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Uniqueness of Meromorphic Function and its Differential Polynomial Concerning Weakly Weighted-Sharing [1]

Hong-Yan Xu, Yi Hu

Abstract

In this paper, we introduce the definition of weakly weighted-sharing which is between "CM" and "IM". Using the notion of weakly weighted-sharing, we investigate problems of meromorphic functions that share a small function with its differential polynomial, and give some results and also answer some questions of Kit-Wing Yu, which were also studied by L.P.Liu and Y.X. Gu[L.P. Liu and Y.X.Gu, Uniqueness of meromorphic functions that share one small function with their derivatives, Kodai. Math. J. 27 (2004), 272-279.], S.H. Lin and W.C. Lin[S.H.Lin and W.C.Lin, Uniqueness of meromorphic functions concerning weakly weighted-sharing, Kodai.Math.J.,29 (2006),269-280.].

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1 Introduction and results

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notation in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f), N(r,f)$ and $m(r,f)$ (see [4] or [10]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly for a set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbf{C}$ which are small functions with respect to f.

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Let $a \in S(f)$, we say that two meromorphic functions f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring (counting) multiplicities.

Mues and Steinmetz [8], Gundersen [3], Yang [12] and Yi [11], and many other authors have obtained elegant results on the uniqueness problems of entire functions that share values CM or IM with their first or n-th derivatives.

In 2003, Yu [9] considered the uniqueness problem of an entire function or meromorphic function when it shares one small function with its derivative and proved the following results.

Theorem A Let $n \geq 1$, let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f, f^{(n)}$ share a CM and $\delta(0,f) > \frac{3}{4}$, then $f \equiv f^{(n)}$.

Theorem B Let $n \geq 1$, let f be a non-constant non-entire meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$, f and a do not have any common pole. If $f, f^{(n)}$ share a CM and $4\delta(0,f) + 2(8+n)\Theta(\infty, f) > 19 + 2n$, then $f \equiv f^{(n)}$.

In the same paper, Yu [9] posed the following open question:

Question A Can a CM shared value be replaced by an IM shared value in Theorem A?

Question B Is the condition $\delta(0,f) > 3/4$ sharp in Theorem A?

Question C Is the condition $4\delta(0,f) + 2(8+n)\Theta(\infty, f) > 19 + 2n$ sharp in Theorem B?

Question D Can the condition f and a do not have any common pole" be deleted in Theorem B?

In 2004, Liu and Gu [7] applied a different method and obtained the following results.

Theorem C Let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f, f^{(n)}$ share a CM, f and a do not have any common pole of same multiplicity and $2\delta(0,f) + 4\Theta(\infty, f) > 5$, then $f \equiv f^{(n)}$.

Theorem D Let $n \geq 1$, let f be a non-constant entire function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If $f, f^{(n)}$ share a CM and $\delta(0,f) > \frac{1}{2}$, then $f \equiv f^{(n)}$.

Let

$$L(f) = f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f, \qquad (*)$$

be a differential polynomial on f, where $a_j (j = 0, 1, \cdots, n-1) \in S(f)$.

Question 1: what happens if $f^{(n)}$ is replaced by $L(f)$ in Theorem C and D?

In order to state our results, we first introduce the definition of weakly weighted-sharing as followed.

Definition 1 *Let k be a positive integer, and let f be a meromorphic function and $a \in S(f)$.*

(i) $\overline{N}(r, a; f| \geq k)$ *denotes the counting function of zeros of $f-a$ whose multiplicities are not greater than k, where each zero is counted only once.*

(ii) $\overline{N}(r, a; f| \leq k)$ *denotes the counting function of zeros of $f-a$ whose multiplicities are not less than k, where each zero is counted only once.*

(iii) $N_p(r.a; f) = \overline{N}(r, a; f) + \sum_{k=2}^{p} \overline{N}(r, a; f| \geq k)$.

Definition 2 [5] *For any complex number $c \in \mathbb{C} \cup \{\infty\}$, We denote by $\delta_p(c, f)$ the quantity*

$$\delta_p(c, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, c; f)}{T(r, f)},$$

where p is a positive integer. Clearly $\delta_p(c, f) \geq \delta(c, f)$.

Let $N_E(r, a)$ be the counting function of all common zeros of $f - a$ and $g - a$ with the same multiplicities, and $N_0(r, a)$ be the counting functions of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. Denotes by $\overline{N}_E(r, a)$ and $\overline{N}_0(r, a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r, a)$ and $N_0(r, a)$, respectively. If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_E(r, a) = S(r, f) + S(r, g),$$

then we say that f and g share a "CM". If

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g),$$

then we say that f and g share a "IM".

Definition 3 *Let f and g be two nonconstant meromorphic functions sharing a "IM", for $a \in S(f) \bigcap S(g)$, and a positive integer k or ∞.*

(i) *$\overline{N}_E^{k)}(r, a)$ denotes the counting function of zeros of $f - a$ whose multiplicities are equal to the corresponding zeros of $g - a$, both of their multiplicities are not greater than k, where each zero is counted only once.*

(ii) *$\overline{N}_0^{(k}(r, a)$ denotes the reduced counting function of zeros of $f - a$ which are zeros of $g - a$, both of their multiplicities are not less than k, where each zero is counted only once.*

(iii) *Let z_0 be the zeros of $f - a$ with multiplicity p and zeros of $g - a$ with multiplicity q. Denote by $\overline{N}_{f>k}(r, a; g)$ the reduced counting function of those zeros of $f - a$ and $g - a$ such that $p > q = k$. $\overline{N}_{g>k}(r, a; f)$ is defined analogously.*

(iv) *$\overline{N}_*(r, a; f, g)$ denotes the reduce counting function of zeros of $f - a$ whose multiplicities differ from the multiplicities of the corresponding zeros of $g - a$.*

Clearly

$\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 4 *For $a \in S(f) \bigcap S(g)$, if k is a positive integer or ∞, and*

$$\overline{N}(r, a; f| \leq k) - \overline{N}_E^{k)}(r, a) = S(r, f), \overline{N}(r, a; f| \geq k + 1) - \overline{N}_0^{(k+1}(r, a) = S(r, f);$$

$$\overline{N}(r, a; g| \leq k) - \overline{N}_E^{k)}(r, a) = S(r, g), \overline{N}(r, a; g| \geq k + 1) - \overline{N}_0^{(k+1}(r, a) = S(r, g).$$

Or if $k = 0$ and

$$\overline{N}(r, a; f) - \overline{N}_0(r, a) = S(r, f), \qquad \overline{N}(r, a; g) - \overline{N}_0(r, a) = S(r, g),$$

where $\overline{N}_0(r, a)$ is the reduce counting functions of all common zeros of $f - a$ and $g - a$ ignoring multiplicities, then we say f and g weakly share a with weight k. Here, we write f, g share "(a, k)" to mean that f, g weakly share a with weight k.

Obviously, if f and g share "(a, k)", then f and $g$share "(a, p)" for any p $(0 \le p \le k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a, 0)$" or "(a, ∞)", respectively.

Question 2: Can a CM shared value be replaced by weakly weighted-sharing in Theorem C and Theorem D?

In this paper, we obtain some uniqueness theorems which answer **Question 1** and **Question 2** as followed.

Theorem 1 *Let $n \ge 1$ and $2 \le k \le \infty$, let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by $(*)$, If $f, L(f)$ share "(a, k)" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5, \tag{1}$$

then $f \equiv L(f)$.

Theorem 2 *Let $n \ge 1$, let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by $(*)$, If $f, L(f)$ share "$(a, 1)$" and*

$$\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + \delta_{2+n}(0, f) > n + 5, \tag{2}$$

then $f \equiv L(f)$.

Theorem 3 *Let $n \ge 1$, let f be a non-constant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose that $L(f)$ is defined by $(*)$, If $f, L(f)$ share "$(a, 0)$" and*

$$(6 + 2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0.f) + 2\delta_{2+n}(0, f) > (2n + 10), \tag{3}$$

then $f \equiv L(f)$.

From Theorem 1.5-1.7 we have

Corollary 1 *Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If $f, L(f)$ share "(a, k)", $k \ge 2$ and $\delta_{2+n}(0, f) > \frac{1}{2}$, or if $f, L(f)$ share "$(a, 1)$" and $\delta_{2+n}(0, f) > \frac{3}{5}$, or if $f, L(f)$ share "$(a, 0)$" and $\delta_{2+n}(0, f) > 2 - \frac{1}{2}(\delta_2(0, f) + 2\Theta(0.f))$, then $f \equiv L(f)$, where $L(f)$ is defined by $(*)$.*

2 Some lemmas

Next, we introduce some notations for the following lemmas.

Lemma 1 *Let f be a transcendental meromorphic function, $L(f)$ be defined by $(*)$, If $L(f) \not\equiv 0$, we have*

(i) $N_2(r, 0; L) \le N_{2+n}(r, 0; f) + n\overline{N}(r, \infty; f) + S(r, f)$;

(ii) $N_2(r, 0; L) \le T(r, L) - T(r, f) + N_{2+n}(r, 0; f) + S(r, f)$.

Proof: By the first fundamental theorem and the lemma of logarithmic derivatives, we get:

$$\begin{aligned}
N_2(r,0;L) &\le N(r,0;L) - \textstyle\sum_{p=3}^{\infty} \overline{N}(r,0;L| \ge p) \\
&= T(r,L) - m(r,\tfrac{1}{L}) - \textstyle\sum_{p=3}^{\infty} \overline{N}(r,0;L| \ge p) + O(1) \\
&\le T(r,L) - m(r,\tfrac{1}{f}) - m(r,\tfrac{L}{f}) - \textstyle\sum_{p=3}^{\infty} \overline{N}(r,0;L| \ge p) + O(1) \\
&\le T(r,L) - T(r,f) + N(r,0;f) - \textstyle\sum_{p=3}^{\infty} \overline{N}(r,0;L| \ge p) + S(r,f) \\
&\le T(r,L) - T(r,f) + N_{2+n}(r,0;f) + \textstyle\sum_{p=3+n}^{\infty} \overline{N}_{(p}(r,0;f) \\
&\quad - \textstyle\sum_{p=3}^{\infty} \overline{N}(r,0;L| \ge p) + S(r,f) \\
&\le T(r,L) - T(r,f) + N_{2+n}(r,0;f) + S(r,f).
\end{aligned}$$

So this proves Lemma (ii).

Since

$$\begin{aligned}
T(r,L) &= m(r,L) + N(r,\infty;L) \\
&\le m(r,f) + m(r,\tfrac{L}{f}) + N(r,\infty;f) + n\overline{N}(r,\infty;f) \\
&= T(r,f) + n\overline{N}(r,\infty;f) + S(r,f).
\end{aligned}$$

From this and Lemma (ii), we can prove Lemma (i).

Lemma 2 [11] *Let k be a nonnegative integer or ∞, F and G be two nonconstant meromorphic functions, F and G share "$(1, k)$". Let*

$$H = \left(\frac{F''}{F'} - 2\frac{F'}{F-1}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G-1}\right).$$

If $H \not\equiv 0$, $2 \le k \le \infty$, then

$$T(r,F) \le N_2(r,\infty;F) + N_2(r,0;F) + N_2(r,\infty;G) + N_2(r,0;G) + S(r,F) + S(r,G).$$

The same inequalities holds for $T(r, G)$.

When f and g share 1 "IM", $\overline{N}_L(r, 1; f)$ denotes the counting function of the 1-points of f whose multiplicities are greater than 1-points of g, where each zero is counted only once. Similarly, we denote $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; f)$ denotes the counting function of those simple 1-points of f and g, and $\overline{N}_E^{(2}(r, 1; f)$ denotes the counting function of those multiplicity 1-points of f and g,each point in these counting functions is counted only once. In the same way ,one can define $N_E^{1)}(r, 1; g), \overline{N}_E^{(2}(r, 1; g)$.

Lemma 3 *If f, g be two nonconstant meromorphic functions such that they share "$(1,1)$", then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g)\le N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 4 *Let f, g share "$(1,1)$". Then*

$$\overline{N}_{f>2}(r,1;g)\le \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}\overline{N}_0(r,0;f')+S(r,f).$$

Lemma 5 *Let f and g be two nonconstant meromorphic functions sharing "$(1,0)$". Then*
$\overline{N}_L(r,1;f)++2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)$
$\le N(r,1;g)-\overline{N}(r,1;g)$.

Lemma 6 *Let f, g share "$(1,0)$". Then*

$$\overline{N}_L(r,1;f)\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 7 *Let f, g share "$(1,0)$". Then*

(i) $\overline{N}_{f>1}(r,1;g)\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f)$;

(ii) $\overline{N}_{g>1}(r,1;f)\le\overline{N}(r,0;g)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,g)$;

Using the method of [1] and [2], we can prove Lemmas 2.3-2.7 easily. Here, we omit them.

3 Proof of main Theorems

Proof of Theorem 1.5

Let

$$F=\frac{f}{a},\qquad G=\frac{L(f)}{a}. \tag{4}$$

From the conditions of Theorem 1.5, we know that F and G share "$(1,k)$", and from (4), we have

$$T(r,F)=O(T(r,f))+S(r,f),\qquad T(r,G)=O(T(r,f))+S(r,f). \tag{5}$$

$$\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)+S(r,f). \tag{6}$$

Let H be defined as in Lemma 2.2. Suppose that $H\not\equiv 0$. It follows from Lemma 2.2 that

$$T(r,G)\le N_2(r,\infty;F)+N_2(r,0;F)+N_2(r,\infty;G)+N_2(r,0;G)+S(r,F)+S(r,G).$$

Using Lemma 2.1, we have

$$\begin{aligned} T(r,L) &\le N_2(r,\infty;f)+N_2(r,0;f)+N_2(r,\infty;L)+N_2(r,0;L)+S(r,f)\\ &\le N_{2+n}(r,0;f)+T(r,L)-T(r,f)+N_{2+n}(r,0;f)\\ &+4\overline{N}(r,\infty;f)+S(r,f), \end{aligned}$$

i.e.

$$T(r,f)\le 2N_{2+n}(r,0;f)+4\overline{N}(r,\infty;f)+S(r,f).$$

Which contradicts the assumption (1) of Theorem 1.5. Thus, $H\equiv 0$. That is

$$\frac{F''}{F'}-2\frac{F'}{F-1}\equiv\frac{G''}{G'}-2\frac{G'}{G-1}.$$

It follows that

$$\frac{1}{F-1}=\frac{A}{G-1}+B,$$

where $A(\neq 0)$ and B are constants. Thus

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)} \tag{7}$$

and $T(r,F)=T(r,G)+S(r,f)$.

Next, we consider three cases.

Case 1. $B\neq 0,-1$. If $A-B-1\neq 0$, then by (7) we know

$$\overline{N}\left(r,\frac{-A+B+1}{B+1};G\right)=\overline{N}(r,0;F).$$

By the Nevanlinna second fundamental theorem and Lemma 2.1 we have

$$\begin{aligned} T(r,G) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\tfrac{-A+B+1}{B+1};G)+S(r,G)\\ &= \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,0;F)+S(r,f), \end{aligned}$$

i.e

$$\begin{aligned} T(r,L) &< \overline{N}(r,\infty;f)+\overline{N}(r,0;L)+\overline{N}(r,0;f)+S(r,f)\\ &\le \overline{N}(r,\infty;f)+T(r,L)-T(r,f)+N_{2+n}(r,0;f)\\ &\quad+\overline{N}(r,0;f)+S(r,f). \end{aligned}$$

Then

$$T(r,f)<\overline{N}(r,\infty;f)+2\overline{N}_{2+n}(r,0;f)+S(r,f).$$

Which contradicts the assumption (1).

If $A-B-1=0$, then by (7) we know $F=((B+1)G)/(BG+1)$. Obviously,

$$\overline{N}\left(r,-\frac{1}{B};G\right)=\overline{N}(r,\infty;F).$$

Similar to the arguments in the above, we also have a contradiction.

Case 2. $B=-1$. Then (7) becomes $F=A/(A+1-G)$.

If $A+1\neq 0$, then $\overline{N}(r,A+1;G)=\overline{N}(r,\infty;F)$. Similarly, we can deduce a contradiction as in Case 1.

If $A+1=0$, then $FG\equiv 1$, that is

$$f\cdot L(f)\equiv a^2. \tag{8}$$

From (8), we have

$$N(r,0;f)+N(r,\infty;f)=S(r,f). \tag{9}$$

Since $N(r,f^{(n)}/f)=S(r,f), m(r,f^{(n)}/f)=S(r,f)$, then $T(r,f^{(n)}/f)=S(r,f)$. From (9), we obtain

$$2T\left(r,\frac{f}{a}\right)=T\left(r,\frac{f^2}{a^2}\right)=T\left(r,\frac{a^2}{f^2}\right)+O(1)=T\left(r,\frac{L}{f}\right)+O(1)+S(r,f).$$

i.e.

$$T(r,f)=S(r,f),$$

we can get a contradiction.

Case 3. $B=0$. Then (7) becomes $F=(G+A-1)/A$.

If $A-1\neq 0$, then $\overline{N}(r,1/(G+A-1))=\overline{N}(r,1/F)$. Similarly, we can again deduce a contradiction as in Case 1.

If $A-1=0$, then $F\equiv G$, that is

$$f\equiv L(f).$$

This completes the proof of the Theorem 1.5.

Proof of Theorem 1.6: Let F,G be given by (4), from the assumption of Theorem 1.6, we know that F and G share "$(1,1)$".

Let H be defined as in Lemma 2.2. Suppose that $H\not\equiv 0$. Since F,G share "$(1,1)$", we can get

$$\begin{aligned}&N(r,\infty;H)\\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}(1;F|\geq 2)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,;G|\geq 2)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f),\end{aligned} \tag{10}$$

and

$$N(r,1;F|=1)\leq N(r,0;H)+S(r,f)\leq N(r,\infty;H)+S(r,f), \tag{11}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

By the second fundamental theorem, we see that

$$\begin{aligned}&T(r,F)+T(r,G)\\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)\\ &+\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_0(r,0;F')\\ &-N_0(r,0;G')+S(r,F)+S(r,G).\end{aligned} \tag{12}$$

Using Lemmas 2.3 and 2.4, (10) and (11) we can get

$$
\begin{aligned}
& \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
\leq\ & N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G) \\
& +\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\
\leq\ & N(r,1;F|=1)+N(r,1;G)-\overline{N}_L(r,1;F) \\
& -\overline{N}_L(r,1;G)+\overline{N}_{F>2}(r,1;G) \\
\leq\ & \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}(r,\infty;F) \\
& +\overline{N}_*(r,1;F,G)+T(r,G)-m(r,1;G)+O(1)+\tfrac{1}{2}\overline{N}(r,\infty;F) \\
& -\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G)+\tfrac{1}{2}\overline{N}(r,0;F) \\
& +N_0(r,0;F')+N_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned} \tag{13}
$$

Combining (12) and (13), we can obtain

$$
\begin{aligned}
T(r,F) &\leq \tfrac{7}{2}\overline{N}(r,\infty;F)+N_2(r,0;F)+N_2(r,0;G)+\tfrac{1}{2}\overline{N}(r,0;F) \\
&\leq \tfrac{7}{2}\overline{N}(r,\infty;F)+\tfrac{3}{2}N_2(r,0;F)+N_2(r,0;G)+S(r,f).
\end{aligned}
$$

By the definition of F,G and Lemma 2.1(i), we have

$$
\begin{aligned}
T(r,f) &\leq \tfrac{7}{2}\overline{N}(r,\infty;f)+\tfrac{3}{2}N_2(r,0;f)+N_2(r,0;L)+S(r,f) \\
&\leq (\tfrac{7}{2}+n)\overline{N}(r,\infty;f)+\tfrac{3}{2}N_2(r,0;f)+N_{2+n}(r,0;f)+S(r,f).
\end{aligned}
$$

By the Lemma 2.1, we have

$$T(r,f)\leq\left(\frac{7}{2}+n\right)\overline{N}(r,\infty;f)+\frac{3}{2}N_2(r,0;f)+N_{2+n}(r,0;f)+S(r,f).$$

So

$$\left(\frac{7}{2}+n\right)\Theta(\infty,f)+\frac{3}{2}\delta_2(0,f)+\delta_{2+n}(0,f)\leq n+5.$$

Which contradicts the assumption (2) of Theorem 1.6. Thus, $H\equiv 0$.

Similar to the arguments in Theorem 1.5, we can prove that the conclusions of Theorem 1.6 hold.

Proof of Theorem 1.7: Let F,G be given by (4), from the assumption of Theorem 1.6, we know that F and G share "$(1,0)$".

Let H be defined as in Lemma 2.2. Suppose that $H\not\equiv 0$. Since F,G share "$(1,0)$", we can get

$$
\begin{aligned}
N(r,\infty;H)\leq\ & \overline{N}(r,\infty;F)+\overline{N}(1;F|\geq 2)+\overline{N}(r,0;F|\geq 2) \\
& +\overline{N}(r,;G|\geq 2)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G) \\
& +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f),
\end{aligned} \tag{14}
$$

and

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)+S(r,f),\ N_E^{(2}(r,1;F)=N_E^{(2}(r,1;G)+S(r,f),$$

$$N_E^{1)}(r,1;F) \le N(r,\infty;H) + S(r,f). \tag{15}$$

Using Lemmas 2.5-2.7 and (14) and (15), we get

$$\begin{aligned}
& \overline{N}(r,1;F) + \overline{N}(r,1;G) \\
\le\ & \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\
\le\ & N_E^{1)}(r,1;F) + N(r,1;G) - \overline{N}_L(r,1;G) \\
& +\overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;G) \\
\le\ & \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + \overline{N}(r,\infty;F) \\
& +\overline{N}_*(r,1;F,G) + T(r,G) - m(r,1;G) + O(1) \\
& -\overline{N}_L(r,1;G) + \overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;G) \\
& +N_0(r,0;F') + N_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned} \tag{16}$$

Combining (12) and (16) and by Lemma 2.1, we can obtain

$$\begin{aligned}
T(r,f) \le\ & 6\overline{N}(r,\infty;f) + N_2(r,0;f) + 2\overline{N}(r,0;f) + 2\overline{N}_2(r,0;L) + S(r,f) \\
\le\ & (2n+6)\overline{N}(r,\infty;f) + N_2(r,0;f) + 2\overline{N}(r,0;f) \\
& +2\overline{N}_{2+n}(r,0;f) + S(r,f)
\end{aligned}$$

So

$$(6+2n)\Theta(\infty,f) + \delta_2(0,f) + 2\Theta(0.f) + 2\delta_{2+n}(0,f) \le (2n+10).$$

Which contradicts the assumption (3) of Theorem 1.7. Thus, $H \equiv 0$.

Similar to the arguments in Theorem 1.5, we can prove that the conclusions of Theorem 1.7 hold.

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Corresponding author: Hong-Yan XU
Jingdezhen Ceramic Institute
Department of Informatics and Engineering
Jingdezhen, Jiangxi, 333403, China
e-mail: xhyhhh@126.com

Yi HU
Jingdezhen Ceramic Institute
Department of Informatics and Engineering
Jingdezhen, Jiangxi, 333403, China
e-mail: hhhaaa123@163.com