z(z(s)))|\, ds \leq \int_{x_0}^{t} L_1 \cdot |y(y(s)) - z(z(s))|\, ds =$$

$$= L_1 \cdot \int_{x_0}^{t} |y(y(s)) - y(z(s)) + y(z(s)) - z(z(s))|\, ds \leq$$

$$L_1 \cdot \int_{x_0}^{t} (|y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))|)ds$$

$$\leq L_1 \cdot \left[L \cdot \int_{x_0}^{t} |y(s) - z(s)|\, ds + \max_{t \in [a,b]} |y(t) - z(t)| \cdot (t - x_0) \right]$$

Now, by applying the norm in last inequalitie, we get

$$||Ay - Az||\,_{C[a,b]} \leq L_1 \cdot (L + 1) \cdot c_{x_0} \cdot ||y - z||\,_{C[a,b]}$$

wich, in view of hypotesis (vi), proves that A is nonexpansive, hence continuous. Now, apply the Schauder's fixed point theorem, a has at least a fixed point. The second part of conclusion is given by Corollary 2.1 and Corollary 2.2.

We consider the equation

$$(2) \qquad y(x) = f(x) + \int_{x_0}^{t} K(x, y(x), y(y(x)))dx,$$

where $y, f \in C[a, b]$ and $K \in C([a, b] \times [a, b] \times [a, b])$ are given.
Denote $c_x = \max\{x - a, b - x\}$, $\forall x \in [a, b]$ and

$$(*) C_L = \{y \in C([a, b] \times [a, b] : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|\,, \forall t_1, t_2 \in [a, b]\}; L > 0$$

Theorem 3.2 *Assume that for* (2), *the following conditions are satisfied:*

(i) $K \in C([a, b] \times [a, b] \times [a, b])$;

(ii) $\exists L_1 > 0, L_2 > 0$ *such that*

$$|K(s,u_1,u_2) - K(s,v_1,v_2)| \le L_1 \cdot (|u_1 - v_1| + |u_2 - v_2|), \;\; \forall s, u_i, v_i \in [a,b], i = 1,2$$

and

$$|f(t_1) - f(t_2)| \le L_2 \cdot |t_1 - t_2|\,, \;\; \forall t_1, t_2 \in [a,b].$$

(iii) If L is the Lipschitz constant involved in (*), then

$$M = \max\{|K(s,u,v)| : s,u,v \in [a,b]\}, \;\; \text{and } M + L_2 \le L$$

(iv) If $\exists x_0 \in [a,b]$ *such that* $|f(x)| \le f(x_0)$ *then, either:*

(a) $M \cdot c_{x_0} \le c_{y_0}, \;\; y_0 = f(x_0)$*; or*

(b) $x_0 = a,\ M \cdot (b-a) \le b - y_0,\ K(s,u) \ge 0,\ \forall s,u \in [a,b]$*;or*

(c) $x_0 = b,\ M \cdot (b-a) \le y_0 - a,\ K(s,u) \ge 0,\ \forall s,u \in [a,b]$*;*

(v) $L_1 \cdot (L+2) \cdot c_{x_0} \le 1$.

Then there exists at least one solutions in C_L *of* (2).

Proof. It is know that C_L is a nonempty convex and compact subset of the Banach space $(C[a,b], \|\cdot\|)$, where $\|\cdot\|$ is the usual sup norm. Consider the integral operator

$$A : C_L \to C[a,b]$$

$$(Ay)(t) = f(t) + \int_{x_0}^{t} K(s, y(s), y(y(s)))ds, \;\; x_0, t \in [a,b], y \in C_L$$

The solutions of equation (2) is the fixed points of integral operator A.
We will show that $A(C_L) \subset C_L$.
In the case (a), we have

$$|(Ay)(t)| = \left| f(t) + \int_{x_0}^{t} K(s,y(s),y(y(s)))ds \right| \le |f(t)| + \int_{x_0}^{t} |K(s,y(s),y(y(s)))|\, ds \le$$

$$\le |f(x_0)| + M \cdot |t - x_0| \le f(x_0) + M \cdot c_{x_0} \le f(x_0) + c_{y_0} \le b$$

$$|(Ay)(t)| \ge |f(t)| - \int_{x_0}^{t} |K(s,y(s),y(y(s)))|\, ds \ge |f(t)| - M \cdot |t - x_0| \ge -f(x_0) - c_{y_0} \ge a$$

So,$Ay \in [a,b]$.Let us prove that $Ay \in C_L,\ \forall y \in C_L$.
Let $t_1, t_2 \in [a,b]$.

$$|(Ay)(t_1) - (Ay)(t_2)| \le |f(t_1) - f(t_2)| + \left| \int_{t_1}^{t_2} K(s,y(s),y(y(s)))ds \right| \le L_2 \cdot |t_1 - t_2| +$$

$$+M \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2| \,.$$

So, $Ay \in C_L, \ \forall y \in C_L$.
Let $y, z \in C[a, b]$

$$|(Ay)(t) - (Az)(t)| \leq \int_{x_0}^{t} |K(s, y(s), y(y(s))) - K(s, z(s), z(z(s)))| \, ds \leq$$

$$\leq L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + |y(y(s)) - z(z(s))|) ds =$$

$$= L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + |y(y(s)) - y(z(s)) + y(z(s)) - z(z(s))|) ds \leq$$

$$L_1 \cdot \left[\int_{x_0}^{t} |y(s) - z(s)| \, ds + \int_{x_0}^{t} (|y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))|) ds \right]$$

$$\leq L_1 \left[\int_{x_0}^{t} |y(s) - z(s)| + L \cdot \int_{x_0}^{t} |y(s) - z(s)| \, ds + \int_{x_0}^{t} \max_{t \in [a,b]} |y(t) - z(t)| \, ds \right].$$

Now, by letting maximum in last inequality, we get

$$\|Ay - Az\|_{C[a,b]} \leq L_1 \cdot (L + 2) \cdot c_{x_0} \cdot \|y - z\|_{C[a,b]}$$

which, in view of condition (v), proves that A is nonexpansive, hence continuos. Applying the Schauder's fixed point theorem we obtain the conclusion of the theorem.

Particular case

If $f(x) \equiv y_0(const.)$, then by Theorem 3.1 we find the results in [2].

References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007.

[2] V. Berinde, *Existence and approximation of solutions of some first order iterative differential equations*, Miskolc Math. Notes,(accepted as found at http://mat76.mat.uni-miskolc.hu/ mnotes/forthcoming.php).

[3] S. R. Bernfeld, V. Lakshmikanthan, *An introduction to nonlinear boundary value problem*, Ed. Academic Press. Inc, 228-232.

[4] A. Buica, *Existence and continuous dependence of solutions of some functional-diferential equations*, Seminar of Fixed Point Theory 3, 1995, 1-14.

[5] C. E. Chidume, *Geometric Properties of Banach spaces and nonlinear Iterations*, Springer Verlag, Berlin, Heidelberg, New York, 2009.

[6] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

Monica Lauran
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: lauranmonica@yahoo.com