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Note on the Grüss Inequality [1]

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Abstract

In the present note, we establish several new Grüss type inequalities which extend some known results.

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1 Introduction

Let f and g be two bounded functions defined on $[a,b]$ with $\gamma_1 \le f(x) \le \Gamma_1$ and $\gamma_2 \le g(x) \le \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants. Then the classic Grüss inequality reads as follows:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx \le \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2).$$

In the years thereafter, numerous generalizations, extensions and variants of Grüss inequality have appeared in the literatures (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]). The purpose of the present note is to establish some new forms of the inequality of Grüss type.

2 Main results and remarks

Theorem 1 *Let $n \ge 1$ and assume that $x_1, \cdots, x_n \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function with $\gamma \le f^{'}(x) \le \Gamma$ for all $x \in [a,b]$, where $\gamma, \Gamma \in \mathbb{R}$.*

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Then we have

$$\begin{aligned}&\gamma F_1(x_1,\cdots,x_n)-\Gamma F_2(x_1,\cdots,x_n)\\&\quad\le\frac{1}{n}\sum_{i=1}^{n}f(x_i)-\frac{1}{b-a}\int_a^b f(x)dx\\&\qquad\le\Gamma F_1(x_1,\cdots,x_n)-\gamma F_2(x_1,\cdots,x_n),\end{aligned}$$

where

$$F_1(x_1,\cdots,x_n)=\sum_{i=1}^{n}\frac{(x_i-a)^2}{2n(b-a)},\ F_2(x_1,\cdots,x_n)=\sum_{i=1}^{n}\frac{(b-x_i)^2}{2n(b-a)}.$$

Remark 1 *From the above result, we can give the following well-known inequalities.*
(1) Taking $x_1=a, x_2=b$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{4}(\Gamma-\gamma)(b-a). \tag{1}$$

(2) Taking $x_1=x_2=\frac{a+b}{2}$, *we obtain a sharper bound than that stated in [5, 8]*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{8}(\Gamma-\gamma)(b-a). \tag{2}$$

Theorem 2 *Assume that* $a\le x_1\le\frac{a+b}{2}\le x_2\le b$ *and* $f:[a,b]\to\mathbb{R}$ *is an absolutely continuous function with* $\gamma\le f'(x)\le\Gamma$ *for all* $x\in[a,b]$, *where* $\gamma,\Gamma\in\mathbb{R}$. *Then we have*

$$\begin{aligned}&\gamma G_1(x_1,x_2)-\Gamma G_2(x_1,x_2)\\&\quad\le\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(x_1)+f(x_2)}{2}\\&\qquad\le\Gamma G_1(x_1,x_2)-\gamma G_2(x_1,x_2),\end{aligned} \tag{3}$$

where

$$G_1(x_1,x_2)=\frac{(x_1-a)^2}{2(b-a)}+\frac{(2x_2-a-b)^2}{8(b-a)},$$
$$G_2(x_1,x_2)=\frac{(b-x_2)^2}{2(b-a)}+\frac{(2x_1-a-b)^2}{8(b-a)}.$$

Remark 2 *If we take* $n=2$, $x_1=x_2=\frac{a+b}{2}$ *or* $x_1=a, x_2=b$, *then the inequality (2) can be obtained, and*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{8}(\Gamma-\gamma)(b-a). \tag{4}$$

From Theorem 2 and 1, we have

Theorem 3 *Assume that $a \le x_1 \le \frac{a+b}{2} \le x_2 \le b$ and $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function with $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$, where $\gamma, \Gamma \in \mathbb{R}$. Then we have*

$$\begin{aligned} &\max\{\gamma G_1(x_1,x_2) - \Gamma G_2(x_1,x_2), \gamma F_1(x_1,x_2) - \Gamma F_2(x_1,x_2)\} \\ \le&\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(x_1)+f(x_2)}{2} \\ \le&\min\{\Gamma G_1(x_1,x_2) - \gamma G_2(x_1,x_2), \Gamma F_1(x_1,x_2) - \gamma F_2(x_1,x_2)\}. \end{aligned} \tag{5}$$

Remark 3 *Here we need explain that the bounds in Theorem 3 is reasonable. (1) and (4) show that Theorem 2 is better than Theorem 1 in some cases. Thus, it is enough to show that the bounds in Theorem 1 is better than ones in Theorem 2. For the convenience, let $a = 0, b = 1$ and $\gamma = 0, \Gamma = 1$. Then*

$$\begin{aligned} &\Gamma G_1(x_1,x_2) - \gamma G_2(x_1,x_2) - (\Gamma F_1(x_1,x_2) - \gamma F_2(x_1,x_2)) \\ =&\frac{(1-x_2)^2}{2} + \frac{(1-2x_1)^2}{8} - \frac{x_1^2}{4} - \frac{x_2^2}{4} \\ =&\frac{5 - 8x_2 - 4x_1 + 2x_2^2 + 2x_1^2}{8} \end{aligned}$$

for any $x_1 \in [0,1/2], x_2 \in [1/2,1]$. If we choose $x_1 = 0$ and $x_2 = 5/8$, then it follows that

$$\frac{5 - 8x_2 - 4x_1 + 2x_2^2 + 2x_1^2}{8} > 0,$$

which implies the desired claim.

Let X be a $\mathbb{R}$-valued random variable and $\mathbb{E}X$ denote the mathematical expectation of X, then we have the following

Theorem 4 *Let X be a $\mathbb{R}$-valued random variable and f, g two functions with $\mathbb{E}|f(X)| < \infty$ and $\phi \le g(X) \le \Phi$ a.e. for some constants ϕ, Φ. Then we have*

$$\begin{aligned} |\mathbb{E}(f(X)g(X)) - \mathbb{E}f(X)\mathbb{E}g(X)| \le&\frac{1}{2}(\Phi - \phi)\mathbb{E}|f(X) - \mathbb{E}f(X)| \\ =&(\Phi - \phi)\mathbb{E}[f(X) - \mathbb{E}f(X)1_{\{f \ge \mathbb{E}f(X)\}}] \\ =&(\Phi - \phi)\mathbb{E}[\mathbb{E}f(X) - f(X)1_{\{f < \mathbb{E}f(X)\}}]. \end{aligned} \tag{6}$$

Remark 4 *(1) If X possesses uniform distribution on the support interval $[a,b]$, then we have*

$$\begin{aligned} &\Big|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\Big| \\ &\qquad\le\frac{1}{2}\left(\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| dx\right)(\Phi - \phi) \end{aligned}$$

which was given in [1].

(2) Let X be a $\mathbb{R}$-valued random variable and $f(X)=(X-\mathbb{E}X)^2$ with $\mathbb{E}X^2<\infty$ and $\phi\le g(X)\le\Phi$ a.e. for some constants ϕ,Φ. Then we have

$$\begin{aligned}
&|\mathbb{E}((X-\mathbb{E}X)^2g(X))-\mathbb{E}[(X-\mathbb{E}X)^2]\mathbb{E}g(X)|\\
\le&\frac{1}{2}(\Phi-\phi)\mathbb{E}|(X-\mathbb{E}X)^2-\mathbb{E}[(X-\mathbb{E}X)^2]|\\
=&(\Phi-\phi)\mathbb{E}\{[(X-\mathbb{E}X)^2-\mathbb{E}[(X-\mathbb{E}X)^2]]1_{\{(X-\mathbb{E}X)^2\ge\mathbb{E}[(X-\mathbb{E}X)^2]\}}\}\\
=&(\Phi-\phi)\mathbb{E}\{[\mathbb{E}[(X-\mathbb{E}X)^2-(X-\mathbb{E}X)^2]]1_{\{(X-\mathbb{E}X)^2<\mathbb{E}[(X-\mathbb{E}X)^2]\}}\}.
\end{aligned}\tag{7}$$

In particular, if X possesses uniform distribution on the support interval $[a,b]$ and $g(x)$ is a twice differentiable mapping on (a,b) with $\phi\le g^{''}(X)\le\Phi$ a.e. for some constants ϕ,Φ, then we have

$$\begin{aligned}
&|\mathbb{E}((X-\mathbb{E}X)^2g(X))-\mathbb{E}[(X-\mathbb{E}X)^2]\mathbb{E}g(X)|\\
=&\frac{1}{b-a}\left|\int_a^b g(x)dx-\frac{(b-a)(g(a)+g(b))}{2}+\frac{1}{12}(b-a)^2(g^{'}(b)-g^{'}(a))\right|\\
\le&\frac{1}{36\sqrt{3}}(\Phi-\phi)(b-a)^2.
\end{aligned}$$

(3) Let X possess uniform distribution on the support interval $[a,b]$ and $f(X)=X$ with $\mathbb{E}|X|<\infty$ and $\phi\le g^{'}(X)\le\Phi$ a.e. for some constants ϕ,Φ. Then we have

$$\left|\frac{(b-a)(g(a)+g(b))}{2}-\int_a^b g(x)dx\right|\le\frac{1}{8}(\Phi-\phi)(b-a)^2.$$

Theorem 5 *Let X be a $\mathbb{R}$-valued random variable and f,g two non-negative functions with $0\le g(X)\le f(X)\le\Gamma$ a.e. for some constant Γ. Then we have*

$$|\mathbb{E}[f(X)g(X)]-\mathbb{E}f(X)\mathbb{E}g(X)|\le\frac{\Gamma^2}{4}.$$

Corollary 1 *Let X be a $\mathbb{R}$-valued random variable and f,g two non-negative functions with $0\le g(X),f(X)\le\Gamma$ a.e. for some constant Γ. In addition, let $F=\{f(X)\ge g(X)\}$, $G=\{f(X)<g(X)\}$, then we have*

$$\left|\mathbb{E}f(X)g(X)-\mathbb{E}f(X)\mathbb{E}g(X)+\mathbb{E}[|f(X)-g(X)|1_G]\mathbb{E}[|f(X)-g(X)|1_F]\right|\le\frac{\Gamma^2}{4}.$$

Remark 5 *In the case that X possesses uniform distribution on the support interval $[0,1]$ and $\Gamma=1$, Mercer gave an upper-bound for above inequality in [9].*

3 Proof of main results

Proof. [Proof of Theorem 1] It is easy to check that for any $t\in[a,b]$,

$$\begin{aligned}
f(t)-\frac{1}{b-a}\int_a^b f(x)dx=&\frac{1}{b-a}\left[\int_a^t\int_x^t f^{'}(y)dydx-\int_t^b\int_t^x f^{'}(y)dydx\right]\\
=&\frac{1}{b-a}\left[\int_a^t(y-a)f^{'}(y)dy-\int_t^b(b-y)f^{'}(y)dy\right].
\end{aligned}\tag{8}$$

Thus we have

$$\begin{aligned}&\sum_{i=1}^{n}\left(f(x_i)-\frac{1}{b-a}\int_a^b f(x)dx\right)\\=&\frac{1}{b-a}\sum_{i=1}^{n}\left[\int_a^{x_i}(y-a)f'(y)dy-\int_{x_i}^{b}(b-y)f'(y)dy\right]\\\le&\frac{1}{b-a}\sum_{i=1}^{n}\left[\frac{\Gamma}{2}(x_i-a)^2-\frac{\gamma}{2}(b-x_i)^2\right]\end{aligned}$$

and

$$\sum_{i=1}^{n}\left(f(x_i)-\frac{1}{b-a}\int_a^b f(x)dx\right)\ge\frac{1}{b-a}\sum_{i=1}^{n}\left[\frac{\gamma}{2}(x_i-a)^2-\frac{\Gamma}{2}(b-x_i)^2\right]$$

which implies the desired result.

Proof. [Proof of Theorem 2] From (8), we have

$$\begin{aligned}&f(x_1)-\frac{1}{b-a}\int_a^b f(x)dx+f(x_2)-\frac{1}{b-a}\int_a^b f(x)dx\\=&\frac{1}{b-a}\left[\int_a^{x_1}(y-a)f'(y)dy-\int_{x_1}^{b}(b-y)f'(y)dy\right]\\&+\frac{1}{b-a}\left[\int_a^{x_2}(y-a)f'(y)dy-\int_{x_2}^{b}(b-y)f'(y)dy\right]\\=&\frac{2}{b-a}\left[\int_a^{x_1}(y-a)f'(y)dy+\int_{x_1}^{\frac{a+b}{2}}\left(y-\frac{b+a}{2}\right)f'(y)dy\right]\\&-\frac{2}{b-a}\left[\int_{\frac{a+b}{2}}^{x_2}\left(\frac{b+a}{2}-y\right)f'(y)dy+\int_{x_2}^{b}(b-y)f'(y)dy\right]\\\le&\Gamma\left(\frac{(x_1-a)^2}{(b-a)}+\frac{(2x_2-a-b)^2}{4(b-a)}\right)-\gamma\left(\frac{(x_2-b)^2}{(b-a)}+\frac{(2x_1-a-b)^2}{4(b-a)}\right).\end{aligned}$$

The remainder of the proof is easy.

Proof. [Proof of Theorem 4] Let $A=\{f\ge\mathbb{E}f(X)\}$, $\bar{A}=\{f<\mathbb{E}f(X)\}$, it follows that

$$\begin{aligned}&\mathbb{E}(f(X)g(X))-\mathbb{E}f(X)\mathbb{E}g(X)\\=&\mathbb{E}[(f(X)-\mathbb{E}f(X))(1_A+1_{\bar{A}})g(X)]\\\le&\Phi\mathbb{E}[(f(X)-\mathbb{E}f(X))1_A]+\phi\mathbb{E}[(f(X)-\mathbb{E}f(X))1_{\bar{A}}]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[(f(X)-\mathbb{E}f(X))(1_A+1_{\bar{A}})g(X)]\\\ge&\phi\mathbb{E}[(f(X)-\mathbb{E}f(X))1_A]+\Phi\mathbb{E}[(f(X)-\mathbb{E}f(X))1_{\bar{A}}].\end{aligned}$$

Since

$$-\mathbb{E}[(f(X)-Ef(X))1_{\bar{A}}]=\mathbb{E}[(f(X)-\mathbb{E}f(X))1_A]=\frac{1}{2}\mathbb{E}|f(X)-\mathbb{E}f(X)|,$$

we have the following desired result

$$|\mathbb{E}(f(X)g(X))-\mathbb{E}f(X)\mathbb{E}g(X)|\le\frac{1}{2}(\Phi-\phi)\mathbb{E}|f(X)-\mathbb{E}f(X)|.$$

Proof. [Proof of Theorem 5] It is easy to see that

$$\mathbb{E}[f(X)g(X)]-\mathbb{E}f(X)\mathbb{E}g(X)\le\Gamma\mathbb{E}g(X)-[\mathbb{E}g(X)]^2=-\left(\mathbb{E}g(X)-\frac{\Gamma}{2}\right)^2+\frac{\Gamma^2}{4}$$

and

$$\mathbb{E}[f(X)g(X)]-\mathbb{E}f(X)\mathbb{E}g(X)\ge\mathbb{E}g^2(X)-\Gamma\mathbb{E}g(X)=\mathbb{E}\left(g(X)-\frac{\Gamma}{2}\right)^2-\frac{\Gamma^2}{4}$$

which yield the desired result.

Proof. [Proof of Corollary 1] It is easy to see that

$$\begin{aligned}&\mathbb{E}f(X)\mathbb{E}g(X)-\mathbb{E}[f(X)\vee g(X)]\mathbb{E}[f(X)\wedge g(X)]\\ =&\big(\mathbb{E}[f(X)1_F]-\mathbb{E}[g(X)1_F]\big)\big(\mathbb{E}[g(X)1_G]-\mathbb{E}[f(X)1_G]\big)\\ =&\mathbb{E}[|f(X)-g(X)|1_F]\mathbb{E}[|f(X)-g(X)|1_G].\end{aligned}\tag{9}$$

From $f(X)g(X)=(f(X)\vee g(X))(f(X)\wedge g(X)),a.e.$ and the equation (9), we have

$$\begin{aligned}&\mathbb{E}f(X)g(X)-\mathbb{E}f(X)\mathbb{E}g(X)+\mathbb{E}[|f(X)-g(X)|1_G]\mathbb{E}[|f(X)-g(X)|1_F]\\ =&\mathbb{E}\big[(f(X)\vee g(X))(f(X)\wedge g(X))\big]-\mathbb{E}[f(X)\vee g(X)]\mathbb{E}[f(X)\wedge g(X)].\end{aligned}$$

Since $0\le(f(X)\wedge g(X))\le(f(X)\vee g(X))\le\Gamma$ and by Theorem 5, the desired inequality is completed.

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