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Characterizations of best approximations in linear 2-normed spaces [1]

S.Elumalai, R.Vijayaragavan

Abstract

In this paper some characterizations of best approximation have been established in terms of 2-semi inner products and normalised duality mapping associated with a linear 2-normed space $(X, \|.,\|)$.

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1 Introduction

The concepts of linear 2-normed space was first introduced by S.Gahler in 1965 [6]. Since 1965, Y.J.Cho, C.R.Diminnie, R.W.Freese, S.Gahler,

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A.White, S.S.Dragomir and many other mathematicians have developed extensively the geometric structure of linear 2-normed space. A.White in his Doctoral dissertation entitled “2-Banach spaces” augments the concepts of a linear 2-normed space by defining Cauchy sequence and convergent sequence for such spaces. Section 2 provides some preliminary definitions and results that are used in the sequel. Some main results of the set of best approximation in the context of bounded linear 2-functionals on real linear 2-normed spaces are established in Section 3. Section 4 deleneates variational characterization of the best approximation elements. Two new characterizations are established in Section 5.

2 Preliminaries

Definition 1 [6] *Let X be a real linear space of dimension greater than one and let $\|.,.\|$ be a real-valued function defined on $X \times X$ satisfying the following for all $x, y, z \in X$.*

(i) $\|x, y\| > 0$ *and* $\|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent,*

(ii) $\|x, y\| = \|y, x\|$,

(iii) $\|\alpha x, y\| = |\alpha| \|x, y\| \quad \alpha \in \mathbb{R}$, *and*

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space.

A concept which is related to a 2-normed space is 2-inner product space as follows:

Definition 2 [1] *Let X be a linear space of dimension greater than one and let $(.,.|.)$ be a real-valued function on $X \times X \times X$ which satisfying the following conditions:*

(i) $(x,x|y) > 0$ *and* $(x,x|y) = 0$ *if and only if* x *and* y *are linearly dependent,*

(ii) $(x,x|y) = (y,y|x)$,

(iii) $(x,y|z) = (y,x|z)$,

(iv) $(\alpha x,y|z) = |\alpha|(x,y|z)$ *for every real* α, *and*

(v) $(x+y,z|b) = (x,z|b) + (y,z|b)$ *for every* $x,y,z \in X$ *and* b *is independent of* x, y *and* z.

Then $(.,.|.)$ is called a 2-inner product on X and $(X,(.,.|.))$ is called a 2-inner product space.

The concept of 2-inner product space was introduced by Diminnie,et.al [1].

The concepts of 2-norm and 2-inner product are 2-dimensional analogue of the concepts of norm and inner product in [1] it was shown that $\|x,y\| = (x,x|y)^{\frac{1}{2}}$ is a 2-norm on $(X,(.,.|.))$, $\|x,y\|$ may be visualized as the area of the parallelogram with vertices at $0,x,y$ and $x+y$.

Example 1 *Let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then, for $x = (a_1,b_1,c_1)$ and $y = (a_2,b_2,c_2)$ in X,*
$\|x,y\| = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2\}^{\frac{1}{2}}$ *and*

$\|x, y\| = |a_1b_2 - a_2b_1| + |b_1c_2 - b_2c_1| + |a_1c_2 - a_2c_1|$ *are 2-norm on* X.

Example 2 *Let* $X = \mathbb{R}^n$. *Then, for* $a = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $b = (\beta_1, \beta_2, \ldots, \beta_n)$ *and* $c = (c_1, c_2, \ldots, c_n)$,

$(a, b|c) = \sum_{i<j} (\alpha_i r_j - \alpha_j r_i)\ (\beta_i r_j - \beta_j r_i)$ *is a 2-inner product and* $(\mathbb{R}^n, (., .|.))$ *is a 2-inner product space.*

Definition 3 [8] *Let* X *be a linear space of dimension greater than one. Then a mapping*

$[., .|.] : X \times X \times X \to \mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) is a 2-semi inner product if the following conditions are satisfied.*

(i) $[x, x|z] > 0$ *and* $[x, x|z] = 0$ *if and only if* x *and* z *are linearly dependent,*

(ii) $[\lambda x, y|z] = \lambda[x, y|z]$ *for all* $\lambda \in \mathbb{K}$, $x, y \in X$, $z \in X\backslash V(x, y)$, *where* $V(x, y)$ *is the subspace of* X *generated by* x *and* y,

(iii) $[x + y, z|b] = [x, z|b] + [y, z|b]$ *for all* $x, y, z \in X$ *and* $b \in X\backslash V(x, y, z)$,

(iv) $|[x, y|z]|^2 \leq [x, x|z][y, y|z]$ *for all* $x, y, z \in X$ *and* $z \notin V(x, y, z)$.

Then $(X, [x, y|z])$ *is a 2-semi inner product space.*

Example 3 *Let* $X = \mathbb{R}^2$ *and let* $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ *and* $c = (c_1, c_2, c_3)$ *in* X. *Then*

$[a, b|c] = (a_1c_2 - a_2c_1)(b_1c_2 - b_2c_1)(c_1^2 + c_2^2)$ *is a 2-semi inner product.*

Definition 4 *Let $(X, \|.,.\|)$ be a linear 2-normed space, G be a linear subspace of X (G a non-empty subset of X), $x \in X\backslash\bar{G}$ and $g_0 \in G$. Then g_0 is said to be a best approximation element of x in G if*

$$\|x - g_0, z\| = \inf_{g \in G} \|x - g, z\|, \text{ for all } z \in X\backslash V(x, G).$$

We shall denote $P_G^z(x)$ by

$$P_G^z(x) = \{g_0 \in G : \|x - g_0, z\| = \inf_{g \in G} \|x - g, z\|\}$$

Lemma 1 [4, 5] *Let $(X, \|.,.\|)$ be a linear 2-normed space, G be a linear subspace of X, $x_0 \in X\backslash\bar{G}$ and $g_0 \in G$. Then $g_0 \in P_G^z(x_0)$ if and only if $x_0 - g_0 \perp_z G$ for every $g \in G$.*

Let $(X, \|.,.\|)$ be a linear 2-normed space. Then, we define

$$(x, y|z)_{s(i)} = \lim_{t \to 0^+} \frac{\|y + tx, z\|^2 - \|y, z\|^2}{2t}, \ x, y \in X \text{ and } Z \in X\backslash V(x, y).$$

The mapping $(.,.|.)_{s(i)}$ will be called supremum (infimum) of 2-semi inner product associated with the norm $\|.,.\|$.

For the sake of completeness we list some of the fundamental properties of $(.,.|.)_{s(i)}$:

(i) $(x, x|y)_p = \|x, y\|^2$ *for all* $x, y \in X$.

(ii) $(\alpha x, \beta y|z)_p = \alpha\beta(x, y|z)_p$ *if* $\alpha\beta \geq 0$ *and* $x, y, z \in X$.

(iii) $(-x, y|z)_p = -(x, y|z)_p = (x, -y|z)_p$ *for all* $x, y, z \in X$.

(iv) For all $x, y, z \in X$ (z is independent of x and y),

$$\begin{aligned} \frac{\|x + ty, z\|^2 - \|x, z\|^2}{2t} &\geq (y, x|z)_s \geq (y, x|z)_i \\ &\geq \frac{\|x + t^*y, z\|^2 - \|x, z\|^2}{2t^*}, \quad t^* < 0 < t. \end{aligned}$$

(v) *The following Schwarz's inequality holds*

$|(x,y|z)_p| \leq \|x,z\| \; \|y,z\|$ *for all* $x,y,z \in X$.

(vi) $(\alpha x + y, x|z)_p = \alpha\|x,z\|^2 + (y,x|z)_p$ *for all* $\alpha \in \mathbb{R}$ *and* $x,y,z \in X$.

(vii) *For all* $x,y,z,b \in X$, $|(y+z,x|b)_p - (z,x|b)_p| \leq \|y,b\| \; \|x,b\|$.

(viii) *For all* $x,y,z \in X$, $x\perp_z(\alpha x + y)(B)$ *if and only if*

$$(y,x|z)_i \leq \alpha\|x,z\|^2 \leq (y,x|z)_s \quad \alpha \in \mathbb{R},$$

and $x\perp_z y(B)$ *if and only if* $(y,x|z)_i \leq 0 \leq (y,x|z)_s$.

(ix) *The norm* $\|.,.\|$ *is Gâteaux differentiable in the space* $(X,\|.,.\|)$ *is smooth if and only if* $(x,y|z)_i = (x,y|z)_S$ *for all* $x,y,z \in X$.

3 Main results

The following theorem gives the characterization of the best approximation element which also gives a possibility of interpolation (estimation) for the bounded linear 2-functionals on real linear 2-normed spaces.

Theorem 1 *Let* $(X,\|.,.\|)$ *be a real linear 2-normed space* X *and* G *be its closed linear subspace of* X, $x_0 \in X\backslash G$ *and* $g_0 \in G$. *Then the following statements are equivalent:*

(i) $g_0 \in P_G^z(x_0)$ $z \in X\backslash V(x_0,G)$.

(ii) *For every* $f \in (G_{x_0} \times [b])^*$, $[b]$ *is the subspace of* $G_{x_0} = G \oplus \mathrm{sp}(x_0)$ *generated by* b *with* $\mathrm{Ker}(f) = G$, *we have*

$$(1) \qquad \|f\|_{G_{x_0}} \left(x, \frac{\lambda_0(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z \right)_i \leq f(x, z) \leq \|f\|_{G_{x_0}} \left(x, \frac{\lambda_0(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z \right)_s,$$

for all $x \in G_{x_0}$, *where*

$$\|f\|_{G_{x_0}} = \sup\left\{ \frac{|f(x,z)|}{\|x,z\|} : \|x,z\| \neq 0, x \in G_{x_0} \mathrm{and} z \in [b] \right\}$$

and $\lambda_0 = \mathrm{sgn} f(x_0, z)$.

To prove this theorem we need the following interesting Lemma.

Lemma 2 *Let* $(X, \|.,.\|)$ *be a linear 2-normed space* $f \in (X \times K)^* \backslash \{0\}, x_0 \in X \backslash \mathrm{Ker}(f)$ *and* $g_0 \in \mathrm{Ker}(f)$, *where* K *is a linear subspace of* X. *Then the following statements are equivalent:*

(i) $g_0 \in P^z_{\mathrm{Ker}(f)}(x_0) \quad z \in X \backslash V(x_0, \mathrm{Ker}(f))$.

(ii) *One has the estimation:*

$$(2) \qquad \|f\| \left(x, \frac{\lambda_0(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z \right)_i \leq f(x, z) \leq \|f\| \left(x, \frac{\lambda_0(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z \right)_s,$$

for all $x \in X$, $z \in X \backslash V(x_0, \mathrm{Ker}(f))$ *and* $\lambda_0 = \mathrm{sgn} f(x_0, z)$.

Proof. (i) $\Rightarrow$ (ii). We shall assume that (i) holds and put $w_0 = x_0 - g_0$. Then $w_0 \neq 0$. Since $g_0 \in P^z_{\mathrm{Ker}(f)}(x_0)$ by Lemma 1, $w_0 \perp_z \mathrm{Ker}(f)(B)$. Then by property (viii),we have

(3)

$$(y, w_0|z)_i \leq 0 \leq (y, w_0|z)_S \text{ for all } y \in \mathrm{Ker}(f) \textit{ and } z \in X \backslash V(x_0, \mathrm{Ker}(f)).$$

Let x be an arbitrary element of X. Then the element $y = f(x,z)w_0 - f(w_0,z)x \in \mathrm{Ker}(f)$, for all $x \in X$. Then by (3), we deduce that

$$(4) \qquad \begin{aligned} &(f(x,z)w_0 - f(w_0,z)x, w_0|z)_i \leq 0 \\ &\leq (f(x,z)w_0 - f(w_0,z)x, w_0|z)_s, \end{aligned}$$

for all $x \in X$.

By the properties of the mappings $(.,.|.)_i$ and $(.,.|.)_S$ we have

$$(f(x,z)w_0 - f(w_0,z)x, w_0|z)_P = f(x,z)\|w_0,z\|^2 + (-f(w_0,z)x, w_0|z)_P, \quad (x \in X)$$

and $p = s$ or $p = i$.

On the other hand, since $w_0 \perp_z \mathrm{Ker}(f)(B)$ and $w_0 \neq 0$, hence $f(w_0,z) \neq 0$. Then we have two cases $f(w_0,z) > 0$ and $f(w_0,z) < 0$.

Case (a): If $f(w_0,z) > 0$, then by (4)

$$\begin{aligned} 0 &\leq f(x,z)\|w_0,z\|^2 + (-f(w_0,z)x, w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 + f(w_0,z)(-x, w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 + (-x, f(w_0,z)w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 - (x, f(w_0,z)w_0|z)_i \end{aligned}$$

whence

(5)
$$f(x,z) \geq \left(x, \frac{f(w_0,z)w_0}{\|w_0,z\|^2} \mid z\right)_i \text{ for all } x \in X \text{ and } z \in X\backslash V(x_0, \mathrm{Ker}(f)).$$

Similarly, by (4) we have

$$\begin{aligned} 0 &\geq f(x,z)\|w_0,z\|^2 + (-f(w_0,z)x, w_0|z)_i \\ &= f(x,z)\|w_0,z\|^2 - (x, f(w_0,z)w_0|z)_s \end{aligned}$$

(6)
$$\Rightarrow \qquad f(x,z) \leq \left(x, \frac{f(w_0,z)w_0}{\|w_0,z\|^2} \mid z\right)_s \text{ for all} x \in X \text{and } z \in X\backslash V(x_0, \mathrm{Ker}(f)).$$

Case (b): Let us first remark that for every $x, y, z \in X$, we have

$$\begin{aligned} -(x,y|z)_i = (-x,y|z)_s &= (-x,-(-y)|z)_s \\ &= (x,-y|z)_s. \end{aligned}$$

If $f(w_0,z) < 0$, then

$$\begin{aligned} 0 &\leq f(x,z)\|w_0,z\|^2 + (-f(w_0,z)x, w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 + (-f(w_0,z))(x, w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 + (x, (-f(w_0,z))w_0|z)_s \\ &= f(x,z)\|w_0,z\|^2 - (x, f(w_0,z)w_0|z)_i \end{aligned}$$

$$\Rightarrow \; f(x,z) \geq \left(x, \frac{f(w_0,z)w_0}{\|w_0,z\|^2} \mid z\right)_i .$$

Similarly for $f(w_0,z) < 0$, we obtain (6).

Hence in both cases we obtain

(7)
$$\left(x, \frac{f(w_0,z)w_0}{\|w_0,z\|^2} \mid z\right)_i \leq f(x,z) \leq \left(x, \frac{f(w_0,z)w_0}{\|w_0,z\|^2} \mid z\right)_s$$

for all $x \in X$and$z \in X\backslash V(x_0, \mathrm{Ker}(f))$

Now, let $u = \dfrac{f(w_0, z)w_0}{\|w_0, z\|^2}$.Then, by (7), we have

$$\begin{aligned} f(x,z) &\geq (x, u|z)_i = -(x, u|z)_s \\ &\geq -\|x, z\| \; \|u, z\| \quad \text{for all} \quad x, z \in X \end{aligned}$$

and $f(x,z) \leq (x, u|z)_s \leq \|x, z\| \; \|u, z\|$ for all $x, z \in X$.

Thus

$$-\|u, z\| \leq \frac{f(x,z)}{\|x, z\|} \leq \|u, z\| \quad \text{for all} \quad x, z \in X$$

That is , $\|f\| \leq \|u, z\|$. On the other hand, we obtain:

$$\|f\| \geq \frac{f(u,z)}{\|u, z\|} \geq \frac{(u, u|z)_i}{\|u, z\|} = \|u, z\|$$

whence $\|f\| = \|u, z\| = \dfrac{|f(w_0, z)|}{\|w_0, z\|}$. But $f(w_0, z) = f(x_0, z)$.
Hence

$$\|f\| = \frac{|f(x_0, z)|}{\|x_0 - g_0, z\|} = \frac{f(x_0, z)|\lambda}{\|x_0 - g_0, z\|}$$

$\Rightarrow f(x_0, z) = \lambda \|f\| \; \|x_0 - g_0, z\|$.

This implies that, by (7), the estimation (ii) holds.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds for all $x \in X$ and $z \in X\backslash V(x_0, \mathrm{Ker}\,(f))$.
Then we have

$$\left(x, \frac{\lambda(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z\right)_i \leq 0 \leq \left(x, \frac{\lambda(x_0 - g_0)}{\|x_0 - g_0, z\|} \mid z\right)_s$$

for all $x \in \mathrm{Ker}(f)$. Then by property (viii), that

$$\frac{\lambda(x_0 - g_0)}{\|x_0 - g_0, z\|} \perp_z \ker\,(f)(B). \tag{8}$$

If $\lambda > 0$, obviously $x_0 - g_0 \perp_z \ker(f)(B)$

$\Rightarrow g_0 \in P^z_{\ker(f)}(x_0)$.

If $\lambda < 0$, then also $-(x_0 - g_0)\perp_z \ker\,(f)(B)$ (or) $(x_0 - g_0)\perp_z(-\ker\,(f))(B)$.

Since $-\ker(f) = \ker(f)$, we have $g_0 \in P^z_{\ker(f)}(x_0)$

Hence the proof.

Proof of the Theorem 1 Proof of the theorem follows by the Lemma 2 applied to the linear 2-normed space $G_{x_0} = G \oplus sp(x_0), (x_0 \notin G)$.

4 Variational characterization

The following theorem gives the variational characterization of the best approximation element.

Theorem 2 *Let $(X, \|.,.\|)$ be a linear 2-normed space and G be a closed linear subspace in X with $G \neq X$, $x_0 \in X \backslash G$ and $g_0 \in G$. Then the following statements are equivalent:*

(i) $g_0 \in P^z_G(x)$.

(ii) For every $f \in (G_{x_0} \times K)^*$, where K is a linear subspace of G_{x_0} with $\ker(f) = G$.

i.e., $G_{x_0} = G \oplus \mathrm{sp}(x_0)$, *the element*

$$u_0 = \frac{f(x_0, z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2}, z \in X \backslash V(x_0, \ker(f)),$$

minimizes the quadratic functional

$$F_f : G_{x_o} \times K \to \mathbb{R}$$

$$F_f(x, z) = \|x, z\|^2 - 2f(x, z).$$

To prove this theorem we need the following lemma.

Lemma 3 *Let $(X, \|.,.\|)$ be a real linear 2-normed space, $f \in (X \times K)^* \backslash \{0\}$ and $w \in X \backslash \{0\}$, where K is a linear subspace of X. Then the following statements are equivalent:*

(i)

$$(9) \qquad (x, w|z)_i \leq f(x, z) \leq (x, w|z)_s \text{ for all } x, z \in X$$

and z is independent of x and w.

(ii) The element w minimizes the quadratic functional

$$F_f = X \times K \to \mathbb{R} \qquad K \in X,$$

$$F_f(u, z) = \|u, z\|^2 - 2f(u, z).$$

Proof. (i) $\Rightarrow$ (ii). Let w satisfy the relation (9).

Then, for $x = w$, we obtain $f(w, z) = \|w, z\|^2$.

Let $u \in X$. Then for z is independent of u and w,

$$\begin{aligned} F_f(u, z) - F_f(w, z) &= \|u, z\|^2 - 2f(u, z) - \|w, z\|^2 + 2f(w, z) \\ &= \|u, z\|^2 - 2f(u, z) + \|w, z\|^2 \\ &\geq \|u, z\|^2 - 2(u, w|z)_s + \|w, z\|^2 \\ &\geq \|u, z\|^2 - 2\|u, z\| \ \|w, z\| + \|w, z\|^2 \\ &= (\|u, z\| - \|w, z\|)^2 \\ &\geq 0. \end{aligned}$$

Which proves that w minimizes the functional F_f.

(ii) $\Rightarrow$ (i). If w minimizes the functional F_f, then for all $u \in X$ and $\lambda \in \mathbb{R}$, we have

$F_f(w + \lambda u, z) - F_f(w, z) > 0$, for $u \in X$, $\lambda \in \mathbb{R}$ and z is independent of u and w.

$$\text{i.e.,} \quad \begin{aligned} F_f(w + \lambda u, z) - F_f(w, z) &= \|w + \lambda u, z\|^2 - \|w, z\|^2 \\ &\quad -2f(w + \lambda u, z) + 2f(w, z) \\ &= \|w + \lambda u, z\|^2 - \|w, z\|^2 - 2\lambda f(u, z). \end{aligned}$$

Therefore

$$(10) \qquad 2\lambda f(u, z) \leq \|w + \lambda u, z\|^2 - \|w, z\|^2 \text{ for all } u, z \in X, \text{ and } \lambda \in \mathbb{R}.$$

Now, Let $\lambda > 0$. Then by (10), we have

$$f(u, z) \leq \frac{\|w + \lambda u, z\|^2 - \|w, z\|^2}{2\lambda}, \qquad u, z \in X.$$

Taking limit as $\lambda \to 0^+$, we obtain

$f(u, z) \leq (u, w|z)_s$ for all $u, z \in X$.

Replacing u by $-u$ in the above relation we obtain

$f(u, z) \geq -(-u, w|z)_s = (u, w|z)_i$ for all $u, z \in X$

Thus the lemma is proved.

Corollary 1 *Let $(X, \|.,.\|)$ be a real linear 2-normed space, $f \in (X \times [b])^* \backslash \{o\}$ and $w \in X \backslash \{o\}$. Then w is a point of smoothness of X and it minimizes the functional F_f if and only if $f(x, z) = (x, w|z)_p$ for all $x \in X$, where $p = s$ or i.*

Proof of the Theorem 2.

(i) $\Rightarrow$ (ii). Let $g_0 \in P_G^z(x_0)$.

Then by Theorem 1, for every $f \in (G_{x_0} \times K)^*$, K is a subspace of G_{x_0}, with ker(f) $= G$. We have the estimation (1). In this relation put $x = \dfrac{\lambda_0(x_0 - g_0)}{\|x_0 - g_0, z\|}$, we obtain

$\|f\|_{G_{x_0}} = \dfrac{|f(x_0, z)|}{\|x_0 - g_0, z\|}$.

Then (1) becomes

$$\left(x, \frac{f(x_0,z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2} \mid z\right)_i \leq x, z) \leq \left(x, \frac{f(x_0,z)(x_0 - g_0)}{\|x_0 - g_0, z\|^2} \mid z\right)_s \tag{11}$$

for all $x \in G_{x_0}$.

Now applying Lemma 3 for $u_0 = f(x_0, z)\dfrac{(x_0 - g_0)}{\|x_0 - g_0, z\|^2}$ on the space G_{x_0}, u_0 minimizes the functional F_f on the space G_{x_0}.

(ii) $\Rightarrow$ (i). If u_0 given above minimizes the functional F_f on G_{x_0}, by Lemma 3, we derive that the estimation (11). Further (1) is valid, that is by Theorem 1, we obtain $g_0 \in P_G^z(x_0)$. Hence the proof.

5 Two new characterization

Let $(X, \|.,.\|)$ be a linear 2-normed space and let X_z^* be the space of all bounded linear 2-functionals defined on $X \times V(z)$ for every non-zero $z \in X$.

Then the mapping $J : X \times V(z) \rightarrow 2^{X_z^*}$ defined by

$$J(x, y) = \{f \in X_z^* : f(x, y) = \|f\|\, \|x, y\|,\ \|f\| = \|x, y\|, x \in X \text{ and } y \in V(z)\}$$

will be called the normalized duality mapping associated with 2-normed space $(X, \|.,.\|)$.

Lemma 4 *Let $(X, \|.,.\|)$ be a real linear 2-normed space. Then for every $\tilde{J}$ a section of the normalized duality mapping one has the representations*

$$(y, x|z)_s = \lim_{t \to 0^+} \langle \tilde{J}(x + ty), y|z \rangle \tag{12}$$

and

$$(y, x|z)_i = \lim_{t \to 0^-} \langle \tilde{J}(x + ty), y|z \rangle \tag{13}$$

for all $x, y, z \in X$ and z is independent of x and y.

Proof. Let $\tilde{J}$ be a section of the duality mapping J. Then, for all $x, y, z \in X$ and z is independent of x and y, $t \in \mathbb{R}$ and $x \neq 0$,

$$\begin{aligned}
\|x + ty, z\| - \|x, z\| &= \frac{\|x + ty, z\| \ \|x, z\| - \|x, z\|^2}{\|x, z\|} \\
&\geq \frac{\langle \tilde{J}x, x + ty|z \rangle - \|x, z\|^2}{\|x, z\|} \\
&= \frac{\langle \tilde{J}x, x|z \rangle + t\langle \tilde{J}x, y|z \rangle - \|x, z\|^2}{\|x, z\|} \\
&= \frac{t\langle \tilde{J}x, y|z \rangle}{\|x, z\|}.
\end{aligned}$$

Whence

$$\|x, z\| \frac{(\|x + ty, z\| - \|x, z\|)}{t} \geq \langle \tilde{J}x, y|z \rangle f \tag{14}$$

or all $x, y \in X$, $z \in X \backslash V(x, y)$ and $t > 0$.

On the other hand, for $t \neq 0$ and $x + ty \neq 0$, we have

$$\begin{aligned}\frac{\|x+ty,z\|-\|x,z\|}{t} &= \frac{\|x+ty,z\|^2-\|x,z\|\ \|x+ty,z\|}{\|x+ty,z\|t}\\ &= \frac{\langle \tilde{J}(x+ty),x+ty|z\rangle-\|x,z\|\ \|x+ty,z\|}{t\|x+ty,z\|}\\ &= \frac{\langle \tilde{J}(x+ty),x|z\rangle+t\langle \tilde{J}(x+ty),y|z\rangle-\|x,z\|\ \|x+ty,z\|}{t\|x+ty,z\|}\\ &\leq \frac{\langle \tilde{J}(x+ty),y|z\rangle}{\|x+ty,z\|}.\end{aligned}$$

Since $\langle \tilde{J}(x+ty),x|z\rangle \leq \|x,z\|\ \|x+ty,z\|$ for all $x,y\in X, z\in X\backslash V(x,y)$ and $t\in\mathbb{R}$.

Consequently we have,

$$\langle \tilde{J}(x+ty),y|z\rangle \geq \|x+ty,z\|\frac{(\|x+ty,z\|-\|x,z\|)}{t} \tag{15}$$

for all $x,y\in X, t>0$ and $z\in X\backslash V(x,y)$.

Replacing x by $x+ty$ in the inequality (14) we have,

$$\|x+ty,z\|\frac{(\|x+2ty,z\|-\|x+ty,z\|)}{t} \geq \langle \tilde{J}(x+ty),y|z\rangle \tag{16}$$

for all $x,y\in X$, $t>0$ and $z\in X\backslash V(x,y)$.

By (15) and (16),we obtain

$$\begin{aligned}\|x+ty,z\|\frac{\|x+ty,z\|-\|x,z\|}{t} &\leq \langle \tilde{J}(x+ty),y|z\rangle\\ &\leq \|x+ty,z\|\frac{\|x+2ty,z\|-\|x+ty,z\|}{t}\end{aligned} \tag{17}$$

for all $x,y\in X$, $t>0$ and $z\in X\backslash V(x,y)$.

Since $(y,x|z)_s=\lim_{t\to 0^+}\left(\|x+ty,z\|\frac{\|x+ty,z\|-\|x,z\|}{t}\right)$, a simple calculation gives

$$\lim_{t\to 0^+}\left(\|x+ty,z\|\frac{\|x+2ty,z\|-\|x+ty,z\|}{t}\right)$$
$$= \|x,z\|\left[2\lim_{t\to 0^+}\left(\|x+ty,z\|\frac{(\|x+2ty,z\|-\|x,z\|)}{2t}\right)\right.$$
$$\left. -\lim_{t\to 0^+}\left(\|x+ty,z\|\frac{(\|x+ty,z\|-\|x,z\|}{t}\right)\right]$$
$$= \|x,z\|\lim_{t\to 0^+}\frac{\|x+ty,z\|-\|x,z\|}{t}$$
$$= (y,x|z)_s \qquad \text{for all } x,y,z\in X.$$

Then by taking limit as $t\to 0^+$ in the inequality (17) we observe that

$\lim_{t\to 0^+}\langle\tilde{J}(x+ty),y|z\rangle$ exists for all $x,y,z\in X$

and $\lim_{t\to 0^+}\langle\tilde{J}(x+ty),y|z\rangle = (y,x|z)_s$ for all $x,y,z\in X$.

Then we have established (12).

On the other hand,

$$\begin{aligned}(y,x|z)_i &= -(-y,x|z)_s\\ &= -\lim_{t\to 0^+}\langle\tilde{J}(x+t(-y)),-y|z\rangle\\ &= \lim_{t\to 0^+}\langle\tilde{J}(x+(-t)y),y|z\rangle\\ &= \lim_{t\to 0^-}\langle\tilde{J}(x+ty),y|z\rangle \quad \text{for all } x,y,z\in X.\end{aligned}$$

Thus (13) is obtained.

Theorem 3 *Let $(X,\|.,.\|)$ be a real linear 2-normed space, G be a linear subspace of X, $x_0\in X\backslash G$ and $g_0\in G$. Then the following statements are equivalent:*

(i) $g_0\in P_G^z(x_0)$.

(ii) For every $f \in (G_{x_o} \times [b])^*)$ with $\ker(f) = G$ we have

$$\frac{f(x_0,z)}{\|x_0-g_0,z\|^2}\lim_{t\to 0^-}\left\langle\frac{\tilde{J}\left(\frac{\lambda_0(x_0-g_0)}{\|x_0-g_0,z\|}+tx\right)-\tilde{J}\left(\frac{\lambda_0(x_0-g_0),}{\|x_0-g_0,z\|}x_0-g_0|z\right)}{t}\right\rangle\leq f(x,z)$$

$$\leq\frac{f(x_0,z)}{\|x_0-g_0,z\|^2}\lim_{t\to 0^+}\left\langle\frac{\tilde{J}\left(\frac{\lambda_0(x_0-g_0)}{\|x_0-g_0,z\|}+tx\right)-\tilde{J}\left(\frac{\lambda_0(x_0-g_0)}{\|x_0-g_0,z\|},x_0-g_0|z\right)}{t}\right\rangle$$

for all $x \in G_{x_0}$ and $\tilde{J}$ a section of the normalized duality mapping J.

To prove this theorem we need the following Lemma.

Lemma 5 *Let $(X, \|.,.\|)$ be a real linear 2-normed space. Then for any $\tilde{J}$ a section of duality mapping J, we have*

$$(y,x|z)_s = \lim_{t\to 0^+}\langle\frac{\tilde{J}(x+ty)-\tilde{J}(x)}{t},x|z\rangle$$

$$(y,x|z)_i = \lim_{t\to 0^-}\langle\frac{\tilde{J}(x+ty)-\tilde{J}(x)}{t},x|z\rangle \text{ for all } x,y,z\in X \text{ and } z\in X\backslash V(x,y).$$

Proof. For every $x, y \in X$, $t \in \mathbb{R}$ with $t \neq 0$ and $z \in X\backslash V(x,y)$,

$$\begin{aligned}\frac{\|x+ty,z\|^2-\|x,z\|^2}{t} &= \frac{\langle\tilde{J}(x+ty),x+ty|z\rangle-\langle\tilde{J}x,x|z\rangle}{t}\\ &= \frac{\langle\tilde{J}(x+ty),x|z\rangle+t\langle\tilde{J}(x+ty,y|z)-\tilde{J}x,x|z\rangle}{t}\\ &= \left\langle\frac{\tilde{J}(x+ty)-\tilde{J}(x,x|z)}{t}\right\rangle+\langle\tilde{J}(x+ty),y|z\rangle\end{aligned}$$

Since $\lim_{t\to 0^+}\frac{\|x+ty,z\|^2-\|x,z\|^2}{t}$

$$\begin{aligned}&=\lim_{t\to 0^+}\frac{\|x+ty,z\|-\|x,z\|}{t}\lim_{t\to 0^+}\frac{\|x+ty,z\|+\|x,z\|}{t}\\ &=2\|x,z\|\lim_{t\to 0^+}\frac{\|x+ty,z\|-\|x,z\|}{t}\\ &=2(y,x|z)_s \quad \text{and}\end{aligned}$$

$\lim_{t\to 0^+}\langle \tilde{J}(x+ty), y|z\rangle = (y, x|z)_s$. Then by the above relation,
$\lim\limits_{t\to 0^+}\left\langle \dfrac{\tilde{J}(x+ty)-\tilde{J}(x),}{t}, x|z\right\rangle$ exists for all $x, y \in X$ and $z \in X\backslash V(x,y)$.
Thus $\lim\limits_{t\to 0^+}\left\langle \dfrac{\tilde{J}(x+ty)-\tilde{J}(x),}{t}, x|z\right\rangle = (y, x|z)_s$

for all $\tilde{J}$ a section of normalized duality mapping.

Proof of the Theorem 3 follows from Theorem 1 and from Lemma 5.

References

[1] C.Diminnie, S.Gahler and A.White, *2-inner product spaces*, Demonstratio Math. 6, 1973, 525-536.

[2] C.Diminnie,*2-inner product spaces II*, Demonstratio Math. 10, 1977, 169-188.

[3] S.S.Dragomir, *A characterization of best approximation elements in real normed spaces* (Romanian),Stud.Mat.(Bucharest), 39, 1987, 497-506.

[4] S.S.Dragomir , *A characterization of the elements of best approximation in real normed spaces*, Studia Univ.Babes-Bolyai Math. 33, 1988,74-80.

[5] S.S.Dragomir , *On best approximation in the sence of Lumer and application*, Riv.Mat.Univ.Parma 15 (1989), 253-263.

[6] S.Gahler, *Linear 2-normierte Raume*, Math. Nachr. 28 (1965), 1-43.

[7] A.L.Garkavi ,*The theory of best approximation in normed linear spaces*, Progress in mathematics, New York 8, 1970, 83-151.

[8] A.H.Siddique and S.M.Rizvi, *2-semi-inner product spaces*, I.Math.Japon. 21, 1976,391-397.

[9] I.Singer , *Best approximation in normed linear spaces by elements of linear subspaces* (Romanian), Ed.Acad.Bucharest,1967.

S.Elumalai
Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chennai - 600005, Tamilnadu, India.

R.Vijayaragavan
Vellore Institute of Technology University
Vellore - 632014, Tamilnadu, India.