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Some corrected optimal quadrature formulas in sense Nikolski and error bounds [1]

Alina Baboş, Ana Maria Acu

Abstract

We consider the corrected quadrature rules of some optimal quadrature formulas in sense Nikolski and the estimations of error involving the second derivative are given. The numerical examples which provides that the approximation in corrected rule of a optimal quadrature formula in sense Nikolski is better than in the original rule are considered.

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1 Introduction

Let $\mathcal{H}$ be the class of sufficiently smooth functions $f : [a, b] \to \mathbb{R}$ and we consider the following quadrature formula with degree of exactness equal $n - 1$

$$\int_a^b f(x)dx = \sum_{i=0}^{m} \sum_{k=0}^{z_i-1} A_{ki} f^{(k)}(x_i) + \mathcal{R}_n(f), \tag{1}$$

where the nodes $a \le x_0 < x_1 < \cdots < x_m \le b$ have the multiplicities z_i, $1 \le z_i \le n$.

The quadrature formula (1) is called **optimal in sense Nikolski** in the space $\mathcal{H}$, if

$$\mathcal{E}_{m,n}(\mathcal{H}, A, X) = \sup_{f \in \mathcal{H}} |\mathcal{R}_n(f)|$$

attains the minimum value with regard to A and X, where $A = \{A_{ki}\}_{i=0}^{m}{}_{k=0}^{z_i-1}$ are the coefficients and $X = (x_0, x_1, \dots, x_m)$ are the nodes of quadrature formula.

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In the last years a number of authors have obtained in many different ways the optimal quadrature formulas ([1], [5], [6], [10], [11], [12]).

Let $I \subset \mathbb{R}$ be an open interval such that $[0,1] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. N. Ujević and L. Mijić [13] constructed the following optimal quadrature formula of close type with 3 nodes:

$$\int_0^1 f(t)dt = \frac{\sqrt{2}}{8} f(0) + \left(1 - \frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \mathcal{R}(f), \tag{2}$$

where

$$|\mathcal{R}(f)| \leq \frac{2-\sqrt{2}}{48} \|f''\|_\infty \tag{3}$$

In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [2], [3], [4], [7], [8], [14]). By corrected quadrature rule we mean the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points. In [13], N. Ujević and L. Mijić constructed the corrected quadrature formula of (2):

$$\begin{aligned} \int_0^1 f(t)dt &= \frac{\sqrt{2}}{8} f(0) + \left(1 - \frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) \\ &+ \frac{1}{96}(4 - 3\sqrt{2}) \left[f'(1) - f'(0)\right] + \mathcal{R}^c(f), \end{aligned} \tag{4}$$

In the mentioned paper inequalities for remainder term of corrected quadrature formula are obtained.

Theorem 1 *[13] Let $f' : [0,1] \to \mathbb{R}$ be an absolutely continous function such that $f'' \in L_1[0,1]$ and there exist real numbers γ, Γ such that $\gamma \leq f''(t) \leq \Gamma$, $t \in [0,1]$. Then*

$$|\mathcal{R}^c(f)| \leq \frac{\Gamma - \gamma}{2} \left(\frac{5}{96}\sqrt{6} - \frac{29}{432}\sqrt{3}\right). \tag{5}$$

If there exists a real number γ such that $\gamma \leq f''(t)$, $t \in [0,1]$ then

$$|\mathcal{R}^c(f)| \leq \left(\frac{1}{12} - \frac{\sqrt{2}}{32}\right)(S - \gamma), \tag{6}$$

where $S = f'(1) - f'(0)$.
If there exists a real number Γ such that $f''(t) \leq \Gamma$, $t \in [0,1]$ then

$$|\mathcal{R}^c(f)| \leq \left(\frac{1}{12} - \frac{\sqrt{2}}{32}\right)(\Gamma - S). \tag{7}$$

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \tag{8}$$

is well known in the literature as the Čebyšev functional. It was proved that $T(f, f) \geq 0$ and the inequality $|T(f, g)| \leq \sqrt{T(f, f)} \cdot \sqrt{T(g, g)}$ holds. Denote by $\sigma(f, a, b) = \sqrt{T(f, f)}$.

In [13], N. Ujević and L. Mijić obtained the following result

Theorem 2 *[13] Let $f' : [0, 1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0, 1]$. Then*

$$|\mathcal{R}^c(f)| \leq \sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}} \cdot \sigma(f''; 0, 1). \tag{9}$$

The inequality (9) is sharp in the sense that the constant $\sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}}$ *cannot be replaced by a smaller one.*

2 Corrected optimal quadrature formulas

In ([9]), D.V. Ionescu gives a model called "the φ-function method" of constructing a quadrature formulas. Suppose that $f \in C^r[a, b]$ and for some given $n \in \mathbb{N}$ consider the nodes $a = x_0 < ... < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = 1, ..., n$, it is considered a function φ_k, $k = 1, ..., n$, with the property that

$$\varphi_k^{(r)} = 1,\ k = 1, ..., n. \tag{10}$$

One defines the function φ as follows

$$\varphi|_{[x_{k-1},x_k]} = \varphi_k,\ k = 1, ..., n, \tag{11}$$

i.e., the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is φ_k.

Using the integration by parts of the integral

$$S(f) := \int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x) f(x)dx,$$

one obtains the following quadrature formula

$$\int_a^b f(x)dx = \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} f^{(j)}(x_k) + \mathcal{R}_n(f), \tag{12}$$

with

$$\mathcal{R}_n(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x)dx \tag{13}$$

and

$$\begin{aligned} A_{0j} &= (-1)^{j+1}\varphi_1^{(r-j-1)}(x_0), \\ A_{kj} &= (-1)^j(\varphi_k - \varphi_{k+1})^{(r-j-1)}(x_k),\ k = 1, \ldots, n-1, \\ A_{nj} &= (-1)^j\varphi_n^{(r-j-1)}(x_n),\ j = 0, 1, \ldots, r-1. \end{aligned} \tag{14}$$

Denote

$$H^{m,2}[a,b] = \left\{ f \in C^{m-1}, f^{(m-1)} \text{ abs. cont. on } [a,b], f^{(m)} \in L_2[a,b] \right\}$$

For $f \in H^{2,2}[0,1]$ one considers the quadrature formula of the form

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + \mathcal{R}_n(f), \tag{15}$$

with $0 = x_0 < x_1 < \cdots < x_n = 1$.

Using the φ-function method, T.Cătinaş and G.Coman [1] obtain the following quadrature formula optimal in sense Nikolski

Theorem 3 *[1] For $f \in H^{2,2}[0,1]$, the quadrature formula of the form (15), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k^* f(x_k^*) + \mathcal{R}_n^*(f), \tag{16}$$

with

$$\begin{aligned} A_0^* &= A_n^* = \frac{3}{4}\mu, \\ A_1^* &= A_{n-1}^* = \frac{5+2\sqrt{6}}{4}m, \\ A_k^* &= \sqrt{6}\mu,\ k = 2, \ldots, n-2, \\ x_k^* &= [2 + (k-1)\sqrt{6}]\mu,\ k = 1, \ldots, n-1, \end{aligned}$$

and

$$|\mathcal{R}_n^*(f)| \le \frac{\mu^2}{2\sqrt{5}}\|f''\|_2,$$

where

$$\mu = \frac{1}{4 + (n-2)\sqrt{6}}.$$

Remark 1 *The remainder term $\mathcal{R}_n^*$ has the following representation*

$$\mathcal{R}_n^*(f) = \int_0^1 \varphi(x) f''(x)dx,$$

with

$$\varphi_1(x) = \frac{x^2}{2} - A_0^* x,$$
$$\varphi_k \equiv \frac{1}{2}\,\tilde{l}_{2,k},\ \ k = 2, \dots, n-1,$$
$$\varphi_n(x) = \frac{(1-x)^2}{2} - A_n^*(1-x),$$

where $\tilde{l}_{2,k}$ is the two degree Legendre polynomial on the interval $[x_{k-1}, x_k]$,

$$\tilde{l}_{2,k}(x) = x^2 - (x_{k-1} + x_k)x + \frac{1}{6}(x_{k-1}^2 + 4x_{k-1}x_k + x_k^2).$$

We consider the following corrected quadrature formula of (16)

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k^* f(x_k^*) + A\left[f'(1) - f'(0)\right] + \mathcal{R}_n^c(f), \tag{17}$$

where $A = \displaystyle\int_0^1 \varphi(t)dt = -\frac{1}{3}\mu^3$.

The remainder term of the corrected quadrature formula can be written

$$\mathcal{R}_n^c(f) = \int_0^1 \tilde{\varphi}(t) f''(t)dt, \text{ where } \tilde{\varphi}(t) = \varphi(t) - A.$$

Remark 2 *For the remainder term of corrected quadrature formula (17) can be established the following estimation*

$$|\mathcal{R}_n^c(f)| \le \sqrt{\left(1 - \frac{20}{9}\mu^2\right)} \frac{\mu^2}{2\sqrt{5}} \|f''\|_2.$$

We note that the estimate of the error in corrected rule is better then in the original rule.

3 Numerical examples

The main purpose of this section is to give estimations of error in corrected optimal quadrature formula. We will consider the particular case $n = 2$ in the corrected quadrature formula, namely

$$\int_0^1 f(x)dx = \sum_{k=0}^{2} A_i^* f(x_i^*) + A\left[f'(1) - f'(0)\right] + \mathcal{R}_2^c(f), \text{ where} \tag{18}$$

(19) $$\mathcal{R}_2^c(f)=\int_0^1\tilde{\varphi}(t)f''(t)dt,\ \text{with}$$

$$\tilde{\varphi}(x)=\begin{cases}\dfrac{x^2}{2}-\dfrac{3}{16}x+\dfrac{1}{192},\ x\in\left[0,\dfrac{1}{2}\right]\\ \dfrac{(1-x)^2}{2}+\dfrac{3}{16}x-\dfrac{35}{192},\ x\in\left(\dfrac{1}{2},1\right]\end{cases}$$

Theorem 4 *Let $f:[0,1]\to\mathbb{R}$ be an absolutely continuous function such that $f''\in L[0,1]$ and there exist real number γ,Γ such that $\gamma\le f''(t)\le\Gamma$, $t\in[0,1]$. Then*

$$|\mathcal{R}_2^c(f)|\le\frac{19}{13824}\sqrt{57}\,\frac{\Gamma-\gamma}{2}.$$

Theorem 5 *Let $f:[0,1]\to\mathbb{R}$ be an absolutely continuous function such that $f''\in L[0,1]$. If there exist a real number γ such that $\gamma\le f''(t)$, $t\in[0,1]$, then*

$$|\mathcal{R}_2^c(f)|\le\frac{7}{192}\cdot\big(f'(1)-f'(0)-\gamma\big).$$

Theorem 6 *Let $f:[0,1]\to\mathbb{R}$ be an absolutely continuous function such that $f''\in L[0,1]$. If there exist a real number Γ such that $f''(t)\le\Gamma$, $t\in[0,1]$, then*

$$|\mathcal{R}_2^c(f)|\le\frac{7}{192}\cdot\big(\Gamma-f'(1)+f'(0)\big).$$

Theorem 7 *Let $f:[0,1]\to\mathbb{R}$ be an absolutely continuous function such that $f''\in L_2[0,1]$. Then*

(20) $$|\mathcal{R}_2^c(f)|\le\frac{1}{960}\sqrt{155}\cdot\sigma(f'';0,1).$$

The inequality (23) is sharp in the sense that the constant $\dfrac{1}{960}\sqrt{155}$ cannot be replaced by a smaller ones.

Remark 3 *In this paper have been considered the following kind of estimations $|\mathcal{R}(f)|\le K_1\dfrac{\Gamma-\gamma}{2}$, $|\mathcal{R}(f)|\le K_2\left(f'(1)-f'(0)-\gamma\right)$, $|\mathcal{R}(f)|\le K_2\left(\Gamma-f'(1)+f'(0)\right)$, $|\mathcal{R}(f)|\le K_3\,\sigma(f'';0,1)$. In the bellow table there are the values of the constants $K_i, i=\overline{1,n}$ which appear in error estimations of quadrature formulas (4) and (18).*

	K_1	K_2	K_3
$\mathcal{R}_2^c(f)$	$\frac{19}{13824}\sqrt{57}\approx 0.0104$	$\frac{7}{192}\approx 0.0365$	$\frac{1}{960}\sqrt{155}\approx 0.013$
$\mathcal{R}^c(f)$	$\frac{5}{96}\sqrt{6}-\frac{29}{432}\sqrt{3}\approx 0.0113$	$\frac{1}{12}-\frac{\sqrt{2}}{32}\approx 0.0391$	$\sqrt{\frac{47}{23040}-\frac{\sqrt{2}}{768}}\approx 0.0141$

Therefore, we can assert that our results are better than Ujević and Mijić's result [13].

For $n = 3$ in the corrected quadrature formula (17) we obtain

$$\int_0^1 f(x)dx = \sum_{k=0}^{3} A_i^* f(x_i^*) + A\left[f'(1) - f'(0)\right] + \mathcal{R}_3^c(f), \text{ where} \tag{21}$$

$$\mathcal{R}_3^c(f) = \int_0^1 \tilde{\varphi}(t) f''(t)dt, \text{ with} \tag{22}$$

$$\tilde{\varphi}(x) = \begin{cases} \dfrac{408x^2 + 162x^2\sqrt{6} - 99x - 36\sqrt{6}x + 2}{6(4+\sqrt{6})^3}, & x \in \left[0, \dfrac{2}{4+\sqrt{6}}\right] \\ \dfrac{408x^2 + 162x^2\sqrt{6} - 408x - 162\sqrt{6}x + 98 + 39\sqrt{6}}{6(4+\sqrt{6})^3}, & x \in \left(\dfrac{2}{4+\sqrt{6}}, \dfrac{2+\sqrt{6}}{4+\sqrt{6}}\right] \\ \dfrac{311 + 126\sqrt{6} - 717x - 288\sqrt{6}x + 408x^2 + 162x^2\sqrt{6}}{6(4+\sqrt{6})^3}, & x \in \left(\dfrac{2+\sqrt{6}}{4+\sqrt{6}}, 1\right] \end{cases}$$

Theorem 8 *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$ and there exist real number γ, Γ such that $\gamma \le f''(t) \le \Gamma$, $t \in [0,1]$. Then*

$$|\mathcal{R}_3^c(f)| \le C\frac{\Gamma - \gamma}{2} \approx 0.0043\frac{\Gamma - \gamma}{2}, \text{ where}$$

$$C = \frac{(639356 + 261009\sqrt{6})\sqrt{14313 + 5832\sqrt{6}} + (19411920 + 7924880\sqrt{6})\sqrt{5}}{54(68 + 27\sqrt{6})^2(4+\sqrt{6})^6}$$

Theorem 9 *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number γ such that $\gamma \le f''(t)$, $t \in [0,1]$, then*

$$|\mathcal{R}_3^c(f)| \le \frac{1}{3} \cdot \frac{226 + 89\sqrt{6}}{(4+\sqrt{6})^5} \cdot \big(f'(1) - f'(0) - \gamma\big) \approx 0.0133\big(f'(1) - f'(0) - \gamma\big).$$

Theorem 10 *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$. If there exist a real number Γ such that $f''(t) \le \Gamma$, $t \in [0,1]$, then*

$$|\mathcal{R}_3^c(f)| \le \frac{1}{3} \cdot \frac{226 + 89\sqrt{6}}{(4+\sqrt{6})^5} \cdot \big(\Gamma - f'(1) + f'(0)\big) \approx 0.0133 \cdot \big(\Gamma - f'(1) + f'(0)\big).$$

Theorem 11 *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$|\mathcal{R}_3^c(f)| \le \frac{\sqrt{10} \cdot \sqrt{89 + 36\sqrt{6}}}{30(4+\sqrt{6})^3} \cdot \sigma(f''; 0, 1) \approx 0.0052 \cdot \sigma(f''; 0, 1). \tag{23}$$

The inequality (23) is sharp in the sense that the constant $\dfrac{\sqrt{10} \cdot \sqrt{89 + 36\sqrt{6}}}{30(4+\sqrt{6})^3}$ cannot be replaced by a smaller ones.

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Alina Baboş
"Nicolae Balcescu" Land Forces Academy
Sibiu, Romania
e-mail: alina_babos_24@yahoo.com

Ana Maria Acu
Lucian Blaga University of Sibiu
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7
RO-550012 Sibiu, Romania
e-mail: acuana77@yahoo.com

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On the approximation of analytic functions by complex Bernstein-type operators in compact discs [1]

Martin-Alexandru Bunescu-Spielhaupter

Abstract

We prove, by using a method involving divided differences, that certain complex Bernstein-type operators converge uniformly to an analytic function in any compact disc $\bar{U}(0,R)$,$R>1$. Some applications will also be given.

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Key words and phrases: Divided differences, complex Bernstein-Type operator.

1 Introduction

In this paper we mean to give a different approach to the problem of the approximation of analytic functions on compact discs by linear operators. In papers [1] , [2] and [5] − [11] the approximation properties of certain operators (Bernstein, Durrmeyer, Kantorovich, Bernstein-Stancu and many others) have been studied, with quantitative estimates for the rate of convergence being given. With our results, we aim to give sufficient conditions, under which the complex variant of a more general Bernstein-type operator converges to the analytic function to be approximated. We first recall a few results that will be used for our proofs

Definition 1 *Let $f:I\to\mathbb{R}$ be a function and $x_0,x_1,...,x_n$ distinct points from the interval I. The divided difference of order n of f with respect to the points $x_0,x_1,...,x_n$ is defined recursively by:*

a) for $n=0$, $[x_0;f]:=f(x_0)$

b) for $n=1, [x_0,x_1;f]:=\frac{f(x_1)-f(x_0)}{x_1-x_0}$

c) for $p\geq 2$,

$$[x_0,x_1,...,x_n;f]:=\frac{[x_1,x_2,...,x_n;f]-[x_0,x_1,...,x_{n-1};f]}{x_n-x_0} \quad (1)$$

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The following Leibniz-type rule was given by T.Popoviciu[14]

Theorem 1 ($Leibniz - Popoviciu\ formula$) *If $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$, then the following equality holds:*

$$[x_0, x_1, ..., x_n; f \cdot g] = \sum_{k=0}^{n} [x_0, x_1, ..., x_k; f]\, [x_k, x_{k+1}, ..., x_n; g] \tag{2}$$

Another useful result is:

Theorem 2 ($Vitali$) *If D is a complex domain and (f_n) is a bounded sequence of functions which converges pointwise to a function f on a set $E \subset D$, which possesses at least one accumulation point in D, then the sequence (f_n) converges uniformly on compacts in D.*

The proof of this theorem can be found in [3]

The Bernstein polynomials attached to a real function f are defined by:

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{3}$$

A very useful representation of the Bernstein polynomials in terms of the powers of x is:

$$(B_n f)(x) = \sum_{k=0}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k \tag{4}$$

2 Main Results

Let f be an analytical function on the complex domain G, which contains the disc $U(0, R)$, $R > 1$. Suppose f has the representation

$$f = \sum_{j=0}^{\infty} c_j z^j \tag{5}$$

We start by giving a new proof for a result of S.N. Bernstein (see [13])

Theorem 3 *The sequence of the complex Bernstein polynomials*

$$(B_n f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$$

converges uniformly on compacts towards f on the set $\overline{U}(0, R)$.

Proof. We shall use the representation (4) for the complex Bernstein polynomial

$$(B_n f)(z) = \sum_{k=0}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] z^k$$

Replacing f by its power series representation, we get

$$(B_n f)(z) = \sum_{j=0}^{\infty} c_j \sum_{k=0}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; e_j\right] z^k. \tag{6}$$

Observe that

$$\sum_{k=0}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; e_j\right] z^k = (B_n e_j)(z),$$

where $e_j : \mathbb{C} \to \mathbb{C}, e_j(z) = z^j$ for each natural number j.

$$|(B_n f)(z)| \leq \sum_{j=0}^{\infty} |c_j| \, |(B_n e_j)(z)| \tag{7}$$

As f is analytical on the open set G containing the unit disc, there exists a certain $R_1 \geq R > 1$ for which the series (5) converges on $U(0, R_1)$. By mathematical induction, we shall prove that

$$|(B_n e_j)(z)| \leq R_1^j, j = 0, 1, ... \tag{8}$$

For $j = 0$ and 1 the inequality is obviously true.

We suppose that the inequality holds for a certain $j \geq 2$. Then, for $j+1$ we have:

$$\begin{aligned} |(B_n e_{j+1})(z)| &= \left| \sum_{k=0}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; e_{j+1}\right] z^k \right| \\ &\leq \sum_{k=1}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; e_{j+1}\right] \left|z^k\right| \\ &\leq \sum_{k=1}^{n} \frac{A_n^k}{n^k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; e_{j+1}\right] R_1^k \\ &= (B_n e_{j+1})(R_1) \end{aligned} \tag{9}$$

By using formula (2) and taking into account the properties of the divided differences, we can write:

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k}{n};e_{j+1}\right]R_1^k \\
=&\sum_{k=1}^{n} \frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k-1}{n};e_j\right]R_1^k+\sum_{k=1}^{n} \frac{A_n^k}{n^k}\cdot\frac{k}{n}\left[0,\frac{1}{n},...,\frac{k}{n};e_j\right]R_1^k \\
=&\sum_{k=1}^{n} \frac{A_n^k}{n^{k+1}}\left[0,\frac{1}{n},...,\frac{k}{n};e_j\right]\left[(n-k)R_1+k\right]R_1^k \\
\leq&\sum_{k=1}^{n} \frac{A_n^k}{n^{k+1}}\left[0,\frac{1}{n},...,\frac{k}{n};e_j\right]nR_1^{k+1}=R_1\left(B_ne_j\right)\left(R_1\right)\leq R_1^{j+1}.
\end{aligned}$$

By this, we get:

$$\left|\left(B_nf\right)(z)\right|\leq\sum_{j=0}^{\infty}\left|c_j\right|R_1^j<\infty$$

Thus the sequence $(B_nf)(z)$ is bounded and using Vitali's theorem and the fact that B_nf converges uniformly on $[0,1]$ to f, we obtain the statement of the theorem.

A more general result can be obtained if we replace the Bernstein operator with a Bernstein-type operator of the form

$$(L_nf)(z)=\sum_{k=0}^{n}p_{n,k}(z)\,\Lambda_{n,k}(f) \tag{10}$$

$\Lambda_{n,k}(f)$ are real linear positive functionals satisfying

$$\Lambda_{n,k}(e_0)=1 \tag{11}$$

and

$$p_{n,k}(z)=\binom{n}{k}z^k(1-z)^{n-k}.$$

Suppose that the sequence (L_nf) converges uniformly to f for any real function $f\in C[0,1]$. For any $t\in[0,1]$ we consider the positive linear functional $\Lambda_{n,nt}:C_{[0,1]}\to\mathbb{R}$. For fixed f we denote by $g, g:[0,1]\to\mathbb{R}$ the function defined by

$$g(t)=\Lambda_{n,nt}(f). \tag{12}$$

One can observe that $g\left(\frac{k}{n}\right)=\Lambda_{n,k}(f)$, $k=\overline{0,n}$. Under these assumptions we can write, for each n

$$(L_nf)(z)=\sum_{k=0}^{n}p_{n,k}(z)\,\Lambda_{n,k}(f)=\sum_{k=0}^{n}p_{n,k}(z)\,g\left(\frac{k}{n}\right)=(B_ng)(z),z\in G. \tag{13}$$

This follows by the identity theorem for holomorphic functions, because L_nf and B_ng are obviously holomorphic and they coincide on a set of points which has an accumulation point in G (recall that G is a complex domain containing the disc $U(0,R), R>1$) This last relationship implies:

$$(L_nf)(z) = (B_ng)(z) = \sum_{k=0}^{n} \frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k}{n};g\right] z^k = \sum_{k=0}^{n} \frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k}{n};\Lambda_{n,nt}(f)\right] z^k. \tag{14}$$

Using the power series representation of f we have

$$(L_nf)(z) = \sum_{j=0}^{\infty} c_j \sum_{k=0}^{n} \frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k}{n};\Lambda_{n,nt}(e_j)\right] z^k. \tag{15}$$

Theorem 4 *Let G be a complex domain containing the disc $\overline{U}(0,R), R>1$ and $f: G\to\mathbb{C}$ an analytic function on G, $f=\sum_{j=0}^{\infty} c_j z^j$. If the function $\Lambda_{n,nt}(e_j)$ is a polynomial of degree j with coefficients $a_{m,j}$ inside the interval $[-1,1]$ for every $m=0,1,...,j$, then*

$$\lim_{n\to\infty}(L_nf)(z)=f(z)$$

uniformly on $\overline{U}(0,R)$.

Proof. We shall show that L_nf is bounded.

Let be $\Lambda_{n,nt}(e_j)=\sum_{m=0}^{j} a_{m,j}t^m$. We have:

$$|(L_nf)(z)| \le \sum_{j=0}^{\infty}|c_j| \sum_{k=0}^{n} \frac{A_n^k}{n^k}\left|\left[0,\frac{1}{n},...,\frac{k}{n};\sum_{m=0}^{j} a_{m,j}t^m\right]\right| R_1^k \tag{16}$$

where $R_1 \ge R > 1$. On the other hand,

$$\left|\left[0,\frac{1}{n},...,\frac{k}{n};\sum_{m=0}^{j} a_{m,j}t^m\right]\right| = \left|\sum_{m=0}^{j} a_{m,j}\left[0,\frac{1}{n},...,\frac{k}{n};t^m\right]\right| \le \sum_{m=0}^{j}|a_{m,j}|\left[0,\frac{1}{n},...,\frac{k}{n};t^m\right].$$

Replacing this in (16), it yields:

$$|(L_nf)(z)| \le \sum_{j=0}^{\infty}|c_j|\left[\sum_{m=1}^{j}|a_{m,j}|\left(\sum_{k=0}^{n}\frac{A_n^k}{n^k}\left[0,\frac{1}{n},...,\frac{k}{n};t^m\right]R_1^k\right)\right].$$

In the proof of theorem 3 we have shown that

$$\sum_{k=0}^{n} \frac{A_n^k}{n^k}\left[0, \frac{1}{n}, ..., \frac{k}{n}; t^m\right] R_1^k \leq R_1^m$$

so

$$|(L_n f)(z)| \leq \sum_{i=0}^{\infty} |c_j| \left[\sum_{m=1}^{j} |a_{m,j}| R_1^m\right] \leq \sum_{i=0}^{\infty} |c_j| \left[\sum_{m=1}^{j} R_1^m\right] \leq \frac{R_1}{R_1 - 1} \sum_{j=0}^{\infty} |c_j| R_1^j.$$

Applying Vitali's theorem we obtain the desired result.

We shall now give a few examples of applications for theorem 4.

1. The complex Kantorovich operators are defined by

$$K_n(f)(z) = (n+1) \sum_{k=0}^{n} p_{n,k}(z) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du. \tag{17}$$

We have in this case $\Lambda_{n,k}(f) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\, du$. It is easy to show that $\Lambda_{n,k}(e_0) = 1$. Calculating $\Lambda_{n,nt}(e_j)$ we get:

$$\Lambda_{n,nt}(e_j) = \frac{n+1}{i+1}\left[\left(\frac{nt+1}{n+1}\right)^{j+1} - \left(\frac{nt}{n+1}\right)^{j+1}\right]. \tag{18}$$

This is obviously a polynomial of degree j. Denoting by S the sum of the coefficients of the polynomial, one obtains:

$$S = \frac{n+1}{j+1}\left[\left(\frac{n+1}{n+1}\right)^{j+1} - \left(\frac{n}{n+1}\right)^{j+1}\right] \leq 1.$$

by applying Lagrange's theorem to the function t^{j+1} on the interval $[n, n+1]$.

As all coefficients are positive, the previous calculation implies that they do not exceed 1. Hence, we have proven:

Corollary 1 *The sequence of the Kantorovich operators $(K_n f)$ converges uniformly towards the analytical function f on $\overline{U}(0, R), R > 1$.*

2. The Durrmeyer operators are defined by

$$(D_n f)(z) = (n+1) \sum_{k=0}^{n} p_{n,k}(z) \int_0^1 p_{n,k}(t) f(t)\, dt. \tag{19}$$

In this case, the functional $\Lambda_{n,k}(f)$ is

$$\Lambda_{n,k}(f) = (n+1) \int_0^1 p_{n,k}(t) f(t)\, dt. \tag{20}$$

The fact that $\Lambda_{n,k}(e_0) = 1$ follows from the properties of the Euler-Beta function.

For the Durrmeyer operators we define $\Lambda_{n,nt}(f)$ by:

$$\Lambda_{n,nt}(f) := \frac{\Gamma(n+2)}{\Gamma(nt+1)\Gamma(n-nt+1)} \int_0^1 u^{nt}(1-u)^{n-nt} f(u)\, du \tag{21}$$

where Γ is the Euler-Gamma function. We have:

$$\Lambda_{n,nt}(e_j) = \frac{(nt+j)(nt+j-1)\ldots(nt+1)}{(n+j+1)(n+j)\ldots(n+2)}.$$

This is a polynomial of degree j and the sum of its coefficients is

$$\frac{(n+j)(n+i-1)\ldots(n+1)}{(n+j+1)(n+j)\ldots(n+2)} \leq 1.$$

Applying theorem 4, we get:

Corollary 2 *The sequence of the Durrmeyer operators* $(D_n f)$ *converges uniformly towards the analytical function* f *on* $\overline{U}(0,R), R > 1$.

3. For $0 \leq \alpha \leq \beta$, the Bernstein-Stancu operators are given by

$$\left(P_n^{\alpha,\beta} f\right)(z) = \sum_{k=0}^{n} p_{n,k}(z) f\left(\frac{k+\alpha}{n+\beta}\right). \tag{22}$$

In this case $\Lambda_{n,k}(f) = f\left(\frac{k+\alpha}{n+\beta}\right)$ and obviously $\Lambda_{n,k}(e_0) = 1$. The function $\Lambda_{n,nt}(f)$ is in this case

$$\Lambda_{n,nt}(f) := \frac{nt+\alpha}{n+\beta} \tag{23}$$

and

$$\Lambda_{n,nt}\left(z^j\right) = \left(\frac{nt+\alpha}{n+\beta}\right)^j = \left(\frac{n}{n+\beta}t + \frac{\alpha}{n+\beta}\right)^j.$$

The requirements of theorem 4 are obviously fulfilled, so we get:

Corollary 3 *The sequence of the Bernstein-Stancu operators* $\left(P_n^{\alpha,\beta} f\right)$ *converges uniformly towards the analytical function* f *on* $\overline{U}(0,R), R > 1$.

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Martin-Alexandru Bunescu-Spielhaupter
Colegiul National "Samuel von Brukenthal",
Sibiu, Romania
e-mail: alexbunescu@yahoo.com

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Sequential optimality conditions for variational inequalities [1]

Liana Cioban

Abstract

In the characterizations of the solutions of the variational inequalities (via gap functions or by means of the subdifferential), the fulfillment of a regularity condition was of great importance. The aim of this paper is to deliver necessary and sufficient sequential characterizations for the solutions of variational inequalities in case no regularity condition is fulfilled. We use as tool the sequential optimality conditions given by Boţ, Csetnek and Wanka. Several examples are illustrating the theoretical aspects.

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1 Introduction

Let us consider the following general variational inequality problem considered in [6] which consists in finding an element $\overline{x} \in \text{dom } \Phi(\cdot, 0)$ such that

$$(VI)^{\Phi} \qquad \langle F(\overline{x}), x - \overline{x}\rangle + \Phi(x, 0) - \Phi(\overline{x}, 0) \geq 0 \ \forall x \in X,$$

where X and Y are real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a proper function fulfilling $0 \in \text{Pr}_Y(\text{dom } \Phi)$ and $F : X \to X^*$ is a given operator.

In [6, Theorem 3.1] the authors used a common approach in order to solve the problem $(VI)^{\Phi}$ by formulating a gap function with the help of the Fenchel-Moreau conjugate of the functions involved in case of convex setting and if a regularity condition is fulfilled. A similar idea was used previously in [1]. In [6, Theorem 3.27]

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the authors also characterize the solutions of the general variational inequalities by means of the properties of the convex subdifferential. For this characterization the fulfilling of a regularity condition is also needed. The aim of this paper is to deliver regularity conditions free conditions in order to characterize the solution of the general variational inequality $(VI)^{\Phi}$. The paper is organized as follows. In Section 2 we give some definitions and results that will be used in the paper. In section 3 we give sequential optimality conditions for the general variational inequality and for some particular cases when we specialize the perturbation function Φ in the definition of $(VI)^{\Phi}$. We give also some examples in order to justify the usefulness of having such characterizations.

2 Preliminaries

Let us mention some notions and results that will be used in the paper (see also [9, 14, 2, 7, 12, 13]).

Consider X a real separated locally convex space and X^* its topological dual space. We denote by $w(X^*, X)$ the weak* topology on X^* induced by X.

We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Let us consider $V \subseteq Y$ (Y being a real separated locally convex space) another nonempty set. The *projection operator* $\mathrm{Pr}_U : U \times V \to U$ is define as $\mathrm{Pr}_U(u, v) = u$ for all $(u, v) \in U \times V$, while the *indicator function* of U, $\delta_U : X \to \overline{\mathbb{R}}$, is defined by $\delta_U(x) = 0$ if $x \in U$ and $+\infty$ otherwise (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line).

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi}\, f = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}\, f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$.

For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the (convex) *sudifferential* of f at x by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle \ \forall y \in X\}$. If $f(x) \in \{\pm\infty\}$ we take by convention $\partial f(x) = \emptyset$. The *normal cone* of U at $x \in X$ is defined by $N_U(x) = \partial\delta_U(x)$, that is $N_U(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0 \ \forall y \in U\}$ when $x \in U$, and $N_U(x) = \emptyset$ if $x \notin U$.

Consider Y another real separated locally convex space. For a vector function $h : X \to Y$ we denote by $h(U) = \{h(u) : u \in U\}$ the *image* of the set $U \subseteq X$ through h, while $h^{-1}(D) = \{x \in X : h(x) \in D\}$ is the *counter image* of the set $D \subseteq Y$ through h. Given a linear continuous mapping $A : X \to Y$, its *adjoint operator* $A^* : Y^* \to X^*$ is defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for all $y^* \in Y^*$ and $x \in X$. For a nonempty *convex cone* $K \subseteq Y$ we denote by $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \ge 0 \ \forall y \in K\}$ its *dual cone*. Further, we denote by $\le_K$ the partial ordering induced by K on Y, defined as $y_1 \le_K y_2 \Leftrightarrow y_2 - y_1 \in K$ for $y_1, y_2 \in Y$. To Y we attach an abstract maximal element with respect to $\le_K$, denoted by ∞_K and let $Y^\bullet := Y \cup \{\infty_K\}$. Then for every $y \in Y$ one has $y \le_K \infty_K$, while on $Y^\bullet$ the following operations are considered: $y + \infty_K = \infty_K + y = \infty_K$ for all $y \in Y$ and $t\infty_K = \infty_K$ for all $t \ge 0$. Moreover, if $\lambda \in K^*$ let $\langle \lambda, \infty_K\rangle := +\infty$.

Some of the above notions given for functions with extended real values can be formulated also for function having their ranges in infinite-dimensional spaces. For a function $h : X \to Y^\bullet$ we denote by dom $h = \{x \in X : h(x) \in Y\}$ its *domain* and by epi ${}_K h = \{(x, y) \in X \times Y : h(x) \leq_K y\}$ its *K-epigraph.* We say that h is *proper* if its domain is a nonempty set. The function h is said to be *K-convex* if for all $x, y \in X$ and $t \in [0, 1]$ we have $h(tx+(1-t)y) \leq_K th(x)+(1-t)h(y)$. Further, for an arbitrary $l \in K^*$ we define the function $lh : X \to \overline{\mathbb{R}}$, by $(lh)(x) = \langle l, h(x)\rangle$ for all $x \in X$. The function h is said to be *star K-lower semicontinuous* at $x \in X$ if for all $\lambda \in K^*$ the function λh is lower semicontinuous at x. The function h is said to be *star C-lower semicontinuous* if it is star C-lower semicontinuous at every $x \in X$ (this was considered first in [11]). Let us notice that other lower semicontinuity notions are the *K-lower semicontinuity* introduced by Penot and Théra, or the *K-epi-closedness* (see [7, Section 2.2.2] and the references therein for more details).

3 Sequential optimality conditions for variational inequalities

Notice that in the characterizations of the solutions of the variational inequalities (via gap functions or by means of the subdifferential, see [6]), the fulfillment of a regularity condition was of great importance. We show in this section that even in the absence of a regularity condition, we can still characterize these solutions. We use as tool the sequential optimality conditions given in [4, 5]. In these papers the authors delivered qualification free sequential optimality conditions for the characterization of the optimal solutions of convex optimization problems. Since the variational inequalities can be reformulated as optimization problems, we obtain similar results when no regularity condition is fulfilled.

Notice that in the relations below the convergence of a sequence in the dual space can be taken with respect to the strong or the weak* topology, see [4, 5] for details.

3.1 Sequential optimality condition for general variational inequality

Consider a so-called perturbation function $\Phi : X \times Y \to \overline{\mathbb{R}}$ which is proper and fulfils the relation $0 \in \mathrm{Pr}_Y(\mathrm{dom}\ \Phi)$, where X and Y are supposed to be real separated locally convex spaces. The general variational inequality problem (see [6]) consists in finding an element $\overline{x} \in X$ such that

$$(VI)^\Phi \qquad \langle F(\overline{x}), x - \overline{x}\rangle + \Phi(x, 0) - \Phi(\overline{x}, 0) \geq 0 \ \forall x \in X,$$

where X and Y are real separated locally convex spaces, $\Phi : X \times Y \to \overline{\mathbb{R}}$ is a proper function fulfilling $0 \in \mathrm{Pr}_Y(\mathrm{dom}\ \Phi)$ and $F : X \to X^*$ is a given operator. Let us mention that we are looking actually for an element $\overline{x} \in \mathrm{dom}\ \Phi(\cdot, 0)$ and it is enough to require that the inequality $(VI)^\Phi$ holds for all $x \in \mathrm{dom}\ \Phi(\cdot, 0)$.

Let $\overline{x} \in \text{dom } \Phi(\cdot, 0)$ be fixed. To the problem $(VI)^{\Phi}$ one can associate the following optimization problem

$$(P^{\Phi}, \overline{x}) \quad \inf_{x \in X} \{\langle F(\overline{x}), x\rangle + \Phi(x, 0)\} - \langle F(\overline{x}), \overline{x}\rangle - \Phi(\overline{x}, 0).$$

It is immediate that $\overline{x}$ is a solution of the variational inequality $(VI)^{\Phi}$ if and only if $v(P^{\Phi}, \overline{x}) = 0$.

In what follows we deliver sequential conditions in order to characterize the solution of the general variational inequality $(VI)^{\Phi}$.

Theorem 1 *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous and $0 \in \text{Pr}_Y(dom\ \Phi)$. Then $\overline{x} \in dom\ \Phi(\cdot, 0)$ solves $(VI)^{\Phi}$ if and only if there exists $(x_n, y_n) \in dom\ \Phi$ and $(x_n^*, y_n^*) \in \partial\Phi(x_n, y_n)$ such that*

$$x_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to 0\, (n \to +\infty)\, and$$

$$\Phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \Phi(\overline{x}, 0) \to 0\, (n \to +\infty).$$

Proof Let us fix an element $\overline{x} \in \text{dom } \Phi(\cdot, 0)$. It is immediate that $\overline{x}$ is a solution of the variational inequality $(VI)^{\Phi}$ if and only if $\overline{x}$ is an optimal solution of the optimization problem:

$$(P^{\Phi}) \quad \{\inf_{x \in X} \langle F(\overline{x}), x\rangle + \Phi(x, 0)\}.$$

We apply now [8, Theorem 3.2] to the function $x \mapsto \langle F(\overline{x}), x\rangle + \Phi(x, 0)$ and obtain the desired conclusion.

3.2 Sequential optimality condition for the case $g + f \circ h$

For this case we work in the following settings: X is a reflexive Banach space and Y is a Banach space partially ordered by the non-empty convex cone $K \subseteq X$, $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous, $h : X \to Y^{\bullet}$ is proper and $K-convex$ and $g : Y^{\bullet} \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous with $g(+\infty_K) = +\infty$. We also suppose that $\text{dom } f \cap \text{dom } h \cap h^{-1}(\text{dom } g) \neq \emptyset$.

The variational inequality in this case consists in finding an element $\overline{x} \in \text{dom } f \cap \text{dom } h \cap h^{-1}(\text{dom } g)$ such that

$$(VI)^{CC} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(h(x)) - f(\overline{x}) - g(h(\overline{x})) \geq 0\ \forall x \in X.$$

To the problem $(VI)^{CC}$ one can associate the following primal problem

$$(P^{CC}, \overline{x}) \quad \inf_{x \in X} \{\langle F(\overline{x}), x\rangle + f(x) + (g \circ h)(x)\} - \langle F(\overline{x}), \overline{x}\rangle - f(\overline{x}) - (g \circ h)(\overline{x}),$$

where $\overline{x}$ is fixed.

3.2.1 The case h is K-epi closed

For this case we assume in additionally that Y is reflexive, h is K-epi-clsed and g is K-increasing on $h(\text{dom } h) + K$. The next theorem is particular case of [8, Theorem 3.8] for $(VI)^{CC}$. In this settings we have the following result.

Theorem 2 *The element $\overline{x} \in dom\ f \cap dom\ h \cap h^{-1}(dom\ g)$ solves $(VI)^{CC}$ if and only if*

$$\begin{cases} \exists (x_n, p_n, q_n, q_n') \in X \times dom\ f \times dom\ g \times Y,\ h(x_n) \leq_K q_n' \\ \exists (u_n^*, e_n^*, u_n'^*, q_n^*), q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), \\ u_n'^* \in \partial (q_n^* h)(x_n), \langle q_n^*, q_n' - h(x_n)\rangle = 0\, \forall n \in \mathbb{N}, \\ u_n^* + u_n'^* \to -F(\overline{x}), e_n^* \to 0, p_n \to \overline{x}, q_n \to h(\overline{x}), q_n' \to h(\overline{x})\ (n \to +\infty) \\ f(p_n) - \langle u_n^*, p_n - x_n\rangle + \langle F(\overline{x}), x_n - \overline{x}\rangle + \langle q_n^*, h(x_n) - h(\overline{x})\rangle - f(\overline{x}) \to 0\ (n \to +\infty)\ and \\ g(q_n) - \langle q_n^*, q_n - h(\overline{x})\rangle - g(h(\overline{x})) \to 0\ (n \to +\infty). \end{cases}$$

3.2.2 The case h is continuous

For this case we consider in addition that $h : X \to Y$ is continuous and $g : Y \to \overline{\mathbb{R}}$ is K-increasing on Y. The next theorem is particular case of [8, Theorem 3.9] for $(VI)^{CC}$.

Theorem 3 *The element $\overline{x} \in dom\ f \cap h^{-1}(dom\ g)$ solves $(VI)^{CC}$ if and only if*

$$\begin{cases} \exists (x_n, y_n) \in dom\ f \times dom\ g, \exists (u_n^*, v_n^*, y_n^*) \in X^* \times X^* \times K^*, \\ u_n^* - F(\overline{x}) \in \partial f(x_n), v_n^* \in \partial (y_n^* h)(x_n), y_n^* \in \partial g(y_n)\, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to \overline{x}, y_n \to h(\overline{x})\ (n \to +\infty), \\ f(x_n) + \langle y_n^*, h(x_n) - h(\overline{x})\rangle + \langle F(\overline{x}), x_n - \overline{x}\rangle - f(\overline{x}) \to 0\ (n \to +\infty)\ and \\ g(y_n) - \langle y_n^*, y_n - h(\overline{x})\rangle - g(h(\overline{x})) \to 0\ (n \to +\infty). \end{cases}$$

3.3 Sequential optimality condition for the case $f + g \circ A$

This is a particular case of $f + g \circ h$ when $h : X \to Y$, $h(x) = Ax, \forall x \in X$ and A is a linear continuous operator. Taking $K = \{0\} \subset Y$ one has that h and g are K-convex and K-increasing, respectively, and $(VI)^{CC}$ is nothing else than finding an element $\overline{x} \in \text{dom } f \cap A^{-1}(\text{dom } g)$ such that

$$(VI)^{f,g,A} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(Ax) - f(\overline{x}) - g(A\overline{x}) \geq 0\ \forall x \in X.$$

The primal problem associated to $(VI)^{f,g,A}$ is

$$(P^{f,g,A}, \overline{x}) \quad \inf_{x \in X} \{\langle F(\overline{x}), x\rangle + f(x) + g(Ax)\} - \langle F(\overline{x}), \overline{x}\rangle - f(\overline{x}) - g(A\overline{x}),$$

where $\overline{x}$ is fixed.

The next theorems are particular cases of Theorem 3.3 and Theorem 3.4 in [8] for $(VI)^{f,g,A}$.

Theorem 4 *Let $A : X \to Y$ be a continuous linear mapping, $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(dom\ f) \cap dom\ g \neq \emptyset$. Then $\overline{x} \in dom\ f \cap A^{-1}(dom\ g)$ solves $(VI)^{f,g,A}$ if and only if*

$$\exists\{\varepsilon_n\} \downarrow 0, \exists x_n^* \in \partial_{\varepsilon_n} f(\overline{x}), \exists y_n^* \in \partial_{\varepsilon_n} g(A\overline{x}) \text{ such that } x_n^* + A^* y_n^* \to -F(\overline{x}),\ (n \to +\infty).$$

Theorem 5 *Let $A : X \to Y$ be a continuous linear mapping, $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(dom\ f) \cap dom\ g \neq \emptyset$. Then $\overline{x} \in dom\ f \cap A^{-1}(dom\ g)$ solves $(VI)^{f,g,A}$ if and only if*

$$\begin{cases} \exists(x_n, y_n) \in dom\ f \times dom\ g,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in \partial g(y_n) \text{ such that} \\ x_n^* + A^* y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to A\overline{x}\,(n \to \infty) \\ f(x_n) - \langle x_n^*, x_n - \overline{x}\rangle - f(\overline{x}) \to 0\,(n \to \infty) \\ g(y_n) - \langle y_n^*, y_n - A\overline{x}\rangle - g(A\overline{x}) \to 0\,(n \to \infty). \end{cases}$$

The next two theorems were considered in [6] and are the sequential characterizations of the solutions of the variational inequalities for the cases $f + g$ and $f + \delta_K$.

If we take $X = Y$ and $A = \text{id}_X$ the variational inequality reduces to finding an element $\overline{x} \in \text{dom } f \cap \text{dom } g$ such that

$$(VI)^{f,g} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) + g(x) - f(\overline{x}) - g(\overline{x}) \geq 0\ \forall x \in X,$$

and we can characterize the solutions of $(VI)^{f,g}$ using the next result.

Theorem 6 *Let X be a reflexive Banach space and $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $dom\ f \cap dom\ g \neq \emptyset$. Then $\overline{x} \in dom\ f \cap dom\ g$ is a solution of the variational inequality $(VI)^{f,g}$ if and only if*

$$\begin{cases} \exists(x_n, y_n) \in dom\ f \times dom\ g,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in \partial g(y_n) \text{ such that} \\ x_n^* + y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to \overline{x}\,(n \to +\infty) \\ f(x_n) - \langle x_n^*, x_n - \overline{x}\rangle - f(\overline{x}) \to 0\,(n \to +\infty) \\ g(y_n) - \langle y_n^*, y_n - \overline{x}\rangle - g(\overline{x}) \to 0\,(n \to +\infty). \end{cases}$$

In case $g = \delta_K$, where $K \subseteq X$ is a nonempty the variational inequality $(VI)^{f,g}$ becomes: find an element $\overline{x} \in \text{dom } f \cap K$ such that

$$(VI)^{f,K} \quad \langle F(\overline{x}), x - \overline{x}\rangle + f(x) - f(\overline{x}) \geq 0\ \forall x \in K,$$

and we can characterize the solutions of $(VI)^{f,K}$ with the next theorem.

Theorem 7 *Let X be a reflexive Banach space, $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $K \subseteq X$ a closed and convex set such that $dom\ f \cap K \neq \emptyset$. Then $\overline{x} \in dom\ f \cap K$ is a solution of the variational inequality $(VI)^{f,K}$ if and only if*

$$\begin{cases} \exists(x_n, y_n) \in dom\ f \times K,\ \exists x_n^* \in \partial f(x_n),\ \exists y_n^* \in N_K(y_n) \text{ such that} \\ x_n^* + y_n^* \to -F(\overline{x}),\ x_n \to \overline{x},\ y_n \to \overline{x}\,(n \to +\infty) \\ f(x_n) - \langle x_n^*, x_n - \overline{x}\rangle - f(\overline{x}) \to 0\,(n \to +\infty) \\ \langle y_n^*, y_n - \overline{x}\rangle \to 0\,(n \to +\infty). \end{cases}$$

In what follows we give some examples to justify the use of the sequential optimality conditions.

Example 1 *Let* $X = \mathbb{R}$, $K = (-\infty, 0]$, $f : \mathbb{R} \to \overline{\mathbb{R}}$ $f(x) = \begin{cases} -\sqrt{x}, & x \geq 0 \\ +\infty, & otherwise, \end{cases}$ $F : \mathbb{R} \to \mathbb{R}$, $F(x) = x$. *In this case the variational inequality* $(VI)^{f,K}$ *consists in finding a point* $\overline{x} \in (-\infty, 0]$ *such that* $\overline{x}(x-\overline{x}) - \sqrt{x} + \sqrt{\overline{x}} \geq 0$, $\forall x \geq 0$ *and* $x \leq 0$, *that means that* $x = 0$ *and* $(VI)^{f,K}$ *reduces to finding* $\overline{x} \in (-\infty, 0]$ *such that* $-\overline{x}^* + \sqrt{\overline{x}} \geq 0$ *which means that* $\overline{x} = 0$ *is the unique optimal solution for* $(VI)^{f,K}$. *It is obviously that* $\partial f(0) = \emptyset$. *For all* $n \in \mathbb{N}$ *we take* $x_n = \frac{1}{n} \in dom\ f = [0, +\infty)$, $y_n = 0 \in K$, $x_n^* = -\frac{\sqrt{n}}{2} \in \partial f(x_n)$, $y_n^* = \frac{\sqrt{n}}{2} \in N_K(0) = [0, +\infty)$. *It holds* $x_n^* + y_n^* = 0 = -F(0)$, $f(x_n) - \langle x_n^*, x_n - \overline{x} \rangle - f(\overline{x}) = -\frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{n}} = -\frac{1}{2\sqrt{n}} \to 0 (n \to +\infty)$, $\langle y_n^*, y_n - \overline{x} \rangle = 0$ *and we can see that all the condition in Theorem 7 are fulfilled.*

In the example above we can apply also [6, Theorem 3.11], i.e. we can characterize the solutions of $(VI)^{f,K}$ by using gap functions. In the following example we underline the advantage of having such sequential characterizations. In the example below we can not apply the characterizations by means of gap functions, however the conditions Theorem 6 is fulfilled.

Example 2 *If we consider the real Hilbert space* $X = \ell^2(\mathbb{N})$ *and the sets*

$$C = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_{2n-1} + x_{2n} = 0\, \forall n \in \mathbb{N}\}$$

and

$$S = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_{2n} + x_{2n+1} = 0\, \forall n \in \mathbb{N}\}$$

which are closed linear subspaces of ℓ^2 *and satisfy* $C \cap S = \{0\}$. *We consider the functions* $f, g : \ell^2 \to \overline{\mathbb{R}}$ *by* $f = \delta_C$ *and* $g(x) = x_1 + \delta_S$, $\forall x = (x_n)_{n \in \mathbb{N}} \in \ell^2$ *respectively, which are proper, convex and lower semicontinuous. We have that* $dom\ f = C$ *and* $dom\ g = S$. *In this case the optimal objective value of the primal problem is* $v(P^{f,g}, \overline{x}) = 0$ *and* $\overline{x} = 0$ *while the optimal objective value of the dual problem is* $v(D^{f,g}, \overline{x}) = -\infty$. *We have that* $S - C$ *is dense in* ℓ^2 *(cf. [10, Example 3.3]), thus* $cl\,(cone\,(dom\ f - dom\ g)) = cl\,(C - S) = \ell^2$ *hence* $0 \in qi\,(dom\ f - dom\ g)$ *(see also [8, Example 2.3], [3, Example 16, Example 25]). In this case and we can not apply [6, Theorem 3.10].*

Let us see now if the conditions in Theorem 6 are fulfilled in this case. One can prove that $f^* = \delta_{C^\perp}$ *and* $g^* = \delta_{e_1 + S^\perp}$ *where*

$$C^\perp = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_{2n-1} = x_{2n} \forall n \in \mathbb{N}\}$$

and

$$S^\perp = \{(x_n)_{n \in \mathbb{N}} \in \ell^2 : x_1 = 0, x_{2n} = x_{2n+1} \forall n \in \mathbb{N}\}$$

In this case Theorem 6 become: $\overline{x} = 0 \in C \cap S = \{0\}$ *solves* $(VI)^{f,g}$ *if and only if* $\exists (x_n, y_n) \in C \times S$, $\exists x_n^* \in C^\perp$, $\exists y_n^* \in e_1 + S^\perp$ *such that* $x_n^* + y_n^* \to 0$, $x_n \to 0$, $y_n \to 0\, (n \to +\infty)$. *We consider* $x_n = y_n = 0$. *It has been proven in [10, Example 3.3] that* $C^\perp + S^\perp$ *is dense in* ℓ^2, *hence exists* x_n^*, y_n^* *satisfying the conditions in Theorem 6.*

3.4 Sequential optimality condition for the case with geometric and cone constraints

Let us particularize now to the function $\Phi^{C_L} : X \times Z \to \overline{\mathbb{R}}$,

$$\Phi^{C_L}(x,z) = \begin{cases} f(x), & \text{if } x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

where X and Z are real separated locally convex spaces, Z partially ordered by a nonempty cone $C \subseteq Z$, $S \subseteq X$ is a nonempty set, $f : X \to \overline{\mathbb{R}}$ is a proper function and $g : X \to Z^\bullet$ is a proper function fulfilling dom $f \cap S \cap g^{-1}(-C) \neq \emptyset$.

The variational inequality $(VI)^{\Phi^{C_L}}$ becomes: find an element $\overline{x} \in$ dom $f \cap \mathcal{A}$ such that

$$(VI)^C \qquad \langle F(\overline{x}), x - \overline{x} \rangle + f(x) - f(\overline{x}) \geq 0 \ \forall x \in \mathcal{A},$$

where $\mathcal{A} = S \cap g^{-1}(C)$.

The primal problem associated to $(VI)^C$ is

$$(P^C, \overline{x}) \quad \inf_{x \in X} \{\langle F(\overline{x}), x \rangle + f(x) + \delta_{\mathcal{A}}(x)\} - \langle F(\overline{x}), \overline{x} \rangle - f(\overline{x}) - \delta_{\mathcal{A}}(\overline{x})$$

where $\overline{x} \in X$ is fixed.

3.4.1 The case g is K-epi closed

We consider X a reflexive Banach space, Y a Banach spaces, and $K \subseteq Y$ a nonempty convex cone inducing a partial ordering on Y, $S \subseteq X$ closed and convex, $f : X \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function and $g : X \to Y^\bullet$ is a K-epi-closed function and K-convex vector-valued function and $S \cap g^{-1}(-K) \cap \text{dom } f \neq \emptyset$.

The next theorem is particular case of [8, Theorem 3.7] for $(VI)^C$. In this settings we can formulate the following result.

Theorem 8 *The element $\overline{x} \in S \cap$ dom $f \cap g^{-1}(-K)$ solves $(VI)^{CC}$ if and only if*

$$\begin{cases} \exists (x_n, p_n, q_n) \in S \times \textit{dom } f \times Y,\ g(x_n) \leq_K q_n, \exists (u_n^*, v_n^*, q_n^*) \in X^* \times X^* \times K^*, \\ u_n^* - F(\overline{x}) \in \partial f(p_n), v_n^* \in \partial (q_n^* g + \delta_S)(x_n), \langle q_n^*, q_n - g(x_n) \rangle = 0 \, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n^* \to \overline{x}, p_n \to \overline{x}, q_n \to 0 \, (n \to +\infty) \, \textit{and} \\ f(p_n) - \langle u_n^*, p_n - x_n \rangle + \langle F(\overline{x}), p_n - \overline{x} \rangle + \langle q_n^*, q_n \rangle - f(\overline{x}) \to 0 \, (n \to +\infty). \end{cases}$$

3.4.2 The case h is continuous

We consider X a reflexive Banach space, Y a Banach spaces, and $K \subseteq Y$ a nonempty closed convex cone inducing a partial ordering on Y, $S \subseteq X$ closed and convex, $f : X \to \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function and $g : X \to Y^\bullet$ is a continuous and K-convex vector-valued function and $S \cap g^{-1}(-K) \cap \text{dom } f \neq \emptyset$.

The next theorem is particular case of [8, Theorem 3.6] for $(VI)^C$.

Theorem 9 *The element* $\overline{x} \in S \cap g^{-1}(-K) \cap dom\ f$ *solves* $(VI)^C$ *if and only if*

$$\begin{cases} \exists (x_n, w_n, t_n) \in dom\ f \times S \times (-K), \exists (u_n^*, v_n^*, w_n^*, q_n^*) \in X^* \times X^* \times X^* \times K^*, \\ u_n^* - F(\overline{x}) \in \partial f(x_n), v_n^* \in \partial (q_n^* g)(x_n), w_n^* \in N_S(w_n), \langle q_n^*, t_n \rangle = 0\, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* + w_n^* \to 0, w_n \to \overline{x}, x_n \to \overline{x}, t_n \to g(\overline{x})\, (n \to +\infty), \\ f(x_n) + \langle q_n^*, q(x_n) \rangle - \langle w_n^*, w_n - x_n \rangle + \langle F(\overline{x}), x_n - \overline{x} \rangle - f(\overline{x}) \to 0\, (n \to +\infty). \end{cases}$$

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Liana Cioban
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
1 Mihail Kogălniceanu, 400084 Cluj-Napoca, Romania
e-mail: liana.cioban@ubbcluj.ro

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Applications of the improved Fejer's sum [1]

Eugen Constantinescu, Adrian Branga

Abstract

In this paper, using the quadrature formulas of Bouzitat type, we present an improvement of Fejer's inequality and also some applications.

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1 The case of Legendre polynomials

Let denote by P_n the Legendre polynomials and by T_n the Chebyshev polynomials of the first kind. We have

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n, \quad T_n(x) = \frac{1}{2^n}\cos(n \arccos x).$$

We know that (see [7]) $|P_n(x)| \leq 1, \quad x \in (-1,1)$ and $|P_n(\pm 1)| = 1$.

Let $(P_n)_{n=0}^{\infty}$ be the sequence of Legendre polynomials, ie

(1) $$P_n(x) = {}_2F_1\left(-n; n+1; 1; \frac{1-x}{2}\right)$$

and we consider the sum

(2) $$S_n(P;x) \equiv \mathcal{F}_n(0;x) = \sum_{k=0}^{n} P_k(x).$$

It is well known that

(3) $$S_n(P;x) \geq 0, (\forall) x \in [-1,1].$$

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Our goal is to improve the inequality (3). The following identity is known as so-called "correction formula":

$$(1+x)^{\beta+\mu} R_n^{(\alpha-\mu,\beta+\mu)}(x) = C_n \int_{-1}^{1} R_n^{(\alpha,\beta)}(t)(1+t)^{\beta}(1-t)^{\mu-1}\,\mathrm{dt}, \tag{4}$$

where it is assumed that $\mu > 0$, $\beta > -1$, $-1 < x \le 1$. The constant factor is specified by

$$C_n = C_n(\alpha,\beta,\mu) = \frac{\binom{n+\beta+\mu}{n}\binom{n+\alpha}{n}}{\binom{n+\alpha-\mu}{n}\binom{n+\beta}{n} B(\mu,\beta+1)},$$

where $B(\cdot,\cdot)$ is the Beta function.

Let consider the following integer numbers

$$m = \left[\frac{n}{2}\right],\ s = n - 2m,\ d = \left[\frac{n+1}{2}\right]. \tag{5}$$

We use the following statement:

Theorem 1 *Suppose that $H \in \Pi_n$ and for $\alpha > -1$ we denote*

$$k_n(\alpha) = \frac{1}{2(n+2\alpha+2)\binom{\left[\frac{n+1}{2}\right]+2\alpha+1}{\left[\frac{n}{2}\right]}},$$

$$A_n(\alpha) = 4 + 4\alpha + s(n-2\alpha-2),$$

$$B_n(\alpha) = s(n+2\alpha+2).$$

If $H \ge 0$ on $[-1,1]$ and $dw_\alpha(t) = \dfrac{(1-t^2)^\alpha}{2^{2\alpha+1}B(\alpha+1,\alpha+1)}dt$ then

$$\int_{-1}^{1} H(t)dw_\alpha(t) \ge k_n(\alpha)(A_n(\alpha)\cdot H(1) + B_n(\alpha)\cdot H(1)) \tag{6}$$

and

$$\int_{-1}^{1} H(t)dw_\alpha(t) \ge k_n(\alpha)(A_n(\alpha)\cdot H(-1) + B_n(\alpha)\cdot H(1)). \tag{7}$$

In addition, considering the following polynomials:

$$h^*(t) = (1-t)^s (R_{[\frac{n}{2}]}^{(\alpha+1,\alpha+s)}(t))^2 \text{ in the inequality (6)} \tag{8}$$

and

$$h_*(t) = (1+t)^s (R_{[\frac{n}{2}]}^{(\alpha+s,\alpha+1)}(t))^2 \text{ in the inequality (7),} \tag{9}$$

the equalities are achieved.

Proof. It is sufficiently to use the quadrature formulas of Bouzitat type: we choose $\alpha = \beta,\ u(x) = H(-x),\ d \in \{0, 1\}$.

We observe that the mentioned inequalities are optimal. Indeed, if we consider the polynomials (8) in the inequality (6), respectively (9) in the inequality (7), then we have equality.

Considering $\alpha = -\dfrac{1}{2}$ in Theorem 1 we find:

Corollary 1 *Let $H \in \Pi_n$ be a non-negative polynomial on $[-1, 1]$. Then*

$$\frac{1}{\pi}\int_{-1}^{1} H(t)\frac{dt}{\sqrt{1-t^2}} \geq \frac{1+s}{2(n+1)}(H(-1)+H(1)) + \frac{1-s}{2(n+1)}(H(-1)-H(1)),$$

where $s = n - 2\left[\dfrac{n}{2}\right]$.

The main result is the following:

Theorem 2 *Let $S_n(P; x)$ defined by relation (2). Then for all $n \in \{0, 1, \ldots\}$ and $x \in [-1, 1]$ we have:*

$$S_n(P;x) \geq \frac{1}{n+1}\max\left\{1-s;\ \frac{1-T_{n+1}(x)}{1-x}\right\}, \tag{10}$$

where $s = n - 2\left[\dfrac{n}{2}\right]$.

Proof. First we observe that $S_n(P; -1) = 1 - s$ and on the other hand the right member from (10) has in the point $x = -1$ the value $\dfrac{1-x}{n+1}$. Further we consider $x \in [-1, 1]$.

We choose in relation (4) $\alpha = \dfrac{1}{2},\ \beta = -\dfrac{1}{2},\ \mu = \dfrac{1}{2}$. Then $C_n\left(\dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}\right) = \dfrac{2n+1}{\pi}$ and from (4) we find the identity:

$$P_k(x) = \frac{1}{\pi}\int_{-1}^{x} \frac{T_k(t) - T_{k+1}(t)}{1-t} \cdot \frac{\mathrm{dt}}{\sqrt{(1+t)(x-t)}}. \tag{11}$$

Summing the equality (11) by $k = 0, ..., n$, we find:

$$S_n(P;x) = \frac{1}{\pi}\int_{-1}^{x} \frac{1 - T_{n+1}(t)}{1-t} \cdot \frac{\mathrm{dt}}{\sqrt{(1+t)(x-t)}},$$

which can be expressed as

$$S_n(P;x) = \frac{1}{\pi}\int_{-1}^{1} \frac{1 - T_{n+1}(\phi(x,t))}{1-\phi(x,t)} \cdot \frac{\mathrm{dt}}{\sqrt{1-t^2}}, \tag{12}$$

where $\phi(x,t)$, $x \in [-1,1]$, $t \in [-1,1]$, is defined by

$$\phi(x,t) = \frac{(1+x)t-(1-x)}{2}.$$

Further we consider in Corollary 1 the polynomial:

$$H_x(t) = \frac{1-T_{n+1}(\phi(x,t))}{1-\phi(x,t)}.$$

Because $H_x(-1) = \dfrac{1+(-1)^n}{2} = 1-s$ and $H_x(1) = \dfrac{1-T_{n+1}(x)}{1-x}$, we find from relation (12) the inequality (10).

Remark 1 *Another significant sum both in the theory of Fourier series and in the approximation theory is the Fejér polynomial*

$$(C_n,1)(x) := \sum_{k=0}^{n} \gamma_k (1-\frac{k}{n+1}) T_k(x),$$

where $\gamma_0 = \dfrac{1}{\pi}$, $\gamma_k = \dfrac{2}{n+1}$, $k \geq 1$. *This sum is also non-negative on* $[-1,1]$. *A simple proof of this issue is based on the identity*

$$(C_n,1)(x) = \frac{1-T_{n+1}(x)}{\pi(n+1)(1-x)}. \tag{13}$$

Taking into account the relations (13) and (10) we conclude with:

Corollary 2 *The inequalities*

$$\frac{1}{\pi} S_n(P;x) \geq (C_n,1)(x),\ n = 0,1,\ldots,$$

are verified on $[-1,1]$.

2 The case of ultraspherical polynomials

In this section we will extend the above results. We consider

$$\mathcal{F}_n(\alpha;x) = \sum_{k=0}^{n} R_k^{(\alpha,\alpha)}(x).$$

It is known that $\mathcal{F}_n(\alpha;x) \geq 0,\ x \in [-1,1]$ (for example see the papers of R. Askey [1], [2], [3] and R. Askey, G. Gasper [4], [5]). If we denote

$$dm_\alpha(t) = \frac{(1+t)^{\alpha-\frac{1}{2}}(1-t)^{-\frac{1}{2}}}{2^\alpha B\left(\frac{1}{2};\alpha+\frac{1}{2}\right)} \ \mathrm{dt}\ ,$$

then the following equalities are verified:

Lemma 1 *Suppose that* $\alpha > -\frac{1}{2},\ x \in (-1,1]$. *Then*

$$(14)\qquad R_k^{(\alpha,\alpha)}(x) = \int_{-1}^{1} \frac{R_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\phi(x,t)) - R_{k+1}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\phi(x,t))}{1-\phi(x,t)}\ dm_\alpha(t)$$

and

$$(15)\qquad \mathcal{F}_n(\alpha;x) = \int_{-1}^{1} \frac{1 - R_{n+1}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\phi(x,t))}{1-\phi(x,t)}\ dm_\alpha(t),$$

where $\phi(x,t),\ x \in (-1,1],\ t \in [-1,1]$, *is defined by*

$$\phi(x,t) = \frac{(1+x)t-(1-x)}{2}.$$

Proof. It is enough to prove the relation (14). This identity follows from (4) and using the equality

$$R_k^{(a+1,b)}(\phi) = \frac{2(a+1)}{2k+a+b+1} \cdot \frac{R_k^{(a,b)}(\phi) - R_{k+1}^{(a,b)}(\phi)}{1-\phi}.$$

We note that according to the inequality (see [7])

$$|R_n^{(a,a)}(x)| \le 1,\ a \ge -\frac{1}{2},\ x \in (-1,1],$$

the right member from (15) is positive for $\alpha \ge 0,\ x \in (-1,1]$.

Remark 2 *In a special case, the equality (15) was found by A. Lupaş [6].*

Theorem 3 *If*

$$\mathcal{F}_n(\alpha;x) = \sum_{k=0}^{n} R_k^{(a,a)}(x),$$

then

$$(16)\qquad \mathcal{F}_n(\alpha;x) \ge \mathcal{I}_n \cdot \frac{1 - R_{n+1}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)}{1-x},\ (\forall)x \in (-1,1),$$

where

$$\mathcal{I}_n := \frac{m!(\frac{1}{2})_d}{(\alpha+\frac{3}{2})_m(\alpha+2)_d},\ d = \left[\frac{n+1}{2}\right],\ m = n-d.$$

Proof. Using the identity (15) and considering $(\alpha,\ \beta) \to \left(\alpha - \frac{1}{2}, -\frac{1}{2}\right)$ and

$$P(x) = \frac{1 - R_{n+1}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\phi(x,t))}{1-\phi(x,t)}$$

it obtains the inequality (16).

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Eugen Constantinescu
"Lucian Blaga" University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
Str. Dr. I. Raţiu, No. 5-7, 550012 - Sibiu, Romania
e-mail: egnconst68@yahoo.com

Adrian Branga
"Lucian Blaga" University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
Str. Dr. I. Raţiu, No. 5-7, 550012 - Sibiu, Romania
e-mail: adrian_branga@yahoo.com

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On the composition and decomposition of positive linear operators IV: Favard-Bernstein operators revisited [1]

Heiner Gonska, Gancho Tachev

Abstract

Following a 1939 article of Favard we consider the composition of classical Bernstein operators and piecewise linear interpolation at mutually distinct knots in $[0, 1]$, not necessarily equidistant. We prove direct theorems in terms of the classical and the Ditzian-Totik modulus of second order.

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Key words and phrases: Favard-Bernstein operator, Bernstein polynomials for arbitrary points of interpolation, piecewise linear interpolation, Bernstein operator, second order modulus of smoothness, Ditzian-Totik modulus, positive linear operator, degree of approximation.

1 Introduction

In the sequel we will pursue a 1939 idea of the famous Jean Favard and deal with - using misleading terminology of G.G. Lorentz - "Bernstein polynomials for arbitrary points of interpolation".

Let $\Delta_n : 0 = x_0 < x_1 < \ldots < x_n = 1$, $n \geq 2$, be a partition of the interval $[0, 1]$ with mesh gauge $\|\Delta_n\| := \max_{0\leq k\leq n-1}(x_{k+1} - x_k)$, and let S_{Δ_n} be the continuous, piecewise linear interpolant for $f \in C[0, 1]$ at $x_i, i = 0, \ldots, n$. Then S_{Δ_n} can be written as

$$S_{\Delta_n}(f, x) = \sum_{k=0}^{n} f(x_k) \cdot N_{k,1}(x), \tag{1}$$

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where $N_{k,1}$ is the normalized B-spline of degree 1 with respect to the sequence x_{k-1}, x_k, x_{k+1}, where we have put $x_{-1} = 0, x_{n+1} = 1$. We avoid the trivial case $n = 1$ because then $S_{\Delta_n} f$ is just the linear interpolant at 0 and 1.

If B_n is the classical Bernstein operator we define - following Favard - the composite mapping

$$C_{\Delta_n} := B_n \circ S_{\Delta_n}. \tag{2}$$

C_{Δ_n} is also a positive and linear operator, reproduces linear functions and is given semi-explicitly by

$$C_{\Delta_n}(f,x) = \sum_{k=0}^{n} f(x_k) \cdot B_n(N_{k,1}, x). \tag{3}$$

One might name C_{Δ_n} a Bernstein-type operator defined for arbitrary different knots $x_k \in [0,1]$. In the case of equidistant knots $x_k = \frac{k}{n}$, C_{Δ_n} reduces to B_n.

The idea to construct the composition operator C_{Δ_n} can be traced to a 1939 paper by Favard [4] (see also his 1950 article [5]) and became known to the authors through the 1953 edition of Lorentz' book [12] on Bernstein polynomials. Thus it is justified to call them Favard-Bernstein operators.

The operator composition $B_n \circ S_{\Delta_n}$ is of interest also because both operators involved constitute certain "extreme" cases of Schoenberg's variation-diminishing splines on $[0,1]$. B_n is a (polynomial) Schoenberg spline operator with no knots in the open interval $(0,1)$, while S_{Δ_n} has minimal piecewise degree 1. For details see, e.g., Beutel et al., in [1], [2], [3].

In this note we show direct pointwise estimates for approximation by C_{Δ_n} in terms of second order moduli of continuity $\omega_2(f,\delta)$ and second order Ditzian-Totik moduli of smoothness $\omega_2^{\varphi}(f,\delta)$. To do this we apply the following two results:

Theorem 1 *(Păltănea - see Th. 3.1, Cor. 3.1 in [13]) If $L : C[a,b] \to C[a,b]$ is a positive linear operator reproducing linear functions, then for $f \in C[a,b], x \in [a,b]$ and each $0 < h \le \frac{b-a}{2}$, the following holds true:*

$$|(Lf)(x) - f(x)| \le \left[1 + \frac{1}{2h^2} \cdot (L(e_1 - x)^2)(x)\right] \cdot \omega_2(f,h). \tag{4}$$

The second general estimate we will use is Theorem 2.5.1 on p. 64 in the book of Păltănea [14]; in slightly weaker form it was given by Gavrea et al. as Th. 15 in [7].

Theorem 2 *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator reproducing linear functions. Then*

$$|L(f,x) - f(x)| \le \left[1 + \frac{3}{2} \cdot \frac{1}{h^2} \cdot \frac{L((e_1 - xe_0)^2, x)}{x(1-x)}\right] \cdot \omega_2^{\varphi}(f,h) \tag{5}$$

for $x \in (0,1), \quad 0 \le h \le \frac{1}{2}$.

In both Theorems 1 and 2 the crucial step in concrete situations is a 'good' estimate (or preferably, an exact representation) for the second moment of the composite operator C_{Δ_n}, i.e., for the quantity $C_{\Delta_n}((e_1-x)^2,x)$.

2 Estimates for the second moments of C_{Δ_n}

Recall a result of D. Kacsó [11] (see also [9]) where it was shown that for two linear operators P,Q with $Qe_i=e_i$, $i=0,1$, one has

$$(PQ)((e_1-x)^2,x)=P^u(Q((e_1-u)^2,u);x)+P((e_1-x)^2;x).$$

In our case this means

$$C_{\Delta_n}((e_1-x)^2;x)=B_n(S_{\Delta_n}((e_1-u)^2;u);x)+\frac{x(1-x)}{n}. \tag{6}$$

We see from (6) that it is not possible to get for the second moment of C_{Δ_n} a quantity smaller than the second moment of B_n itself, but now we have the opportunity to choose an arbitrary set of knots x_k. We denote by β_k that knot x_μ from the partition Δ_n for which

$$\beta_k=x_\mu\le\frac{k}{n}<x_{\mu+1}\quad ,0\le k\le n,$$

and define $\overline{\beta}_k:=x_{\mu+1}$. Moreover, let $m_n:=\max_{0\le k\le n}(\overline{\beta}_k-\frac{k}{n})(\frac{k}{n}-\beta_k)$.

Temporarily we will write $\beta_{n,\nu}$ instead of β_ν, etc., below in order to stress the dependence on n. C_{Δ_n} is thus given explicitly by

$$C_{\Delta_n}(f,x)=\sum_{\nu=0}^{n}\frac{1}{\overline{\beta}_{n,\nu}-\beta_{n,\nu}}\{f(\overline{\beta}_{n,\nu})(\frac{\nu}{n}-\beta_{n,\nu})+f(\beta_{n,\nu})(\overline{\beta}_{n,\nu}-\frac{\nu}{n})\}\binom{n}{\nu}x^\nu(1-x)^{n-\nu},$$

whence

$$\begin{aligned}C_{\Delta_n}(e_2,x)&=\sum_{\nu=0}^{n}\frac{1}{\overline{\beta}_{n,\nu}-\beta_{n,\nu}}\{\overline{\beta}^2_{n,\nu}(\frac{\nu}{n}-\beta_{n,\nu})+\beta^2_{n,\nu}(\overline{\beta}_{n,\nu}(\overline{\beta}_{n,\nu}-\frac{\nu}{n})\}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\\&=\sum_{\nu=0}^{n}\frac{(\overline{\beta}^2_{n,\nu}\frac{\nu}{n}-\overline{\beta}^2_{n,\nu}\beta_{n,\nu}+\beta^2_{n,\nu}\overline{\beta}_{n,\nu}-\beta^2_{n,\nu}\frac{\nu}{n}}{\overline{\beta}_{n,\nu}-\beta_{n,\nu}}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\\&=\sum_{\nu=0}^{n}\frac{(\overline{\beta}_{n,\nu}\frac{\nu}{n}-\overline{\beta}_{n,\nu}\beta_{n,\nu}-\beta_{n,\nu}\frac{\nu}{n})(\overline{\beta}_{n,\nu}-\beta_{n,\nu})}{\overline{\beta}_{n,\nu}-\beta_{n,\nu}}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}.\end{aligned}$$

Thus

$$\begin{aligned}
C_{\Delta_n}(e_2,x)-x^2 &= C_{\Delta_n}(e_2,x)-B_n(e_2,x)+B_n(e_2,x)-x^2\\
&= \sum_{\nu=0}^{n}[\overline{\beta}_{n,\nu}(\frac{\nu}{n}-\beta_{n,\nu})+\beta_{n,\nu}\frac{\nu}{n}]\binom{n}{\nu}x^\nu(1-x)^{n-\nu}\\
&\qquad -\sum_{\nu=0}^{n}\left(\frac{\nu}{n}\right)^2\binom{n}{\nu}x^\nu(1-x)^{n-\nu}+\frac{x(1-x)}{n}\\
&= \sum_{\nu=0}^{n}[(\overline{\beta}_{n,\nu}-\frac{\nu}{n})(\frac{\nu}{n}-\beta_{n,\nu})]\binom{n}{\nu}x^\nu(1-x)^{n-\nu}+\frac{x(1-x)}{n}\\
&\le \max_{0\le\nu\le n}\{(\overline{\beta}_{n,\nu}-\frac{\nu}{n})(\frac{\nu}{n}-\beta_{n,\nu})\}+\frac{x(1-x)}{n}\\
&= m_n+\frac{x(1-x)}{n}\le\frac{1}{2}.
\end{aligned}$$

We summarize these observations in

Lemma 1 *For the second moments of C_{Δ_n} the following inequality holds:*

$$C_{\Delta_n}((e_1-x)^2;x)=C_{\Delta_n}(e_2;x)-x^2\le m_n+\frac{x(1-x)}{n}\le\frac{1}{2}$$

for $n\ge 1$ and $x\in[0,1]$.

Theorem 1 immediately implies

Theorem 3 *For each $f\in C[0,1]$ and $x\in[0,1]$ we have*

$$|C_{\Delta_n}(f,x)-f(x)|\le\frac{3}{2}\cdot\omega_2\left(f,\sqrt{m_n+\frac{x(1-x)}{n}}\right). \tag{7}$$

Proof. Put $h^2=m_n+\frac{x(1-x)}{n}$ in Theorem 1.

If we assume $m_n\to 0$ for $n\to\infty$, then the inequality of Theorem 3 implies uniform convergence for each $f\in C[0,1]$. This is one of the results of Favard (see [4], p. 104 f.). Theorem 3 was given in slightly weaker form by the first author in [8].

Our main goal is to replace the term m_n in (7) by a term $C(x(1-x))^\gamma\cdot\epsilon_n^\beta$ for some $\gamma>0,\beta>0$ and $C>0$ independent of n,x and $\epsilon_n\to 0$ when $n\to\infty$. Thus we would have (7) in a pointwise form expressing the fact that $C_{\Delta_n}f$ interpolates f at 0 and 1. The latter property of C_{Δ_n} is clear from interpolation at the endpoints by both B_nf and $S_{\Delta_n}f$.

Recalling that

$$\|\Delta_n\|=\max_{0\le k\le n-1}(x_{k+1}-x_k), \tag{8}$$

it easily follows that

(9) $$m_n \leq \frac{\|\Delta_n\|^2}{4}.$$

To have uniform convergence for each $f \in C[0,1]$ by the sequence $(C_n f)$ it is natural to suppose

(10) $$\lim_{n\to\infty} \|\Delta_n\| = 0.$$

In fact, this condition is necessary and sufficient. Moreover, it is known that

$$S_{\Delta_n}(e_2; x) - x^2 = (x - x_k)(x_{k+1} - x) \ \text{ for } \ x \in [x_k, x_{k+1}].$$

For $x \in [0,1]$ we define

(11) $$t_{\Delta_n}(x) = (x - x_k)(x_{k+1} - x) \ \text{ if } \ x \in [x_k, x_{k+1}], 0 \leq k \leq n-1.$$

Therefore

(12) $$C_{\Delta_n}((e_1 - x)^2; x) = B_n(t_{\Delta_n}(\cdot); x) + B_n((e_1 - x)^2; x).$$

If Δ_n is such that

(13) $$t_{\Delta_n}(u) \leq A \cdot u(1-u) \cdot \epsilon_n^2 + B \cdot \epsilon_n^4$$

(A and B should not depend on n), then

(14) $$B_n(t_{\Delta_n})(\cdot); x) \leq A \cdot \epsilon_n^2 \cdot B_n(u(1-u); x) + B \cdot \epsilon_n^4.$$

Therefore from (12)-(14) we get

(15) $$C_{\Delta_n}((e_1 - x)^2; x) \leq \left[A\epsilon_n^2 + \frac{1}{n}\right] \cdot x(1-x) + B \cdot \epsilon_n^4.$$

Now using Theorem 1 we obtain a direct pointwise estimate similar to (7).

The upper bound for the second moment of S_{Δ_n} given in (15) is not quite satisfactory due to the second summand $B\epsilon_n^4$ which does not take into account the position of x. Motivated by the last fact, in 1999 the first author raised the following question:

Suppose that $S_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operator satisfying

(i) $S_n e_i = e_i, i = 0, 1,$

(i') $S_n(f; 0) = f(0)$ and $S_n(f; 1) = f(1)$ for all $f \in C[0,1]$ (this follows from (i)),

(ii) $S_n((e_1 - x)^2; x) \leq Ax(1-x)\epsilon_n^2 + B\epsilon_n^4$, where $\epsilon_n \to 0$ and A and B are independent of x and n.

The question was whether it is possible to derive an inequality of the form

$$S_n((e_1 - x)^2; x) \leq C(x(1-x))^\gamma \cdot \epsilon_n^\beta, \tag{16}$$

where C, γ and β are suitably chosen?

The affirmative answer to this question was given by Tachev in [15] (see Th. 4.1 - p. 951). There it was proved that (16) holds with

$$C = \max(2A, \sqrt{2B}), \quad \beta = 2, \quad \gamma = \frac{1}{2}. \tag{17}$$

Consequently we have from (16) - (17) that

$$S_n((e_1 - x)^2; x) \leq C\sqrt{x(1-x)} \cdot \epsilon_n^2. \tag{18}$$

Using that, if g is a concave function, then $B_n(g, x) \leq g(x)$, we obtain

$$B_n^u(S_{\Delta_n}((e_1 - u)^2; u); x) \leq C\sqrt{x(1-x)} \cdot \epsilon_n^2. \tag{19}$$

Now the estimates (6) and (19) imply

$$\begin{aligned} C_{\Delta_n}((e_1 - x)^2; x) &\leq C\sqrt{x(1-x)} \cdot \epsilon_n^2 + \frac{x(1-x)}{n} && (20)\\ &\leq \sqrt{x(1-x)} \cdot \left[C \cdot \epsilon_n^2 + \frac{1}{2n}\right] =: \sqrt{x(1-x)} \cdot d_n^2. && (21) \end{aligned}$$

If we put

$$h^2 := \sqrt{x(1-x)} \cdot d_n^2 \tag{22}$$

in (4), also assuming that $h \leq \frac{1}{2}$, we arrive at the proof of

Theorem 4 *Let $\Delta_n : 0 = x_0 < x_1 < \ldots < x_n = 1$ be a partition of the interval $[0, 1]$ such that there exist positive numbers A, B, ϵ_n satisfying (ii) for the piecewise linear interpolant S_{Δ_n}. Then for each $f \in C[0, 1]$ we have*

$$|C_{\Delta_n}(f, x) - f(x)| \leq \frac{3}{2} \cdot \omega_2(f, h), \tag{23}$$

where h is defined in (22).

Remark 1 *The estimate (22), although in pointwise form, is not quite satisfactory because we have $\sqrt[4]{x(1-x)}$ in the argument of $\omega_2(f, h)$, while in the case of $B_n f$ (i.e., when C_{Δ_n} reduces to B_n) we have only $\sqrt{x(1-x)}$ in the argument of ω_2.*

Motivated by Remark 1 we are interested to obtain upper bounds in (23) with $x(1-x)$ instead of $\sqrt{x(1-x)}$. In many cases this is possible. We only list some of them:

- In Gavrea et al. (see [6] - Theorem 12) the knots x_k are defined in the following manner

$$x_k = \sin^2 \theta_k, \qquad k = 0, \ldots, n, \text{ where}$$
$$\theta_{k+1} - \theta_k \le \tfrac{c}{n}, \quad k = 0, \ldots, n-1.$$

  It was proved for this case that

$$S_{\Delta_n}((e_1 - x)^2; x) \le \pi^2 \cdot \frac{x(1-x)}{n^2}, \tag{24}$$

  so the condition (ii) holds with $B = 0, A = \pi^2$ and $\epsilon_n = \frac{1}{n}$. From (6) we get

$$C_{\Delta_n}((e_1 - x)^2; x) \le \frac{x(1-x)}{n}\left[1 + \frac{\pi^2}{n}\right]. \tag{25}$$

- Tachev's Example B (see [15] - p. 949) deals with the following situation. We have now $n = 2m + 1$ knots given as follows:

$$x_k = \frac{y_k + 1}{2}, \text{ with } y_j = 1 - \frac{(m-j)^2}{m^2}, j = 0, \ldots, m$$

  $y_{-j} := -y_j, j = 1, \ldots, m$. Then

$$S_{\Delta_n}((e_1 - x)^2; x) \le 64 \cdot \frac{x(1-x)}{n^2}. \tag{26}$$

- For the case of equidistant knots $x_k = \frac{k}{n}, 0 \le k \le n$, it is possible to prove

$$S_{\Delta_n}((e_1 - x)^2; x) \le \frac{2x(1-x)}{n}, x \in [0, 1]. \tag{27}$$

  This was shown in [1], see Theorem 2 there.

The last three examples motivate the following:

Proposition 1 *Let $\Delta_n : 0 = x_0 < x_1 < \ldots < x_n = 1$ be a partition of $[0, 1]$ and let S_{Δ_n} be the piecewise linear interpolant for $f \in C[0, 1]$ at x_i, $i = 0, \ldots, n$. Then*

$$S_{\Delta_n}((e_1 - x)^2; x) \le \|\Delta_n\| \cdot x(1-x) \text{ for all } x \in [0, 1]. \tag{28}$$

Proof. We know that

$$S_{\Delta_n}((e_1 - x)^2; x) = (x - x_k)(x_{k+1} - x) \text{ for } x \in [x_k, x_{k+1}], 0 \le k \le n-1.$$

To verify (28) we need to show

$$\frac{(x - x_k)(x_{k+1} - x)}{x(1-x)} \le (x_{k+1} - x_k) \le \|\Delta_n\| \text{ for all } x \in [x_k, x_{k+1}]. \tag{29}$$

The last is equivalent to

$$-x^2 + (x_{k+1} + x_k)x - x_k \cdot x_{k+1} \leq (x_{k+1} - x_k)x - (x_{k+1} - x_k)x^2$$

or to

$$g(x) := \underbrace{(1 + x_k - x_{k+1})}_{:=a} x^2 - 2x_k \cdot x + x_k \cdot x_{k+1} \geq 0.$$

Here $a = 1 - x_{k+1} + x_k > 0$ since $n \geq 2$. The discriminant of the last term is

$$\begin{aligned} D &:= 4\{x_k^2 - x_k x_{k+1}(1 + x_k - x_{k+1})\} \\ &= 4(1 - x_{k+1})x_k(x_k - x_{k+1}) \leq 0. \end{aligned}$$

Since $a > 0$, this means that $g(x) \geq 0, x \in [0,1]$, and hence (29) holds.

3 Main results

Taking into account (28) and Theorems 1 and 2 we obtain the following two direct theorems for approximation by Favard-Bernstein operators C_{Δ_n}:

Theorem 5 *For each $f \in C[0,1]$ and each partition Δ_n we have for $n \geq 2$*

$$|C_{\Delta_n}(f,x) - f(x)| \leq \frac{3}{2} \cdot \omega_2(f,h), x \in [0,1]. \tag{30}$$

Here $h = \sqrt{x(1-x)} \cdot \sqrt{\frac{1}{n} + \|\Delta_n\|} \leq \frac{1}{2}$.

In terms of the Ditzian-Totik modulus we arrive at the following assertion.

Theorem 6 *If $\|\Delta_n\| \leq \frac{1}{2}$, then we have for $n \geq 2$*

$$\|C_{\Delta_n}(f) - f\|_\infty \leq 3 \cdot \omega_2^\varphi(f, \sqrt{\|\Delta_n\|}), \tag{31}$$

where $\|\cdot\|_\infty$ denotes the max norm on $[0,1]$.

4 Concluding remarks

1. In the 1950 article [5] Favard mentioned the facts that C_{Δ_n} preserves monotonicity and convexity (concavity), a simple consequence of the corresponding behavior of S_{Δ_n} and B_n. This raises the natural question for simultaneous approximation of (at least) the first and second order derivatives provided f is smooth enough. To have an idea of how to approach this problem see the recent paper [10].
2. Intuitively the term "Bernstein polynomials for arbitrary points of interpolation" suggests an operator D_n of the form

$$D_n(f,x) = \sum_{k=0}^{n} f(x_k) \cdot \binom{n}{k} x^k (1-x)^{n-k}. \tag{32}$$

However, D_n reproduces linear functions if and only if $x_k = \frac{k}{n}, 0 \leq k \leq n$. It is nonetheless possible to prove a meaningful convergence assertion provided the knots x_k become dense in $[0,1]$.

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Heiner Gonska
University of Duisburg-Essen
Faculty for Mathematics
47048 Duisburg, Germany
e-mail: heiner.gonska@uni-due.de

Gancho Tachev
University for Architecture
Department of Mathematics
1046 Sofia, Bulgaria
e-mail: gtt_fte@uacg.bg

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Bounds to a sequence which use Euler's constant [1]

Nicuşor Minculete, Petrică Dicu

Abstract

The aim of this paper is to improve an inequality which characterizes the order of convergence of the sequence $E_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$ to the limit e.

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1 Introduction

In 1690, while studying a problem related to compound interest, Jacob Bernoulli (1654-1705) discovered the sequence $e_n = \left(1 + \frac{1}{n}\right)^n$ and its limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \cong 2.71828... \tag{1}$$

Leonhard Euler (1707-1783) demonstrated in 1737 that the number e is irrational and Charles Hermite (1822-1901) demonstrated in 1873 that the number e is transcendent, which means that the number e is not the root of any non-zero polynomial with rational coefficients.

Euler established a more general formula, namely

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \tag{2}$$

for every real number x.

The properties of the sequence $e_n = \left(1 + \frac{1}{n}\right)^n$ can be found in many works, including [2], [3] and [5].

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Let the sequence $E_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$, where $n \geq 1$.

In 1728, Daniel Bernoulli (1700-1782) demonstrated the following relation:

$$e = \lim_{n\to\infty} E_n = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{3}$$

In [6] is mentioned an inequality related to "the speed" of convergence of the sequence $(E_n)_{n\geq 1}$ to the number e,

$$\frac{1}{n!(n+1)} < e - E_n < \frac{1}{n!n}, \ (\forall)\ n \geq 1. \tag{4}$$

Hence, it results the limit

$$\lim_{n\to\infty} (n+1)!(e - E_n) = 1. \tag{5}$$

In [1], in order to study of convergence of E_n, the following inequality is established:

$$\frac{n+1}{n!(n^2+n+1)} < e - E_n < \frac{1}{n!n}, \ (\forall)\ n \geq 1. \tag{6}$$

2 Main results

Next, we will improve the inequality (4) in two ways which will help us characterize the sequence E_n.

Theorem 1 *For every $n \geq 1$, the following inequality*

$$\frac{1}{n!(n+1)} < \frac{n+3}{(n+2)!} < e - E_n < \frac{(n+5)(n+1)}{(n+3)!} + \frac{2e}{(n+3)!} < \frac{1}{n!n}, \tag{7}$$

holds.

Proof. Let $I_n = \int_0^1 x^n e^{-x} dx$, with $n \geq 0$ be a sequence of integrals, which implies the recurrence relation

$$I_n = -\frac{1}{e} + nI_{n-1}, \tag{8}$$

for every $n \geq 1$ and $I_0 = 1 - \frac{1}{e}$.

In [6] also appears the relation

$$I_n = \frac{n!}{e}(e - E_n). \tag{9}$$

We are studying the expression

$$I_{n+1} + I_n = \int_0^1 x^n e^{-x}(x+1)dx.$$

The function $f : [0,1] \to \mathbb{R}$, defined by $f(x) = e^{-x}(x+1)$ is decreasing on the interval $[0,1]$, it result that $f(x) \geq f(1) = \frac{2}{e}$, so

$$I_{n+1} + I_n \geq \frac{2}{e}\int_0^1 x^n dx = \frac{2}{e(n+1)}. \tag{10}$$

From relations (8), (9) and (10) we obtain the following relation:

$$I_{n+1}+I_n = -\frac{1}{e}+(n+1)I_n+I_n = -\frac{1}{e}+(n+2)I_n = -\frac{1}{e}+\frac{(n+2)n!}{e}(e-E_n) \geq \frac{2}{e(n+1)},$$

that is

$$\frac{n+3}{(n+2)!} < e - E_n,$$

and

$$\frac{1}{n!(n+1)} = \frac{n+2}{(n+2)!} < \frac{n+3}{(n+2)!} < e - E_n.$$

For the two part of the inequality we use an inequality that is easy to demonstrate, namely

$$e^x \geq 1 + x + \frac{x^2}{2}, \tag{11}$$

so

$$1 - \frac{e^{-x}x^2}{2} \geq e^{-x}(1+x),$$

which means that $\int_0^1 x^n \left(1 - \frac{e^{-x}x^2}{2}\right) dx \geq \int_0^1 x^n e^{-x}(1+x)dx = I_{n+1} + I_n$.

It results

$$\frac{1}{n+1} - \frac{1}{2}I_{n+2} \geq I_{n+1} + I_n,$$

which is equivalent to

$$\frac{1}{2}I_{n+2} + I_{n+1} + I_n \leq \frac{1}{n+1}. \tag{12}$$

But

$$I_{n+2} = -\frac{1}{e}+(n+2)I_{n+1} = -\frac{1}{e}+(n+2)\left[-\frac{1}{e} + (n+1)I_n\right] = -\frac{n+3}{e}+(n+2)(n+1)I_n,$$

so, relation (12) becomes

$$-\frac{n+3}{2e} + \frac{(n+2)(n+1)}{2}I_n - \frac{1}{e} + (n+1)I_n + I_n \leq \frac{1}{n+1},$$

and finally we have

$$I_n \leq \frac{2}{(n+1)(n+2)(n+3)} + \frac{n+5}{e(n+2)(n+3)}. \tag{13}$$

From relations (9) and (13), we deduce the following relation

$$\frac{n!}{e}(e - E_n) \leq \frac{2}{(n+1)(n+2)(n+3)} + \frac{n+5}{e(n+2)(n+3)},$$

that is

$$e - E_n < \frac{(n+5)(n+1)}{(n+3)!} + \frac{2e}{(n+3)!} < \frac{1}{n!n},$$

because

$$n[2e + (n+5)(n+1)] < (n+1)(n+2)(n+3),$$

which means that

$$2en + n^3 + 6n^2 + 5n < n^3 + 6n^2 + 11n + 6,$$

which is true, because $2e + 5 < 11$. The demonstration of the theorem is now concluded.

Remark 1 *Due to the fact that*

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + ...,$$

it results that

$$e - E_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ..., \tag{14}$$

and the demonstration of the left parts of the inequalities (4), (6), and (7) becomes obvious.

Further, we will apply a method to establish some classifications of some sequences, similar to the method in [4].

Theorem 2 *For every* $n \geq 1$ *and* $k \geq 2$*, the following inequality*

$$\begin{aligned} &\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... + \frac{1}{(n+k)!} < e - E_n \\ &< \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... + \frac{1}{(n+k-1)!} + \frac{1}{(n+k-1)!(n+k-1)}, \end{aligned} \tag{15}$$

holds.

Proof. Let $k \geq 2$ be a fixed natural number. We chose the sequence

$$x_n = e - E_n - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - ... - \frac{1}{(n+k-1)!},\ n \geq 1.$$

Using relation (4), we deduce the equality

$$x_n - \frac{1}{(n+k)!} = \frac{1}{(n+k+1)!} + \frac{1}{(n+k+2)!}... > 0,\ (\forall)\ n \geq 1,$$

which is the left part of the inequality (15).

Let

$$y_n = e - E_n - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - ... - \frac{1}{(n+k-1)!} - \frac{1}{(n+k+1)!(n+k-1)},\ n \geq 1$$

be a sequence. We will study the monotony of the sequence $(y_n)_{n\geq 1}$. Therefore, we have the relation

$$y_{n+1} - y_n = \frac{1}{(n+k)!(n+k)(n+k-1)} > 0,$$

from where results that the sequence $(y_n)_{n\geq 1}$ is increasing. But $\lim\limits_{n\to\infty} y_n = 0$, which means that $y_n < 0,\ (\forall)\ n \geq 1$, so we will obtain inequality

$$e - E_n < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... + \frac{1}{(n+k-1)!} + \frac{1}{(n+k-1)!(n+k-1)},\ n \geq 1,$$

which is the right part of inequality (15).

Remark 2 *With the aid of inequality (15) and using the Sandwich Property, we deduce the following limit:*

$$\lim_{n\to\infty} (n+k)! \left[e - E_n - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - ... - \frac{1}{(n+k-1)!}\right] = 1. \tag{16}$$

This limit can also be calculated in another way. Above we have defined the sequence: $x_n = e - E_n - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - ... - \frac{1}{(n+k-1)!},\ n \geq 1$. *For this sequence we will show that*

$$\lim_{n\to\infty} (n+k)! x_n = 1. \tag{17}$$

Now that we have the limit, we will use Cesaro-Stolz's lemma for the case $\frac{0}{0}$ *and we will obtain relation (17).*

Next, we will generalize the sequence E_n, thus: let the sequence $E_n(t) = 1 + \frac{t}{1!} + \frac{t^2}{2!} + ... + \frac{t^n}{n!}$, where $n \geq 1$ and $t \geq 0$.

Theorem 3 *For every $n \geq 1$ and $t \geq 0$, the following inequality*

$$\frac{(n+t+2)t^{n+1}}{(n+2)!} < e^t - E_n(t) < \frac{(n+t+4)(n+1)t^{n+1}}{(n+3)!} + \frac{2e^t t^{n+1}}{(n+3)!}, \tag{18}$$

holds.

Proof. We consider the sequence $I_n(t) = \int_0^t x^n e^{-x} dx$, with $n \geq 0$ and $t \geq 0$, which implies the recurrence relation

$$I_n(t) = -\frac{t^n}{e^t} + nI_{n-1}(t), \tag{19}$$

for every $n \geq 1$ and $I_0(t) = 1 - \frac{1}{e^t}$.

Multiplying relation (19) by $\frac{1}{n!}$ and by simple calculations we obtain the relation

$$I_n(t) = \frac{n!}{e^t}(e^t - E_n(t)). \tag{20}$$

Similar to the proof of theorem 1 we have the following inequality:

$$I_{n+1} + I_n \geq \frac{(t+1)t^{n+1}}{e^t(n+1)}. \tag{21}$$

Using relations (20) and (21) we deduce the following :

$$\frac{(n+t+2)t^{n+1}}{(n+2)!} < e^t - E_n(t).$$

For the two part of the inequality is easy to demonstrate that,

$$\frac{1}{2}I_{n+2}(t) + I_{n+1}(t) + I_n(t) \leq \frac{t^{n+1}}{n+1}. \tag{22}$$

Combining relations (20) and (22) we deduce the following inequality:

$$e^t - E_n(t) < \frac{(n+t+4)(n+1)t^{n+1}}{(n+3)!} + \frac{2e^t t^{n+1}}{(n+3)!}.$$

Thus, the proof of the theorem is complete.

Remark 3 *For $t = 1$ in relation (18), we obtain the important part of inequality (7).*

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Nicuşor Minculete
"Dimitrie Cantemir" University of Brasov
Str. Bisericii Române, no.107Braşov, România
e-mail: minculeten@yahoo.com

Petrică Dicu
"Lucian Blaga" University of Sibiu
Str. Dr. I. Raţiu, no.5-7Sibiu, România
e-mail: petrica.dicu@ulbsibiu.ro

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Complete monotonicity and estimates for Gamma Function. A Survey [1]

Cristinel Mortici

Abstract

This work is a survey of the authors previous results [New sharp inequalities for approximating the factorial function and the digamma function, Miskolc Math. Notes, 11, 2010, no. 1, 79-86] and [Sharp inequalities and complete monotonicity for the Wallis ratio, Bulletin of the Belgian Mathematical Society Simon Stevin, 17, 2010, 929-936]. The aim is to show how the theory of completely monotonic functions can be used to improve or establish new estimates for gamma and related functions.

2010 Mathematics Subject Classification: 30E15, 26D07, 41A60 .

Key words and phrases: Gamma function, Polygamma function, Complete monotonicity, Asymptotic series, Approximations.

1 Completely Monotonic Functions

We recall that a function $z : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic, if z has derivatives of all orders and satisfies,

$$(-1)^n z^{(n)}(x) \geq 0 \ , \quad \text{for every} \ \ n = 0, 1, 2, ... \quad \text{and} \quad x \in (0,\infty) .$$

In 1939 J. Dubourdieu proved that if a non-constant function z is completely monotonic, then

$$(-1)^n z^{(n)}(x) > 0 \ , \quad \text{for every} \ \ n = 0, 1, 2, ... \quad \text{and} \quad x \in (0,\infty) .$$

Theorem 1 (Hausdorff-Bernstein-Widder) *The function z is completely monotonic if and only if*

$$z(x) = \int_0^\infty e^{-tx} d\mu(t) ,$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$.

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For a function $t:(0,\infty)\to\mathbb{R}$, we say that t is *logaritmically completely monotonic* if $\ln t$ is completely monotonic.

Completely monotonic functions z are of great importance because they provide sharp bounds for z and their derivatives.

As $z^{(n)}$ is monotonic (its derivative $z^{(n+1)}$ keeps same sign), $z^{(n)}(x)$ lies between $z^{(n)}(a)$ and $z^{(n)}(b)$, for every $x\in[a,b]$.

Many functions involving the gamma function Γ, digamma function ψ,

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt \quad , \qquad \psi(x)=\frac{d}{dx}(\ln\Gamma(x))=\frac{\Gamma'(x)}{\Gamma(x)}, \qquad x>0,$$

are completely monotonic. Also for the derivatives ψ', ψ'', ..., known as polygamma functions.

In order to use Hausdorff-Bernstein-Widder theorem, we use the following representations, for $n=1,2,3,...$:

$$\psi^{(n)}(x)=(-1)^{n-1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}dt \tag{1}$$

and

$$\frac{1}{x^n}=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt. \tag{2}$$

2 An estimate of Robbins

In order to show our method, we discuss here a result of H. Robbins [3], who proved

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{1}{12n+1}}<n!<\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{1}{12n}}\ , \qquad n=1,2,3... \quad .$$

The standard procedure consists in considering the corresponding functions

$$x\longmapsto\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\cdot e^{\frac{1}{12x}}} \quad , \qquad x\longmapsto\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\cdot e^{\frac{1}{12x+1}}}$$

and study if they are logaritmically completely monotonic.

In this sense, define

$$f(x)=\ln\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x-\frac{1}{12x}$$

and

$$g(x)=\ln\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x-\frac{1}{12x+1}.$$

Such functions are not only decreasing or convex, but even completely monotonic.

More precisely, $-f$ and g are completely monotonic. For proofs and other details, see [1].

As $-f$ is completely monotonic, we have that f is increasing, so

$$\omega \cdot \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}} \leq \Gamma(x+1) < \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x}}. \tag{3}$$

But g is completely monotonic, so g is decreasing and it follows

$$\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x+1}} < \Gamma(x+1) \leq \eta \cdot \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} e^{\frac{1}{12x+1}}. \tag{4}$$

The constants

$$\omega = \frac{1}{\sqrt{2\pi}} e^{\frac{11}{12}} = 0.99773... \quad , \qquad \eta = \frac{1}{\sqrt{2\pi}} e^{\frac{12}{13}} = 1.00414...$$

are the best possible such that inequalities (3)-(4) are sharp.

Estimates (3)-(4) follows from

$$f(1) \leq f(x) < f(\infty),$$

respective

$$g(\infty) < g(x) \leq g(1).$$

Furthermore, by using the monotonicity of first derivatives, we can establish estimates for the digamma function. We mean that $-f$ is completely monotonic, so f' is decreasing, which is

$$f'(\infty) < f'(x) \leq f'(1). \tag{5}$$

Similarly, g is completely monotonic, so g' is increasing,

$$g'(1) \leq g'(x) < g'(\infty). \tag{6}$$

Inequalities (5)-(6) can be written in the form

$$\ln x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \tau$$

$$\ln x - \frac{1}{2x} - \frac{1}{12\left(x+\frac{1}{12}\right)^2} - \nu \leq \psi(x) \leq \ln x - \frac{1}{2x} - \frac{1}{12\left(x+\frac{1}{12}\right)^2},$$

respectively, where

$$\tau = -\gamma + \frac{7}{12} = 0.006117... \quad , \qquad \nu = \frac{193}{338} - \gamma = 6209... \ .$$

3 A result on Wallis ratio

The Wallis ratio

$$P_n = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}$$

has important applications in pure mathematics (combinatorics, number theory, probabilities) or in other branches of science such as applied statistics, statistical physiscs and quantum mechanics. It is closely related to the Euler gamma function, since

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+1\right)}.$$

Many authors gave in the past estimates for P_n and we mention here the following:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \qquad \text{(D. K. Kazarinoff)},$$

or

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/3}\right)}} \qquad (ZhaoDeJun).$$

Zhao and Wu improved the upper bound of the previous inequality, showing that for $0 < \varepsilon < 1/2$,

$$P_n < \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-1/2+\varepsilon}\right)}} \quad , \qquad n \geq n^*\left(\varepsilon\right),$$

where $n^*\left(\varepsilon\right)$ is the maximal root of the equation

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

These inequalities show us that the best approximations of the form

$$P_n \approx \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-a}\right)}} \quad , \quad a \in \mathbb{R}$$

are obtained for $a = 1/2$. Moreover, if we are interested to obtain further accurate approximations of the form

$$P_n \approx \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\theta_n}\right)}} \quad , \quad \theta_n \in \mathbb{R},$$

then θ_n should tends to $1/2$, as n approaches infinity.

Equivalent:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+1\right)} \approx \frac{1}{\sqrt{n\left(1+\frac{1}{4n-a}\right)}}$$

and we are entitled to study the logarithmically completely monotonicity of the functions

$$g_a\left(x\right) = \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x+1\right)}\sqrt{x\left(1+\frac{1}{4x-a}\right)}, \quad a \in \mathbb{R}.$$

More precisely, it is proven in [2] that for all $1/2 \le a \le 2$, the function g_a is logarithmically completely monotonic on $(0,\infty)$. Afterwards, by exploiting $a = 1/2$ case, the following double inequality is obtained

$$\frac{\alpha}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \le \frac{\beta}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}},$$

where the constants

$$\alpha = 1 \quad \text{and} \qquad \beta = \frac{3\sqrt{7\pi}}{14} = 1.0049\ldots$$

are the best possible.

4 The technique

We show in this final section the method of proving the complete monotonicity of the function $f_a = \ln g_a$. For a detailed proof, see [2]. In this sense, we have

$$f_a\left(x\right) = \ln\Gamma\left(x+\frac{1}{2}\right) - \ln\Gamma\left(x+1\right) + \frac{1}{2}\ln x + \frac{1}{2}\ln\left(1+\frac{1}{4x-a}\right).$$

By differetiating twice, we get

$$f_a''\left(x\right) = \psi'\left(x+\frac{1}{2}\right) - \psi'\left(x+1\right) - \frac{1}{2x^2} + \frac{1}{2\left(x-\frac{a}{4}\right)^2} - \frac{1}{2\left(x+\frac{1-a}{4}\right)^2}.$$

Then, by using the integral representations (1)-(2), we deduce

$$\begin{aligned} f_a''\left(x\right) &= \int_0^\infty \frac{te^{-t\left(x+\frac{1}{2}\right)}}{1-e^{-t}}dt - \int_0^\infty \frac{te^{-t(x+1)}}{1-e^{-t}}dt \\ &\quad -\frac{1}{2}\int_0^\infty te^{-tx}dt + \frac{1}{2}\int_0^\infty te^{-t\left(x-\frac{a}{4}\right)}dt - \frac{1}{2}\int_0^\infty te^{-t\left(x+\frac{1-a}{4}\right)}dt \\ &= \frac{1}{2}\int_0^\infty \varphi\left(\frac{t}{4}\right)\frac{te^{-tx}}{e^t-1}dt, \end{aligned}$$

where $\varphi(t) = 2e^{2t} - e^{at} - e^{4t} + e^{(a-1)t} - e^{(a+3)t} + e^{(a+4)t} - 1$. But

$$\varphi(t) = e^{-t} \sum_{n=3}^{\infty} x_n(a) \frac{t^n}{n!} > 0,$$

where for $n \geq 3$, $a \in [1/2, 2]$

$$x_n(a) = (a+5)^n + a^n + 2 \cdot 3^n - (a+1)^n - (a+4)^n - 5^n - 1 \geq 0.$$

See [2].

By Hausdorff-Bernstein-Widder theorem, f_a'' is completely monotonic.

Now f_a' is strictly increasing (since $f_a'' > 0$) with $f_a'(\infty) = 0$, so $f_a' < 0$.

Finally, f_a is strictly decreasing (since $f_a' < 0$) with $f_a(\infty) = 0$, so $f_a > 0$ and consequently f_a is completely monotonic.

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Cristinel Mortici
Valahia University of Târgovişte
Department of Science
Bd. Unirii 18, 130086 Târgovişte, Romania
e-mail: cmortici@valahia.ro

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Some approximations properties of Schurer-Bernstein operators based on q integers [1]

Carmen Violeta Muraru, Ana Maria Acu

Abstract

We consider the class of generalized Schurer-Bernstein operators in q calculus for which some properties of monotonicity and convexity are studied. The paper contain also numerical representation of these operators, based on Matlab algorithms and iterative relations of basic functions.

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Key words and phrases: generalized Bernstein-Schurer operator, q-integers, monotonicity, convexity.

1 Introduction

We mention in the following paragraph some basic definitions of q-calculus of which applications occur in many domains as physics, quantum theory, number theory, etc. In recent years, there are many preoccupations in construction and studied generalized version in q-calculus of well-known linear and positive operators starting with the class of Bernstein type. Lupas [4] introduced in 1987 a q-type of the Bernstein operators and in 1996 another generalization of these operators based on q-integers was introduced by Phillips [9]. After this, many authors studied new classes of q-generalized operators.

For any fixed real number, the q-integer $[k]$, for all nonnegative integers k, is defined as

$$[k] = \begin{cases} (1-q^k)/(1-q), \ q \neq 1 \\ k, \ q = 1. \end{cases}$$

The q-factorial $[k]!$ and q-binomial are also defined

$$[k]! = \begin{cases} [k][k-1]\cdots[1], \ k = 1, 2, \ldots, \\ 1, \ k = 0 \end{cases}$$

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$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{[n]!}{[k]![n-k]!} \ (n \geq k \geq 0).$$

Let $\Delta_q^0 f_j = f_j$ for $j = 0, 1, \ldots, n$ and recursively

$$\Delta_q^{k+1} f_j = \Delta_q^k f_{j+1} - q^k \Delta_q^k f_j, \text{ for } k = 0, 1, \ldots, n-j-1 \text{ and } f_j = f\left(\frac{[j]}{[n]}\right).$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1, & n = 0, \\ (x-a)(x-qa)\ldots(x-q^{n-1}a), & n \geq 1. \end{cases}$$

Let $p \in N$ be fixed. In 1962 Schurer introduced and studied the Bernstein-Schurer operators $\tilde{B}_{m,p} : C([0, p+1]) \to C([0, 1])$ defined for any $m \in N$ and any function $f \in C([0, p+1])$ as follows

$$\text{(1)} \qquad \tilde{B}_{m,p}(f; x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right).$$

For any $m \in N$, $f \in C([0, p+1])$, p be fixed C. Muraru constructs the class of generalized q-Bernstein-Schurer operators ([5]), as follows:

$$\text{(2)} \qquad \tilde{B}_{m,p}(f; q; x) = \sum_{k=0}^{m+p} \left[\begin{array}{c} m+p \\ k \end{array}\right] x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) f\left(\frac{[k]}{[m]}\right), \ x \in [0, 1]$$

where the function is evaluated at intervals which are in geometric progression.

We noticed that the classical Schurer-Bernstein is retrieve when we set the parameter $q = 1$.

Lemma 1 *[5] The operator defined by (2) is linear and positive.*

Lemma 2 *[5] For the polynomials defined above satisfy the following properties:*
1.$\tilde{B}_{m,p}(e_0; q; x) = 1$,
2.$\tilde{B}_{m,p}(e_1; q; x) = \dfrac{x[m+p]}{[m]}$,
3.$\tilde{B}_{m,p}(e_2; q; x) = \dfrac{[m+p]}{[m]^2}\big([m+p]x^2 + x(1-x)\big)$,
where we note by $e_j(x) = x^j$, $j = 0, 1, 2$, the test functions.

Theorem 1 *[5] Let $q = q_m$ satisfy $0 < q_m < 1$ and $\lim_{m\to\infty} q_m^p = 1$ and $\lim_{m\to\infty} q_m^m = a$, $a < 1$. Then for any $f \in C([0, p+1])$ the next results holds*

$$\lim_{m\to\infty} \tilde{B}_{m,p}(f; q_m) = f \text{ uniformly on [0,1].}$$

Theorem 2 *[5] If* $f \in C([0, p+1])$, *then*

$$\left|\tilde{B}_{m,p}(f; q_m; x) - f(x)\right| \leq 2\omega_f(\delta_m)$$

holds, where

$$\delta_m = \frac{1}{\sqrt{[m]}}\left(p + \frac{1}{\sqrt{1-q^m}}\right), \; q \in (0,1).$$

2 Monotonicity of q-Schurer Bernstein polynomials

It is known ([7]) that, when the function f is convex on $[0,1]$, its q-Bernstein polynomials are monotonic decreasing. In this section we will study the monotonicity of q-Schurer Bernstein polynomials.

Theorem 3 *Let* f *be convex and increasing on* $[0, p+1]$. *Then, for* $0 < q \leq 1$,

$$\tilde{B}_{m-1,p}(f; q; x) \geq \tilde{B}_{m,p}(f; q; x) \tag{3}$$

for $0 \leq x \leq 1$ *and all* $m \geq 2$.

Proof. For $0 < q < 1$ we begin by writing

$$\prod_{s=0}^{m+p-1}(1-q^s x)^{-1}\left[\tilde{B}_{m-1,p}(f; q; x) - \tilde{B}_{m,p}(f; q; x)\right]$$

$$= \sum_{k=0}^{m+p-1}\left[\begin{array}{c} m+p-1 \\ k \end{array}\right] x^k \prod_{s=m+p-k-1}^{m+p-1}(1-q^s x)^{-1} f\left(\frac{[k]}{[m-1]}\right)$$

$$-\sum_{k=0}^{m+p}\left[\begin{array}{c} m+p \\ k \end{array}\right] x^k \prod_{s=m+p-k}^{m+p-1}(1-q^s x)^{-1} f\left(\frac{[k]}{[m]}\right).$$

Denote by

$$\psi_k(x) = x^k \prod_{s=m+p-k}^{m+p-1}(1-q^s x)^{-1} \tag{4}$$

and using the following relation

$$x^k \prod_{s=m+p-k-1}^{m+p-1}(1-q^s x)^{-1} = \psi_k(x) + q^{m+p-k-1}\psi_{k+1}(x)$$

we find

$$\prod_{s=0}^{m+p-1}(1-q^s x)^{-1}\left[\tilde{B}_{m-1,p}(f; q; x) - \tilde{B}_{m,p}(f; q; x)\right]$$

$$= \sum_{k=0}^{m+p-1} f\left(\frac{[k]}{[m-1]}\right) \left[\begin{array}{c} m+p-1 \\ k \end{array} \right] \left\{ \psi_k(x) + q^{m+p-k-1} \psi_{k+1}(x) \right\}$$

$$- \sum_{k=0}^{m+p} f\left(\frac{[k]}{[m]}\right) \left[\begin{array}{c} m+p \\ k \end{array} \right] \psi_k(x)$$

$$= \sum_{k=0}^{m+p-1} f\left(\frac{[k]}{[m-1]}\right) \left[\begin{array}{c} m+p-1 \\ k \end{array} \right] \psi_k(x) + \sum_{k=1}^{m+p} q^{m+p-k} f\left(\frac{[k-1]}{[m-1]}\right) \left[\begin{array}{c} m+p-1 \\ k-1 \end{array} \right] \psi_k(x)$$

$$- \sum_{k=0}^{m+p} f\left(\frac{[k]}{[m]}\right) \left[\begin{array}{c} m+p \\ k \end{array} \right] \psi_k(x)$$

$$= \sum_{k=1}^{m+p-1} \left\{ f\left(\frac{[k]}{[m-1]}\right) \left[\begin{array}{c} m+p-1 \\ k \end{array} \right] + q^{m+p-k} f\left(\frac{[k-1]}{[m-1]}\right) \left[\begin{array}{c} m+p-1 \\ k-1 \end{array} \right] \right.$$

$$\left. - f\left(\frac{[k]}{[m]}\right) \left[\begin{array}{c} m+p \\ k \end{array} \right] \right\} \psi_k(x) + \left\{ f\left(\frac{[m+p-1]}{[m-1]}\right) - f\left(\frac{[m+p]}{[m]}\right) \right\} \psi_{m+p}(x)$$

$$= \sum_{k=1}^{m+p-1} \left[\begin{array}{c} m+p \\ k \end{array} \right] a_k \psi_k(x) + \left\{ f\left(\frac{[m+p-1]}{[m-1]}\right) - f\left(\frac{[m+p]}{[m]}\right) \right\}, \text{ where}$$

$$a_k = f\left(\frac{[k]}{[m-1]}\right) \frac{[m+p-k]}{[m+p]} + q^{m+p-k} f\left(\frac{[k-1]}{[m-1]}\right) \frac{[k]}{[m+p]} - f\left(\frac{[k]}{[m]}\right).$$

From (4) it is clear that each $\psi_k(x)$ is non-negative on $[0,1]$ for $0 \le q \le 1$ and thus, it suffices to show that each a_k is non-negative.

Since f is convex on $[0, p+1]$ it follows for any t_0, t_1 such that $0 \le t_0 < t_1 \le p+1$ and any λ, $0 < \lambda < 1$,

$$f(\lambda t_0 + (1-\lambda)t_1) \le \lambda f(t_0) + (1-\lambda) f(t_1). \tag{5}$$

If we choose in (5) $t_0 = \dfrac{[k-1]}{[m-1]}$, $t_1 = \dfrac{[k]}{[m-1]}$ and $\lambda = q^{m+p-k} \dfrac{[k]}{[m+p]}$, then $0 \le t_0 < t_1 \le p+1$ and $0 < \lambda < 1$ for $1 \le k \le m+p-1$ and, it follows

(6)
$$q^{m+p-k} \frac{[k]}{[m+p]} f\left(\frac{[k-1]}{[m-1]}\right) + \frac{[m+p-k]}{[m+p]} f\left(\frac{[k]}{[m-1]}\right) - f\left(\frac{[k]}{[m+p]} \cdot \frac{[m+p-1]}{[m-1]}\right) \ge 0.$$

Since f is increasing on $[0, p+1]$ and

$$a_k = f\left(\frac{[k]}{[m-1]}\right) \frac{[m+p-k]}{[m+p]} + q^{m+p-k} f\left(\frac{[k-1]}{[m-1]}\right) \frac{[k]}{[m+p]} - f\left(\frac{[k]}{[m+p]} \cdot \frac{[m+p-1]}{[m-1]}\right)$$

$$+ \left\{ f\left(\frac{[k]}{[m+p]} \cdot \frac{[m+p-1]}{[m-1]}\right) - f\left(\frac{[k]}{[m]}\right) \right\},$$

using the relation (6) we obtain $a_k \geq 0,\ k = \overline{1, m+p-1}$. Thus $\tilde{B}_{m-1,p}(f;q;x) \geq \tilde{B}_{m,p}(f;q;x)$.

For $q = 1$ and $0 \leq x < 1$ in a similar way the property (3) is verified.

For $q = 1$ and $x = 1$ we have

$$\tilde{B}_{m-1,p}(f;1;1) - \tilde{B}_{m,p}(f;1;1) = f\left(\frac{m+p-1}{m-1}\right) - f\left(\frac{m+p}{m}\right) \geq 0.$$

Theorem 4 *If f is convex and increasing on $[0, p+1]$, then*

$$\tilde{B}_{m,p}(f;q;x) \geq f(x),\ 0 \leq x \leq 1,$$

for all $n \geq 1$ and for $0 < q \leq 1$.

Proof. We consider the knots $x_k = \dfrac{[k]}{[m]}$ and $\lambda_k = \left[\begin{array}{c} m+p \\ k \end{array}\right] x^k \prod_{s=0}^{m+p-k-1}(1 - q^s x)$, $0 \leq k \leq m+p$.

From Lemma 2 it follows

$$\lambda_0 + \lambda_1 + \cdots + \lambda_{m+p} = 1,\ x_0\lambda_0 + x_1\lambda_1 + \cdots + x_{m+p}\lambda_{m+p} = x\frac{[m+p]}{[m]}.$$

Then using the convexity of function f we have

$$\tilde{B}_{m,p}(f;q;x) = \sum_{k=0}^{m+p} \lambda_k f(x_k) \geq f\left(\sum_{k=0}^{m+p} \lambda_k x_k\right) = f\left(x\frac{[m+p]}{[m]}\right).$$

From the increasing property of function we obtain for $x\dfrac{[m+p]}{[m]} \geq x$, $0 \leq x \leq 1$ the following inequality $f\left(x\dfrac{[m+p]}{[m]}\right) \geq f(x)$.

Finally it is obtained that $\tilde{B}_{m,p}(f;q;x) \geq f(x)$.

The result can be verified also using representation of function and associated operator for the case of convex and increasing function.

3 Properties of q-Shurer-Bernstein polynomials basis

In the parametric representation of a curve or surface it is important which basis is used in order to preserve the shape of the form. It is well known that Bernstein polynomials due to their remarkable properties play a significant role in CAGD. In [8] it is studied the q-Bernstein basis and their shape preserving properties using the concept of total positivity and a Bezier curve is constructed based on a Casteljau type algorithm.

The basis functions which appear in form of q-Schurer Bernstein operators are

$$p_{m,p,q}^{k}(x) = \left[\begin{array}{c} m+p \\ k \end{array}\right] x^k \prod_{s=0}^{m+p-k-1}(1 - q^s x),\ k = 0, 1, \ldots, m+p$$

and satisfy the following recurrence relations

$$\begin{aligned} p^k_{m,p,q}(x) &= q^{m+p-k}xp^{k-1}_{m-1,p,q}(x) + (1-q^{m+p-k-1}x)p^k_{m-1,p,q}(x) \\ p^k_{m,p,q}(x) &= xp^{k-1}_{m-1,p,q}(x) + (q^k - q^{m+p-1})p^k_{m-1,p,q}(x) \end{aligned}$$

which can be deduced using

$$\begin{bmatrix} m+p \\ k \end{bmatrix} = q^{m+p-k}\begin{bmatrix} m+p-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} m+p-1 \\ k \end{bmatrix}$$

$$\begin{bmatrix} m+p \\ k \end{bmatrix} = \begin{bmatrix} m+p-1 \\ k-1 \end{bmatrix} + q^k\begin{bmatrix} m+p-1 \\ k \end{bmatrix}$$

From Lemma 2 it is obvious that the q-Schurer-Bernstein basis forms a partition of unity because

$$\sum_{k=0}^{m+p}\begin{bmatrix} m+p \\ k \end{bmatrix} x^k \prod_{s=0}^{m+p-k-1}(1-q^s x) = 1,$$

but for any real numbers a and b, generalized Schurer-Bernstein polynomials doesn't reproduce linear functions, that is $\tilde{B}_{m,p}(ax+b;x) = ax\dfrac{[m+p]}{[m]} + b$.

If the q-Bernstein operators satisfies the end point interpolation conditions $B_{n,q}(f,0) = f(0)$ and $B_{n,q}(f,1) = f(1)$, for the q-Schurer-Bernstein operators we have the following conditions $\tilde{B}_{m,p}(f;q;0) = f(0),\ \tilde{B}_{m,p}(f;q;1) = f\left(\dfrac{[m+p]}{[m]}\right)$.

4 Numerical Examples

In [3] it is shown that for any convex function f, the classical Bernstein polynomial is convex and the sequence of Bernstein polynomials is monotonic decreasing. In [7] were been discussed that this result extends to the Bernstein polynomials in q-calculus for $0 < q \leq 1$. In this section we will verify numerical these properties of the q-Schurer-Bernstein operator.

The next figure illustrates the monotonocity and convexity of $f(x) = x^3\exp(x+1)$ on [0,1] and that the generalized Schurer-Bernstein operators preserve these properties also. In the above figure we remark that when m is increases the operators interpolates also in 1.

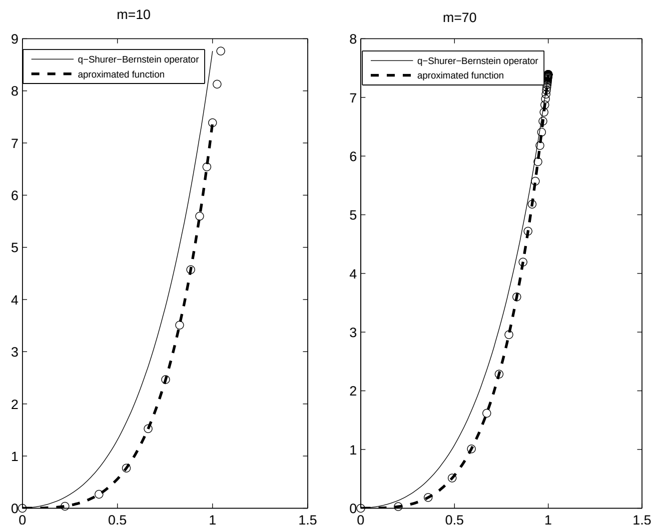

Figure 1.

The numerical results based on the Matlab algorithms of representation for the q-Schurer Bernstein polynomials confirm also that in 1 end the polynomial will be interpolator as m increases for $f(x) = \exp(x) * \sin(x)$, $x \in [0, 1]$.

The order m	The function value in 1	The polynomial value in 1
$m = 10$	2.87355287	2.378564
$m = 40$	2.87355287	2.8745515440
$m = 70$	2.873552871	2.8735541079
$m = 150$	2.873552871	2.873552871

In the next figures we can see the behavior of the generalized Schurer-Bernstein polynomials as q and m varies:

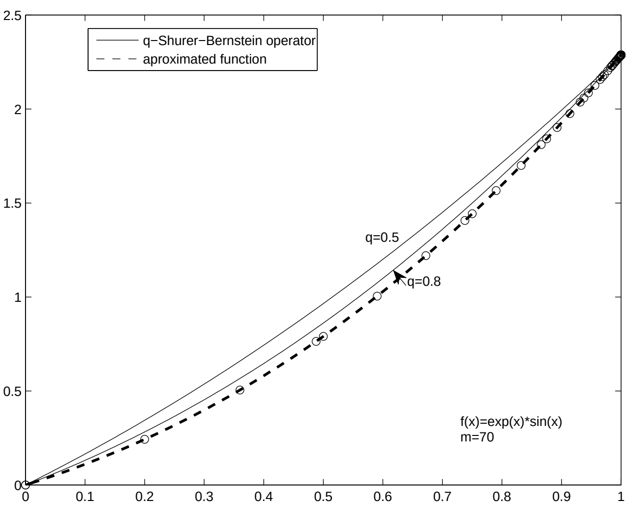

Figure 2.

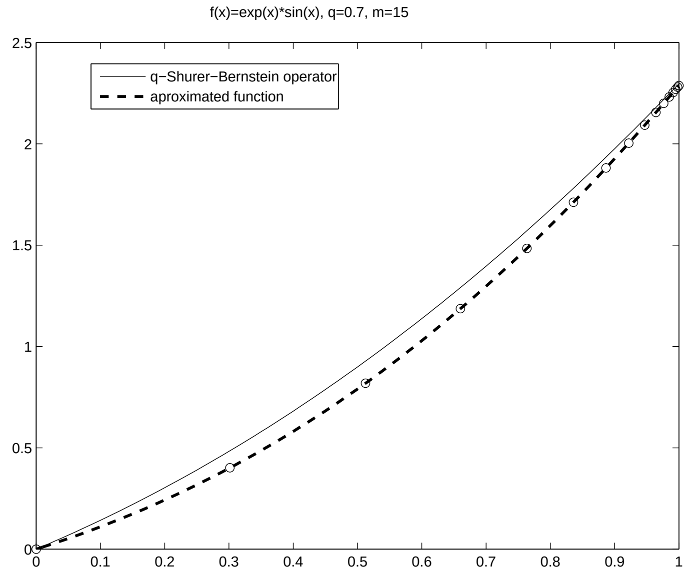

Figure 3.

The bellow figures illustrate the variation of q-Schurer-Bernstein polynomials when q increases for function $f(x) = \sin(2\pi x)$, respectively $f(x) = 1 - \sin(\pi x)$, $x \in [0, 1]$ We note that if f is increasing (decreasing) on $[0, 1]$, then the operator is also increasing (decreasing) on $[0, 1]$, for $0 < q < 1$.

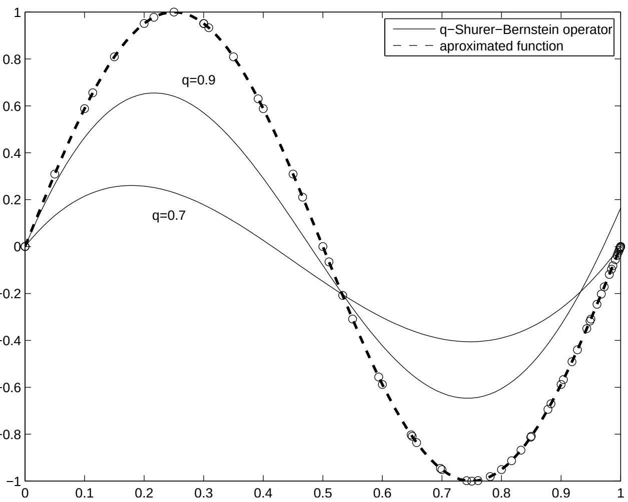

Figure 4.

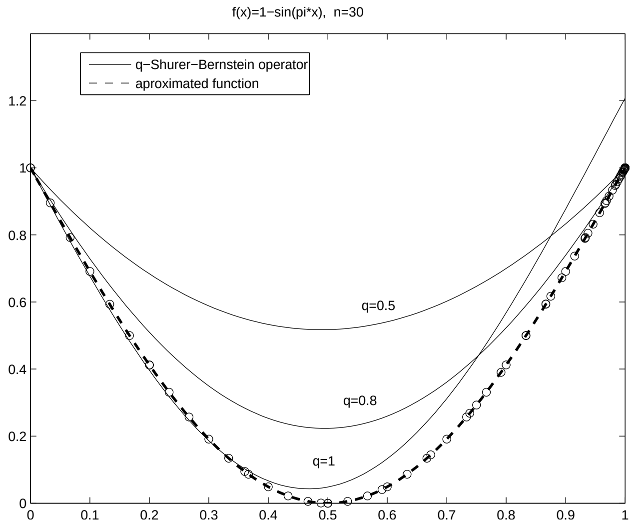

Figure 5.

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Muraru Carmen Violeta
University "Vasile Alecsandri" of Bacau
Department of Mathematics and Informatics
e-mail: carmen_7419@yahoo.com

Acu Ana Maria
University "Lucian Blaga" of Sibiu
Department of Mathematics and Informatics
e-mail: acuana77@yahoo.com

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Generalized Bernstein-Durrmeyer operators on a simplex [1]

Radu Păltănea

Abstract

We study the uniform approximation and the uniform convergence on compact sets by a sequence of general Bernstein-Durrmeyer-type operators on a simplex.

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Key words and phrases: Bernstein-Durrmeyer type operators on a simplex, uniform approximation, uniform approximation on compact sets.

1 Introduction

We make the following notations. Set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix $m \in \mathbb{N}$.
Define

$$\begin{aligned} S &:= \{\mathbf{x} = (x_1, \ldots, x_m) |\, x_i \geq 0,\ (1 \leq i \leq m),\ x_1 + \ldots + x_m \leq 1\}. \\ \Delta &:= \{\mathbf{y} = (y_0, y_1, \ldots, y_m) |\, y_i \in \mathbb{R}_+,\ (0 \leq i \leq m),\ y_0 + \ldots + y_m = 1\}. \end{aligned}$$

We have the bijection $\Psi : S \to \Delta$ and $\Psi^{-1} : \Delta \to S$, given by:

$$\begin{aligned} \Psi(x_1, \ldots, x_m) &= (1 - x_1 - \ldots - x_m, x_1, \ldots, x_m) \\ \Psi^{-1}(y_0, \ldots, y_m) &= (y_1, \ldots, y_m). \end{aligned}$$

For any function $f \in C(S)$, define $\overline{f} \in C(\Delta)$, by $\overline{f} = f \circ \Psi^{-1}$.
If $\mathbf{y} = (y_0, \ldots, y_m) \in \Delta$ and $\mathbf{a} = (a_0, \ldots, a_m) \in \mathbb{R}^{m+1}$ put

$$|\mathbf{a}| := a_0 + \ldots a_m, \quad \mathbf{y}^{\mathbf{a}} := y_0^{a_0} \cdot \ldots \cdot y_m^{a_m}, \quad B(\mathbf{a}) := \frac{\Gamma(\mathbf{a})}{\Gamma(|\mathbf{a}|)} := \frac{\Gamma(a_0) \ldots \Gamma(a_m)}{\Gamma(|\mathbf{a}|)}.$$

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We denote $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^{m+1}$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^{m+1}$. If $0 \le i \le m$, denote $(\mathbf{1})_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^{m+1}$, where digit 1 appears at position i . Let the function $e_0(\mathbf{x}) = 1$, $\mathbf{x} \in S$. If $\mathbf{a} = (a_0, \dots, a_m)$ we write $\mathbf{a} > \mathbf{0}$ if $a_i > 0$, $0 \le i \le m$. We have:

$$\int_S (\Psi(\mathbf{t}))^{\mathbf{a}} \mathrm{d}\mathbf{t} = B(\mathbf{a} + \mathbf{1}), \text{ when } \mathbf{a} + \mathbf{1} > \mathbf{0}. \tag{1}$$

For any $n \in \mathbb{N}$, $n \ge 1$ denote

$$I_n := \{\mathbf{k} = (k_0, k_1, \dots, k_m) |\ k_i \in \mathbb{N}_0,\ (0 \le i \le m),\ k_0 + \dots + k_m = n\}.$$

If $\mathbf{k} \in I_n$, $\quad \mathbf{y} = (y_0, \dots, y_m) \in \Delta$ define

$$\binom{n}{\mathbf{k}} = \binom{n}{k_0 \dots k_m} = \frac{n!}{k_0! k_1! \dots k_m!} \text{ and } \overline{p}_{n,\mathbf{k}}(\mathbf{y}) = \binom{n}{\mathbf{k}} \mathbf{y}^{\mathbf{k}}.$$

We have the following formula

$$\sum_{\mathbf{k} \in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) = 1,\ \mathbf{y} \in \Delta,\ n \in \mathbb{N}. \tag{2}$$

Also, for $\mathbf{x} \in S$ and $\mathbf{k} \in I_n$ we denote

$$p_{n,\mathbf{k}}(\mathbf{x}) = \overline{p}_{n,\mathbf{k}}(\Psi(\mathbf{x})).$$

Consider μ a Borel positive measure on S. We define operator L_n^{μ}, by:

$$L_n^{\mu}(f, \mathbf{x}) = \sum_{\mathbf{k} \in I_n} \frac{\int_S f(\mathbf{t}) p_{n,\mathbf{k}}(\mathbf{t}) \mathrm{d}\mu(\mathbf{t})}{\int_S p_{n,\mathbf{k}}(\mathbf{t}) \mathrm{d}\mu(\mathbf{t})} \cdot p_{n,\mathbf{k}}(\mathbf{x}), \tag{3}$$

where f is a μ-integrable function, $\mathbf{x} \in S$ and $n \in \mathbb{N}$.

In what follows we consider only the particular case when the measure μ is constructed by the aid of a *strictly positive* continuous function $\rho : S \to \mathbf{R}_+$ and a vector $\mathbf{a} = (a_0, \dots, a_m) \in \mathbf{R}^{m+1}$, where $a_i > -1$, $0 \le i \le m$, namely $\mu(A) = \int_A (\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathrm{d}\mathbf{t}$, $A \subset S$, where the integral is taken in Lebesgue sense. We obtain operators $L_n^{\rho,\mathbf{a}}$, given by:

$$L_n^{\rho,\mathbf{a}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in I_n} \frac{\int_S f(\mathbf{t}) p_{n,\mathbf{k}}(\mathbf{t}) (\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathrm{d}\mathbf{t}}{\int_S p_{n,\mathbf{k}}(\mathbf{t}) (\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathrm{d}\mathbf{t}} \cdot p_{n,\mathbf{k}}(\mathbf{x}),\ \mathbf{x} \in S,\ n \in \mathbb{N}, \tag{4}$$

where the function $f : S \to \mathbb{R}$ is continuous on $\overset{\circ}{S}$ and such that integral $\int_S f(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathbf{dt}$ exists in the Lebesgue sense. Consequently integrals $\int_S f(\mathbf{t}) p_{n,\mathbf{k}}(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathrm{d}\mathbf{t}$ and integrals $\int_S |f(\mathbf{t})| p_{n,\mathbf{k}}(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}} \rho(\mathbf{t}) \mathrm{d}\mathbf{t}$ exist for each $\mathbf{k} \in I_n$.

These operators were considered for $\rho = e_0$, by Derriennic [3], for $m = 1$ in our paper [6] and for $\rho = e_0$ and $m = 1$, previously in our paper [5]. If we take $\rho = e_0$ and we pass to limit $a_i \to -1$ one obtains the genuine Bernstein-Durrmeyer operators on the simplex, first considered by Waldron [7].

2 The corresponding operators on $C(\Delta)$

Let ρ be a continuos strictly positive function on S and let the vector $\mathbf{a}$ satisfying condition $\mathbf{a} > -\mathbf{1}$. We asign to operator $L_n^{\rho,\mathbf{a}}$ given in (4) a corresponding operator $\overline{L}_n^{\overline{\rho},\mathbf{a}}$ in the following mode. Denote $\overline{\rho} = \rho \circ \Psi^{-1}$ and define:

$$(5) \qquad \overline{L}_n^{\overline{\rho},\mathbf{a}}(f,\mathbf{y}) = \sum_{\mathbf{k}\in I_n} \frac{\int_\Delta f(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})} \cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}),\ \mathbf{y}\in\Delta,\ n\in\mathbb{N},$$

where the integrals are surface integrals and $f : \Delta \to \mathbb{R}$ is continuous on the relative interior of Δ in the hypersurface $y_0 + \ldots + y_m = 1$ in $\mathbb{R}^{m+1}$ and is such that the integral $\int_\Delta |f(\mathbf{u})|\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})$ exists. Consequently all the integrals in formula (5) exist.

Theorem 1 *We have*

$$(6) \qquad L_n^{\rho,\mathbf{a}}(f,\mathbf{x}) = \overline{L}_n^{\overline{\rho},\mathbf{a}}(f\circ\Psi^{-1},\Psi(\mathbf{x})),\ f\in C(S),\ \mathbf{x}\in S.$$

Proof. The hypersurface $\Delta \subset \mathbb{R}^{m+1}$ is parametrized by the function $\Psi : S \to \Delta$. If $g \in C(\Delta)$, the formula for computation of surface integrals yields

$$\int_\Delta g(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}) = \int_S (g\circ\Psi)(\mathbf{t})\Theta(\mathbf{t})\mathrm{d}\mathbf{t},$$

where $\Theta(\mathbf{t}) = \sqrt{\det[< \Psi_{t_i}\cdot\Psi_{t_j}]_{1\le i,j\le m}}$. We have $\Psi_{t_i} = (-1,0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^{m+1}$, $(1 \le i \le m)$, where digit 1 appears at position of index i. Hence

$$\det[< \Psi_{t_i}\cdot\Psi_{t_j}]_{1\le i,j\le m} = \det\begin{pmatrix} 2 & 1 & \ldots & 1\\ 1 & 2 & \ldots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & 2\end{pmatrix} = m+1.$$

For $\mathbf{k} \in I_n$ it follows

$$\begin{aligned}&\int_S f(\mathbf{t})p_{n,\mathbf{k}}(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}}\rho(\mathbf{t})\mathrm{d}\mathbf{t}\\ &= \frac{1}{\sqrt{m+1}}\int_\Delta (f\circ\Psi^{-1})(\mathbf{u})(p_{n,\mathbf{k}}\circ\Psi^{-1})(\mathbf{u})\mathbf{u}^{\mathbf{a}}(\rho\circ\Psi^{-1})(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})\\ &= \frac{1}{\sqrt{m+1}}\int_\Delta (f\circ\Psi^{-1})(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}).\end{aligned}$$

In a similar mode we obtain

$$\int_S p_{n,\mathbf{k}}(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}}\rho(\mathbf{t})\mathrm{d}\mathbf{t} = \frac{1}{\sqrt{m+1}}\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}).$$

Consequently, we obtain relation (6).

Remark 1 Operators $\overline{L}_n^{\overline{\mu}}$ are more convenient for computations then operators L_n^{μ} since functions $\overline{p}_{n,\mathbf{k}}$, $\mathbf{k} \in I_n$ are symmetric.

Remark 2 Using the same method as in Theorem 1 and using formula (1) we obtain

$$\text{(7)} \qquad \frac{1}{\sqrt{m+1}} \int_{\Delta} \mathbf{u}^{\mathbf{a}} \mathrm{d}\sigma(\mathbf{u}) = B(\mathbf{a}+\mathbf{1}), \text{ when } \mathbf{a}+\mathbf{1} > \mathbf{0}.$$

3 Uniform approximation

In this section we consider the particular case when operators $L_n^{\rho,\mathbf{a}}$ are applied to functions $f \in C(S)$.

Theorem 2 *For all $f \in C(S)$ we have*

$$\text{(8)} \qquad \lim_{n\to\infty} L_n^{\rho,\mathbf{a}}(f,\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in S,\ \textit{uniformly.}$$

Proof. Consider the following functions from $C(S)$: $e_0(\mathbf{x}) = 1$, $\pi_i(\mathbf{x}) = x_i$, $(1 \le i \le m)$, $\mathbf{x} = (x_1, \dots, x_m) \in S$ and $\varphi := \pi_1^2 + \dots + \pi_n^2$. It is well known that the set $A := \{e_0, \pi_1, \dots, \pi_n, \varphi\}$ is a Korovkin set on $C(S)$. So, in order to prove relation (8) for all $f \in C(S)$ it suffices to prove it only for the test functions $g \in A$. From Theorem 2 it follows that this is equivalent with to prove

$$\text{(9)} \qquad \lim_{n\to\infty} \overline{L}_n^{\overline{\rho},\mathbf{a}}(\overline{g})(\mathbf{y}) = \overline{g}(\mathbf{y}),\ \mathbf{y} \in \Delta,\ \text{uniformly, for all } \overline{g} \in B,$$

where $B := \{\overline{g} |\ g \in A\}$ and $\overline{g} := g \circ \Psi^{-1}$, when $g \in A$.

For $\mathbf{y} = (y_0, \dots, y_m) \in \Delta$ we have $\overline{e}_0(\mathbf{y}) = 1$, $\overline{\pi}_j(\mathbf{y}) = y_j$, $(1 \le j \le m)$ and $\overline{\varphi}(\mathbf{y}) = y_1^2 + \dots + y_m^2$. Because $\overline{L}_n^{\overline{\rho},\mathbf{a}}(\overline{e}_0) = \overline{e}_0$ it remains to prove relation (9) for the rest of the test functions.

Since function $\overline{\rho}$ is strictly positive on the compact set Δ there is a number $\eta > 0$ such that $\overline{\rho}(\mathbf{y}) \ge \eta$, $\mathbf{y} \in \Delta$. Denote by B_q, the Bernstein operator of order q on Δ, namely

$$B_q(h,\mathbf{y}) = \sum_{\mathbf{i}\in I_q} h\left(\frac{\mathbf{i}}{q}\right) \cdot \overline{p}_{q,\mathbf{i}}(\mathbf{y}),\ h \in C(\Delta),\ \mathbf{y} \in \Delta.$$

Let $0 < \varepsilon < 1$, arbitrarily chosen. Take $0 < \eta < \frac{\varepsilon}{5}$. It follows that $1 - \frac{\varepsilon}{2} < \frac{1-\eta}{1+\eta}$ and $\frac{1+\eta}{1-\eta} < 1 + \frac{\varepsilon}{2}$. Since $\lim_{q\to\infty} B_q(\overline{\rho}) = \overline{\rho}$ uniformly, we can find an index q such that $\|B_q(\overline{\rho}) - \overline{\rho}\|_\infty < \eta$. Denote $\overline{\rho}^\star := B_q(\overline{\rho})$.

Choose a function $\overline{g} \in B$. Denote $< \overline{g}, \overline{p}_{n,\mathbf{k}} >= \int_\Delta \overline{g}(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})$ and $< e_0, \overline{p}_{n,\mathbf{k}} >= \int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})$. We have

$$L_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{g},\mathbf{y}) = \sum_{\mathbf{k}\in I_n} \frac{< \overline{g}, \overline{p}_{n,\mathbf{k}} > + \int_\Delta \overline{g}(\mathbf{u})\mathbf{u}^{\mathbf{a}}[B_q(\overline{\rho},\mathbf{u}) - \overline{\rho}(\mathbf{u})]\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{< e_0, \overline{p}_{n,\mathbf{k}} > + \int_\Delta \mathbf{u}^{\mathbf{a}}[B_q(\overline{\rho},\mathbf{u}) - \overline{\rho}(\mathbf{u})]\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})} \cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}).$$

From the inequality $-\eta\overline{\rho}(\mathbf{u}) \le B_q(\overline{\rho},\mathbf{u}) - \overline{\rho}(\mathbf{u}) \le \eta\overline{\rho}(\mathbf{u})$, $\mathbf{u} \in \Delta$ we obtain for $\mathbf{y} \in \Delta$:

$$\overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{g},\mathbf{y}) < \frac{1+\eta}{1-\eta} \cdot \overline{L}_n^{\overline{\rho},\mathbf{a}}(g,\mathbf{y}) < \left(1+\frac{\varepsilon}{2}\right)\overline{L}_n^{\overline{\rho},\mathbf{a}}(\overline{g},\mathbf{y})$$

and

$$\overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{g},\mathbf{y}) > \frac{1-\eta}{1+\eta} \cdot L_n^{\rho,\mathbf{a}}(\overline{g},\mathbf{y}) > \left(1-\frac{\varepsilon}{2}\right)\overline{L}_n^{\overline{\rho},\mathbf{a}}(\overline{g},\mathbf{y})$$

Hence, for all $n \in \mathbb{N}$ and $\mathbf{y} \in \Delta$ we have:

$$|\overline{L}_n^{\overline{\rho},\mathbf{a}}(\overline{g},\mathbf{y}) - \overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{g},\mathbf{y})| < \frac{\varepsilon}{2}|\overline{L}_n^{\rho,\mathbf{a}}(\overline{g},\mathbf{y})| \le \frac{\varepsilon}{2}. \tag{10}$$

In the last inequality we taken into account that $\|\overline{\pi_j}\|_\infty \le 1$, $0 \le j \le m$ and also $\|\overline{\varphi}\|_\infty \le 1$.

Put $D_{\mathbf{i}} := \binom{q}{\mathbf{i}}\overline{\rho}\left(\frac{\mathbf{i}}{q}\right)$, for $\mathbf{i} \in I_q$. We have, for $\mathbf{y} \in \Delta$:

$$\begin{aligned}
\overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{g},\mathbf{y}) &= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\int_\Delta \overline{g}(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}B_n(\overline{\rho},\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}B_n(\overline{\rho},\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}\\
&= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \int_\Delta \overline{g}(\mathbf{u})\mathbf{u}^{\mathbf{k}+\mathbf{i}+\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})}{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \int_\Delta \mathbf{u}^{\mathbf{k}+\mathbf{i}+\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})}\\
&= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \int_\Delta \overline{g}(\mathbf{u})\mathbf{u}^{\mathbf{k}+\mathbf{i}+\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})}{\sqrt{m+1}\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}.
\end{aligned}$$

If we take $\overline{g} = \overline{\pi_j}$, $1 \le j \le m$, we obtain

$$\begin{aligned}
\overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{\pi_j},\mathbf{y}) &= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1}+(\mathbf{1})_j)}{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}\\
&= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})(k_j+i_j+a_j+1)}{(n+q+|\mathbf{a}|+m+1)\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}\\
&= \frac{1}{n+q+|\mathbf{a}|+m+1}\sum_{\mathbf{k}\in I_n} k_j \cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}) + \sum_{\mathbf{k}\in I_n} c_{n,\mathbf{k}}^j\overline{p}_{n,\mathbf{k}}(\mathbf{y}),
\end{aligned}$$

where

$$c_{n,\mathbf{k}}^j = \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})(i_j+a_j+1)}{(n+q+|\mathbf{a}|+m+1)\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}}B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}.$$

Using formula (2) we have

$$\sum_{\mathbf{k}\in I_n} k_j \cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}) = ny_j \sum_{\mathbf{l}\in I_{n-1}} \mathbf{y}^{\mathbf{l}}\binom{n-1}{\mathbf{l}} = ny_j.$$

On the other hand we have

$$c_{n,\mathbf{k}}^j \le \frac{q+|\mathbf{a}|+1}{n+q+|\mathbf{a}|+m+1}.$$

Consequently, for $1 \le j \le m$ we have:

$$\lim_{n\to\infty} \overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{\pi_j},\mathbf{y}) = \overline{\pi_j}(\mathbf{y}).\ \mathbf{y}\in\Delta,\ \text{uniformly.} \tag{11}$$

Let now $\overline{g} = \overline{\varphi}$. For $\mathbf{y}\in\Delta$ we have:

$$\overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{\varphi},\mathbf{y}) = \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \sum_{j=1}^m B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1}+2(\mathbf{1})_j)}{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}$$

$$= \sum_{\mathbf{k}\in I_n} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \cdot \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \sum_{j=1}^m B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})(k_j+i_j+a_j+1)(k_j+i_j+a_j+2)}{(n+q+|\mathbf{a}|+m+1)(n+q+|\mathbf{a}|+m+2)\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}$$

$$= \sum_{j=1}^m \frac{\sum_{\mathbf{k}\in I_n} k_j(k_j-1)\overline{p}_{n,\mathbf{k}}(\mathbf{y})}{(n+q+|\mathbf{a}|+m+1)(n+q+|\mathbf{a}|+m+2)} + \sum_{\mathbf{k}\in I_n} d_{n,\mathbf{k}} \cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}),$$

where

$$d_{n,\mathbf{k}} = \frac{\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} \sum_{j=1}^m B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})(k_j(2i_j+2a_j+4)+(i_j+a_j+1)(i_j+a_j+2))}{(n+q+|\mathbf{a}|+m+1)(n+q+|\mathbf{a}|+m+2)\sum_{\mathbf{i}\in I_q} D_{\mathbf{i}} B(\mathbf{k}+\mathbf{i}+\mathbf{a}+\mathbf{1})}.$$

Using formula (2) we have

$$\sum_{\mathbf{k}\in I_n} k_j(k_j-1)\cdot \overline{p}_{n,\mathbf{k}}(\mathbf{y}) = n(n-1)y_j^2 \sum_{\mathbf{l}\in I_{n-2}} \mathbf{y}^{\mathbf{l}} \binom{n-2}{\mathbf{l}} = n(n-1)y_j^2.$$

Also we have

$$d_{n,\mathbf{k}} \le m \cdot \frac{2n(q+|\mathbf{a}|+2)+(q+|a|+1)(q+|\mathbf{a}|+2)}{(n+q+|\mathbf{a}|+m+1)(n+q+|\mathbf{a}|+m+2)}.$$

Consequently,

$$\lim_{n\to\infty} \overline{L}_n^{\overline{\rho}^\star,\mathbf{a}}(\overline{\varphi},\mathbf{y}) = \overline{\varphi}(\mathbf{y}),\ \mathbf{y}\in\Delta,\ \text{uniformly.} \tag{12}$$

From relations (10), (11) and (12) we deduce that for any $\overline{g}\in B$, there is $n_\varepsilon \in \mathbb{N}$, such that $|\overline{L}_n^{\rho,\mathbf{a}}(\overline{g},\mathbf{y}) - \overline{g}(\mathbf{y})| < \varepsilon$, for all, $\mathbf{y}\in\Delta$ and $n\in\mathbb{N}$, $n \ge n_\varepsilon$. So that we can apply Korovkin's theorem and we obtain relation (9).

4 Uniform approximation on compact sets

Example 1 Let $m = 1$. Then $S = [0,1]$. Let $\rho(x) = 1$, $x\in[0,1]$ and let $\mathbf{a} = \mathbf{0}$. If we take the function $f(x) = \frac{1}{\sqrt{x}}$, $x \in (0,1)$ we cannot apply Theorem 2 to it. But we can consider the problem of uniform approximation on compact sets of f by operators $L_n^{e_0,\mathbf{0}}$.

Our main result is the following:

Theorem 3 *Let $\rho \in C(S)$ a strictly positive function and a vector $\mathbf{a} \in \mathbb{R}^{m+1}$, such that $\mathbf{a}+\mathbf{1} > \mathbf{0}$. If $f : S \to \mathbb{R}$ satisfies conditions:*

i) f is continue on $\overset{\circ}{S}$;

ii) integral $\int_S f(\mathbf{t})(\Psi(\mathbf{t}))^{\mathbf{a}}\rho(\mathbf{t})\mathbf{dt}$ exists,

then we have

$$\lim_{n\to\infty} L_n^{\rho,\mathbf{a}}(f,\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in K,\ \textit{uniformly}, \tag{13}$$

for any compact set $K \subset \overset{\circ}{S}$.

Proof. Denote $\overset{\circ}{\Delta}$ the relative interior of Δ in the hypersurface $y_0 + \ldots + y_m = 1$. Denote $g := f \circ \Psi^{-1}$. Using the notations in the previous sections, it suffices to show that for any function $g : \Delta \to \mathbb{R}$, which is continue in $\overset{\circ}{\Delta}$ and for which the integral $\int_\Delta g(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})$ exists, we have

$$\lim_{n\to\infty} \overline{L}_n^{\overline{\rho},\mathbf{a}}(g,\mathbf{y}) = g(\mathbf{y}),\ \mathbf{y} \in \overline{K},\ \text{uniformly}, \tag{14}$$

for all compact set $\overline{K} \subset \overset{\circ}{\Delta}$.

Let $\overline{K} \subset \overset{\circ}{\Delta}$. There is a number $0 < q < 1$, such that $y_i \geq q$, $0 \leq i \leq m$, for all $\mathbf{y} = (y_0, \ldots, y_m) \in \overline{K}$. Take a number $0 < r < 1$, not precised at this moment and denote

$$I_{n,r} := \{\mathbf{k} \in I_n |\ \frac{1}{n}\mathbf{k} \in \Delta_r\} = \{\mathbf{k} = (k_0, \ldots, k_m) \in I_n |\ k_i \geq nr,\ 0 \leq i \leq m\}.$$

Then choose a number $0 < s < \frac{1}{2}q$, not precised at this moment and denote

$$\Delta_s := \{\mathbf{y} = (y_0, \ldots, y_m) \in \Delta | y_i \geq s,\ 0 \leq i \leq m\}.$$

We have $\overline{K} \subset \overset{\circ}{\Delta}_s \subset \Delta_s \subset \overset{\circ}{\Delta}$. We decompose:

$$\overline{L}_n^{\overline{\rho},\mathbf{a}}(g,\mathbf{y}) = L_n^1(g,\mathbf{y}) + L_n^2(g,\mathbf{y}) + L_n^3(g,\mathbf{y}),\ \mathbf{y} \in \overline{K},$$

where

$$\begin{aligned}
L_n^1(g,\mathbf{y}) &= \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \frac{\int_{\Delta_s} g(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}, \\
L_n^2(g,\mathbf{y}) &= \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \frac{\int_{\Delta\setminus\Delta_s} g(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}, \\
L_n^3(g,\mathbf{y}) &= \sum_{\mathbf{k}\in I_n\setminus I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \frac{\int_{\Delta} g(\mathbf{u})\overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}.
\end{aligned}$$

From the hypothesis we can define

$$M := \int_\Delta |g(\mathbf{u})|\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}), \quad \eta := \min_{\mathbf{y}\in\Delta}\overline{\rho}(\mathbf{y}). \tag{15}$$

Choose an integer p such that $p \geq a_i$, for $0 \leq i \leq m$. There is a constant $C > 0$, independent of n and r, such that

$$\frac{M}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\overline{\rho}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})} \leq Cn^{mp+p+m}, \text{ for all } \mathbf{k} \in I_n. \tag{16}$$

Indeed, we have

$$\begin{aligned}
\frac{M}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\overline{\rho}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})} &\leq \frac{M}{\eta\binom{n}{\mathbf{k}}\int_\Delta \mathbf{u}^{\mathbf{a}+\mathbf{k}}\mathrm{d}\sigma(\mathbf{u})} \\
&\leq \frac{M}{\eta\binom{n}{\mathbf{k}}\int_\Delta \mathbf{u}^{p\mathbf{1}+\mathbf{k}}\mathrm{d}\sigma(\mathbf{u})} \\
&= \frac{M}{\eta\sqrt{m+1}\binom{n}{\mathbf{k}}B((p+1)\mathbf{1}+\mathbf{k})} \\
&= \frac{M}{\eta\sqrt{m+1}}\cdot\frac{(n+mp+p+m)!}{n!}\frac{k_0!}{(k_0+p)!}\cdots\frac{k_m!}{(k_m+p)!} \\
&\leq Cn^{mp+p+m}.
\end{aligned}$$

We have

$$I_n \setminus I_{n,r} = \bigcup_{0\leq j\leq m} J_j, \tag{17}$$

where

$$J_j = \{\mathbf{k} = (k_0, \dots, k_m) \in I_n |\ k_j < rn\}.$$

Since $\overline{p}_{n,\mathbf{k}}(\mathbf{u}) \leq 1$, $\mathbf{u} \in \Delta$ we obtain from relations (15), (16) and (17):

$$\begin{aligned}
\left|L_n^3(g,\mathbf{y})\right| &\leq \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y})\frac{M}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\overline{\rho}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})} \\
&\leq Cn^{mp+p+m}\sum_{j=0}^{m+1}\sum_{\mathbf{k}\in J_j}\overline{p}_{n,\mathbf{k}}(\mathbf{y}).
\end{aligned}$$

Set $\nu = [nr]$, where $[\cdot]$ denotes the integer part. Then

$$\begin{aligned}
\sum_{\mathbf{k}\in J_0}\overline{p}_{n,\mathbf{k}}(\mathbf{y}) &= \sum_{k_0=0}^{\nu}\sum_{k_1+\ldots+k_m=n-k_0}\overline{p}_{n,\mathbf{k}}(\mathbf{y}) \\
&= \sum_{k_0=0}^{\nu}\binom{n}{k_0}y_0^{k_0}\sum_{k_1+\ldots+k_m=n-k_0}\binom{n-k_0}{k_1\ldots k_m}y_1^{k_1}\ldots y_m^{k_m} \\
&= \sum_{k_0=0}^{\nu}\binom{n}{k_0}y_0^{k_0}(1-y_0)^{n-k_0}.
\end{aligned}$$

If $\mathbf{y} \in \overline{K}$, then $y_0 \geq q$. For $0 \leq k_0 \leq \nu$ we have $y_0 - \frac{k_0}{n} \geq q - r > \frac{q}{2}$. Then

$$\begin{aligned}\sum_{\mathbf{k}\in J_0} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) &\leq \frac{2}{q}\sum_{k_0=0}^{\nu}\binom{n}{k_0} y_0^{k_0}(1-y_0)^{n-k_0}\left(y_0 - \frac{k_0}{n}\right)\\ &= \frac{2}{q}\sum_{k_0=0}^{\nu}\left[\binom{n}{k_0} y_0^{k_0+1}(1-y_0)^{n-k_0} - \binom{n-1}{k_0-1} y_0^{k_0}(1-y_0)^{n-k_0}\right]\\ &= \frac{2}{q}\binom{n}{\nu} y_0^{\nu+1}(1-y_0)^{n-\nu}\\ &\leq \frac{2}{q}\binom{n}{\nu}(1-q)^{n-\nu}\end{aligned}$$

From the symmetry we obtain similar inequalities if we take an arbitrary set J_j, $0 \leq j \leq m$ instead of J_0. Hence

$$\left|L_n^3(g,\mathbf{y})\right| \leq C_1 n^{mp+p+m}\binom{n}{\nu}(1-q)^{n-\nu},$$

where $C_1 = \frac{2(m+1)C}{q}$ is independent on n and r. Set $t = \frac{n}{\nu}$ and $d = mp + m + p - \frac{1}{2}$. Hence $t > \frac{1}{r}$. Using the Stirling formula, there is a constant C_2 independent on t, ν and $\mathbf{y}$, and hence on n and r, such that

$$\begin{aligned}\left|L_n^3(g,\mathbf{y})\right| &\leq C_2 \nu^d \frac{t^{d+1}}{\sqrt{t-1}}\left[\frac{t^t(1-q)^{t-1}}{(t-1)^{t-1}}\right]^{\nu}\\ &\leq C_2 \nu^d \frac{t^{d+1}}{\sqrt{t-1}}\left[\mathrm{e}\cdot t(1-q)^{t-1}\right]^{\nu}.\end{aligned}$$

There is $t_0 > 1$, such that for any $t > t_0$, we have $\mathrm{e}\cdot t(1-q)^{t-1} < 1$ and hence

$$\lim_{\nu\to\infty} \nu^d \frac{t^{d+1}}{\sqrt{t-1}}\left[\mathrm{e}\cdot t(1-q)^{t-1)}\right]^{\nu} = 0.$$

Consequently, if we take $0 < r < \frac{1}{t_0}$, then

$$\lim_{n\to\infty} L_n^3(g,\mathbf{y}) = 0,\ \mathbf{y} \in \overline{K},\ \text{uniformly}. \tag{18}$$

In the next step we estimate $|L_n^2(g,\mathbf{y})|$, for $\mathbf{y} \in \overline{K}$, with r is fixed as above. First we have

$$|L_n^2(g,\mathbf{y})| \leq \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \frac{\int_{\Delta\setminus\Delta_s} |g(\mathbf{u})|\binom{n}{\mathbf{k}}\mathbf{u}^{\mathbf{a}+\mathbf{k}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_{\Delta} \overline{p}_{n,\mathbf{k}}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})} \tag{19}$$

We decompose

$$\Delta \setminus \Delta_s = \bigcup_{j=0}^{m} Q_j, \tag{20}$$

where

$$Q_j := \{\mathbf{y} = (y_0, \ldots, y_m) \in \Delta |\ y_j < s\}.$$

For $\mathbf{k} \in I_{n,r}$ we obtain

$$\int_{Q_j} |g(\mathbf{u})|\mathbf{u}^{\mathbf{a}+\mathbf{k}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}) \leq s^{k_j} \int_{Q_j} |g(\mathbf{u})|\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u}) \leq s^{k_j} M \leq s^{nr} M,$$

where M is defined in (15). Using also relations (16), (19), (20) we get

$$\begin{aligned} |L_n^2(g, \mathbf{y})| &\leq \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \sum_{j=0}^{m} \frac{\int_{Q_j} |g(\mathbf{u})|\binom{n}{\mathbf{k}}\mathbf{u}^{\mathbf{a}+\mathbf{k}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\overline{\rho}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})} \\ &\leq \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) \frac{(m+1)s^{nr}M\binom{n}{\mathbf{k}}}{\int_\Delta \overline{p}_{n,\mathbf{k}}(\mathbf{u})\overline{\rho}(\mathbf{u})\mathbf{u}^{\mathbf{a}}\mathrm{d}\sigma(\mathbf{u})} \\ &\leq \sum_{\mathbf{k}\in I_{n,r}} \overline{p}_{n,\mathbf{k}}(\mathbf{y}) C(m+1)n^{mp+p+m}s^{nr}\binom{n}{\mathbf{k}} \\ &\leq C(m+1)n^{mp+p+m}s^{nr}\max_{\mathbf{k}\in I_{n,r}}\binom{n}{\mathbf{k}}. \end{aligned}$$

Using the Stirling formula we have a constant $C_3 > 0$, such for any $\mathbf{k} \in I_n$ we have $\binom{n}{\mathbf{k}} \leq C_3 \frac{n^n}{k_0^{k_0}\ldots k_m^{k_m}}$ and from this we obtain $\binom{n}{\mathbf{k}} \leq C_3(m+1)^n$ and then

$$|L_n^2(g, \mathbf{y})| \leq CC_3 n^{mp+p+m}(m+1)^{n+1}s^{nr}.$$

Now we choose $0 < s < \frac{1}{2}q$, such that $(m+1)s^r < 1$. With this choice we obtain:

(21) $$\lim_{n\to\infty} L_n^2(g, \mathbf{y}) = 0,\ \mathbf{y} \in \overline{K},\ \text{uniformly}.$$

Finally, with r and s fixed as above, show that

(22) $$\lim_{n\to\infty} L_n^1(g, \mathbf{y}) = g(\mathbf{y}),\ \mathbf{y} \in \overline{K},\ \text{uniformly}.$$

We can regard L_n^1 as positive linear operators $L_n^1 : C(\Delta_s) \to C(\overline{K})$. We show that the sequence $(L_n^1)_n$ satisfies the conditions of Korovkin's theorem. Let $h \in \{\overline{e_0}, \overline{\pi}_0, \ldots, \overline{\pi}_m, \varphi\}$, where the notations are the same as in Theorem 2. We have:

$$L_n^1(h|_{\Delta_s}) = \overline{L}_n^{\overline{\rho},\mathbf{a}}(h) - L_n^2(h) - L_n^3(h).$$

From Theorem 2 we have

$$\lim_{n\to\infty} \overline{L}_n^{\overline{\rho},\mathbf{a}}(h) = h(\mathbf{y}),\ \mathbf{y} \in \Delta,\ \text{uniformly}$$

and consequently this limit exists unifomly for $\mathbf{y} \in \overline{K}$. Also, since, clearly h is continue on $\overset{\circ}{\Delta}$ and integral $\int_\Delta h(\mathbf{u})\mathbf{u}^{\mathbf{a}}\overline{\rho}(\mathbf{u})\mathrm{d}\sigma(\mathbf{u})$ exists, we can apply relations (18), (21), for g replaced by h. We obtain

$$\lim_{n\to\infty} L_n^1(h,\mathbf{y}) = h(\mathbf{y}),\ \mathbf{y} \in \overline{K},\ \text{uniformly.}$$

Applying the theorem of Korovkin we obtain (22). From relations (18), (21) and (22) it follows relation (13).

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Radu Păltănea
Transilvania University of Braşov
Department of Mathematics and Informatics,
Str. Maniu Iuliu, No. 50, Braşov, 500 091, Romania
e-mail: radupaltanea@yahoo.com

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Some remarks for relative interior in set-valued optimization [1]

Emilia-Loredana Pop

Abstract

Considering the notion of relative interior we formulate new set relations with the help of a convex cone introduced by D. Kuroiwa. Then by using the idea of A. Grad we deliver duality results and optimality conditions for a set-valued optimization problem.

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1 Introduction and preliminaries

The notion of relative interior is very important in the optimization theory and represents the interior which results when the set is regarded as a subset of its affine hull. For example some properties for the relative interior can be found in [3, 4, 5]. Let X be a separated locally convex space and $U \subseteq X$. The *relative interior* of U is

$$\operatorname{ri} U = \{x \in \operatorname{aff} U : \text{ there exists } \varepsilon > 0 \text{ such that } B(x,\varepsilon) \cap \operatorname{aff} U \subseteq U\},$$

where $B(x,\varepsilon)$ is the closed ball centered at x with radius ε in the Euclidean norm and $\operatorname{aff} U$ is the affine hull of U.

Lemma 1 *(D. Kuroiwa [4]) Let $U \subseteq X$, $x_0 \in U$, $\overline{x} \in \operatorname{ri} U$, $\alpha \in \mathbb{R}$ and $\lambda \in (0,1]$.*
(a) Then $\alpha \operatorname{ri} U = \operatorname{ri}(\alpha U)$;
(b) If U is convex, then $(1-\lambda)x_0 + \lambda\overline{x} \in \operatorname{ri} U$;
(c) If K is a convex cone of X, then $K + \operatorname{ri} K \subseteq \operatorname{ri} K$.

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From Lemma 1 follows that if K is a convex cone, then $\operatorname{ri} K \cup \{0\}$ is a convex cone, too. The separation theorem for relative interior follows.

Lemma 2 *(see [3]) Let $U \subseteq X$ be a closed-convex set with $\operatorname{ri} U \neq \emptyset$. If $0 \notin \operatorname{ri} U$, then there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x, x^* \rangle \geq 0$ for each $x \in U$.*

In finite dimensional spaces we have that the quasi interior and the relative interior for a set are equal (see, for example [1]).

In [2] were formulated some properties for quasi interior and were given duality results for qi −efficient solutions of a set-valued optimization problem. The *quasi interior* of $U \subseteq X$ is the set $\operatorname{qi} U = \{x \in U : \operatorname{cl}(\operatorname{cone}(U - x)) = X\}$.

Theorem 1 *Let U be a convex subset of a separated locally convex space X and let $x \in U$. Then $x \in \operatorname{qi} U$ if and only if $N_U(x) = \{0\}$, where $N_U(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0,$ for all $y \in U\}$.*

Next we introduce some set relations with respect to the relative interior with the help of a convex cone introduced by D. Kuroiwa and by using the idea of A. Grad. We formulate some properties for these set relations. By using the perturbation theory we give duality results for ri −efficient solutions attached to a set-valued optimization problem and also optimality conditions.

2 Set relations for relative interior

Let X be a topological vector space, Y be a separated locally convex space and $K \subset Y$ a pointed, convex cone with $\operatorname{ri} K \neq \emptyset$. We recall that $\mathcal{P}(Y) = \{A : A \subseteq Y\}$ and $\mathcal{P}_0(Y) = \{A : A \subseteq Y \text{ and } A \neq \emptyset\}$. Let $F : X \to \mathcal{P}(Y)$ a proper function, which means that $\operatorname{dom} F = \{x \in X : F(x) \neq \emptyset\} \neq \emptyset$, and also the set-valued optimization problem

$$(P) \qquad \operatorname*{Min}_{x \in X} F(x).$$

For a convex cone $K \subset X$, $A, B \in \mathcal{P}_0(Y)$, D. Kuroiwa in [4] introduced some set relations:

(i) $A \leq^l B$ if $B \subseteq A + K$;

(ii) $A \leq^u B$ if $A \subseteq B - K$;

(iii) $A \sim^l B$ if $A \leq^l B$ and $B \leq^l A$.

It was proved by D. Kuroiwa that $\sim^l$ is an equivalence relation on $\mathcal{P}_0(Y)$. Starting from here A. Grad [2] established some relations with respect to the quasi interior and proved some results. For $A, B \in \mathcal{P}_0(Y)$ the following statements are true (see [2]):

(a) $A \sim^l B$ if and only if $A + K = B + K$.

(b) If $A \sim^l B$ and $y \in Y$, then $A + y \sim^l B + y$.

In what follows we formulate some remarks on the idea of A. Grad but with respect to relative interior. First, we define the following relations:

(i) $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ if $B \subseteq A + \mathrm{ri}\,K$;

(ii) $A \trianglelefteq^u_{\mathrm{ri}\,K} B$ if $A \subseteq B - \mathrm{ri}\,K$;

and we remark that

(a) The relation $\trianglelefteq^l_{\mathrm{ri}\,K}$ is transitive, i.e. if $A, B, C \in \mathcal{P}_0(Y)$ satisfy $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ and $B \trianglelefteq^l_{\mathrm{ri}\,K} C$, then $A \trianglelefteq^l_{\mathrm{ri}\,K} C$. Indeed, since K is a convex cone, taking into consideration Lemma 1 (c) it follows that $\mathrm{ri}\,K + \mathrm{ri}\,K \subseteq \mathrm{ri}\,K$. Thus we obtain the following chain of inclusions $C \subseteq B + \mathrm{ri}\,K \subseteq A + \mathrm{ri}\,K + \mathrm{ri}\,K \subseteq A + \mathrm{ri}\,K$. This mean that $A \trianglelefteq^l_{\mathrm{ri}\,K} C$.

(b) The relation $\trianglelefteq^u_{\mathrm{ri}\,K}$ is also transitive and the proof is similar to the proof of (a).

The following proposition present some properties of the relations introduced above and some connections between $\trianglelefteq^l_{\mathrm{ri}\,K}$, $\trianglelefteq^u_{\mathrm{ri}\,K}$ and $\sim^l$ that are used in the proofs of the weak and strong duality theorems.

Proposition 1 *Let $A, B \in \mathcal{P}_0(Y)$. Then the following statements are true:*

(a) *If $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ and $B \trianglelefteq^l_{\mathrm{ri}\,K} A$, then $A \sim^l B$.*

(b) *If $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ and $B \leq^l A$, then $B \trianglelefteq^l_{\mathrm{ri}\,K} A$.*

(c) *$A \trianglelefteq^l_{\mathrm{ri}\,K} B$ if and only if $-B \trianglelefteq^u_{\mathrm{ri}\,K} -A$.*

(d) *If $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ and $y \in Y$, then $A + y \trianglelefteq^l_{\mathrm{ri}\,K} B + y$.*

Proof. (a) From $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ it follows that $B \subseteq A + \mathrm{ri}\,K \subseteq A + K$, i.e. $A \leq^l B$. Similarly, from $B \trianglelefteq^l_{\mathrm{ri}\,K} A$ we have that $A \subseteq B + \mathrm{ri}\,K \subseteq B + K$, i.e. $B \leq^l A$. Consequently, $A \sim^l B$.

(b) From $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ we have that $B \subseteq A + \mathrm{ri}\,K$, and from $B \leq^l A$ we have that $A \subseteq B + K$. Then, by considering Lemma 1 (c) we obtain the following chain of inclusions $A \subseteq B + K \subseteq A + \mathrm{ri}\,K + K \subseteq A + \mathrm{ri}\,K \subseteq B + K + \mathrm{ri}\,K \subseteq B + \mathrm{ri}\,K$. This mean that $B \trianglelefteq^l_{\mathrm{ri}\,K} A$.

(c) We have that $B \subseteq A + \mathrm{ri}\,K$ if and only if $-B \subseteq -A - \mathrm{ri}\,K$. This means that $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ is equivalent to $-B \trianglelefteq^u_{\mathrm{ri}\,K} -A$.

(d) From $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ we have that $B \subseteq A + \mathrm{ri}\,K$. Let now $y \in Y$ and we obtain in the previous relation that $B + y \subseteq A + y + \mathrm{ri}\,K$, which is nothing else than $A + y \trianglelefteq^l_{\mathrm{ri}\,K} B + y$.

Remark 1 *In Proposition 1 (a) the converse affirmation does not hold, i.e. $A \sim^l B \not\Rightarrow A \trianglelefteq^l_{\mathrm{ri}\,K} B$ and $B \trianglelefteq^l_{\mathrm{ri}\,K} A$. Indeed, for $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ one has that $\mathrm{ri}\,\mathbb{R}_+ = (0, +\infty)$. Choosing $A = \{-1\}$ and $B = [-1, 0)$ we have that $A + K = [-1, +\infty) = B + K$, so $A \sim^l B$. But, $A \nsubseteq B + \mathrm{ri}\,\mathbb{R}_+ = (-1, +\infty)$ and $B \nsubseteq A + \mathrm{ri}\,\mathbb{R}_+ = (-1, +\infty)$.*

Using the relations introduced above, one can define new efficiency notions for sets.

Definition 1 *Let $S \subseteq \mathcal{P}_0(Y)$ and $A \in S$. The set A is called*

(i) *$l - \mathrm{Min}_{\mathrm{ri}}$**-efficient set** of S if for each set $B \in S$ with $B \trianglelefteq^l_{\mathrm{ri}\,K} A$ the relation $A \trianglelefteq^l_{\mathrm{ri}\,K} B$ holds.*

(ii) *$u - \mathrm{Min}_{\mathrm{ri}}$**-efficient set** of S if for each set $B \in S$ with $B \trianglelefteq^u_{\mathrm{ri}\,K} A$ the relation $A \trianglelefteq^u_{\mathrm{ri}\,K} B$ holds.*

Remark 2 (a) *In the same manner as in the previous definition one can introduce the notions of* $l-\mathrm{Max}_{\mathrm{ri}}$***-efficient set*** *and* $u-\mathrm{Max}_{\mathrm{ri}}$***-efficient set****.*

(b) *The sets of all* $l-\mathrm{Min}_{\mathrm{ri}}$***-efficient****,* $u-\mathrm{Min}_{\mathrm{ri}}$***-efficient****,* $l-\mathrm{Max}_{\mathrm{ri}}$***-efficient*** *and* $u-\mathrm{Max}_{\mathrm{ri}}$***-efficient*** *are denoted by* $l-\mathrm{Min}_{\mathrm{ri}}\, S$*,* $u-\mathrm{Min}_{\mathrm{ri}}\, S$*,* $l-\mathrm{Max}_{\mathrm{ri}}\, S$ *and* $u-\mathrm{Max}_{\mathrm{ri}}\, S$*.*

(c) *From the previous definition and Proposition 1 (d) it follows that* $y+l-\mathrm{Min}_{\mathrm{ri}}\, S = l-\mathrm{Min}_{\mathrm{ri}}(y+S)$ *for all* $y \in Y$ *and similar for* $u-\mathrm{Min}_{\mathrm{ri}}\, S$*,* $l-\mathrm{Max}_{\mathrm{ri}}\, S$ *and* $u-\mathrm{Max}_{\mathrm{ri}}\, S$*.*

Proposition 2 *Let* $S \subseteq \mathcal{P}_0(Y)$ *and* $-S = \{-A: \ A \in S\}$*. Then* $l-\mathrm{Min}_{\mathrm{ri}}(-S) = -u-\mathrm{Max}_{\mathrm{ri}}\, S$*.*

The proof follows the lines of the corresponding one from [2] but with respect to relative interior.

Next, we introduce the conjugate function of F in the same manner as in [2]. The $\mathrm{ri}-$conjugate function of F is the set-valued function $F^*_{\mathrm{ri}\,K} : \mathcal{L}(X,Y) \to \mathcal{P}(\mathcal{P}(Y))$ defined by

$$F^*_{\mathrm{ri}\,K}(x^*) = u-\underset{\mathrm{ri}}{\mathrm{Max}}\{\langle x^*, x\rangle - F(x): \ x \in X\} \text{ for all } x^* \in \mathcal{L}(X,Y).$$

From Definition 1 one has that $F^*_{\mathrm{ri}\,K}(x^*) = u-\mathrm{Max}_{\mathrm{ri}}\{\langle x^*, x\rangle - F(x): \ x \in \mathrm{dom}\, F\}$. By this definition and the one of $\trianglelefteq^l_{\mathrm{ri}\,K}$ we obtain an extension of the Young-Fenchel inequality.

Theorem 2 *Let us consider* $x_0, x_1 \in \mathrm{dom}\, F$ *and* $x^* \in \mathcal{L}(X,Y)$ *such that*

$$F(x_1) - \langle x^*, x_1\rangle \in -F^*_{\mathrm{ri}\,K}(x^*). \tag{1}$$

Then the following holds

(i) *If* $F(x_0)-\langle x^*, x_0\rangle \trianglelefteq^l_{\mathrm{ri}\,K} F(x_1)-\langle x^*, x_1\rangle$*, then* $F(x_1)-\langle x^*, x_1\rangle \trianglelefteq^l_{\mathrm{ri}\,K} F(x_0)-\langle x^*, x_0\rangle$*.*

(ii) *If* $F(x_0)-\langle x^*, x_0\rangle \trianglelefteq^l_{\mathrm{ri}\,K} F(x_1)-\langle x^*, x_1\rangle$*, then* $F(x_1)-\langle x^*, x_1\rangle \sim^l_{\mathrm{ri}\,K} F(x_0)-\langle x^*, x_0\rangle$*.*

The proof of this theorem follows the lines of the proof given by A. Grad in [2], but with respect to relative interior.

The notions of the subgradient and subdifferential of the set-valued functions follows similarly as in the case of $\mathrm{qi}-$subgradients from [2].

Definition 2 *Let* $\overline{x} \in \mathrm{dom}\, F$*. An operator* $x^* \in \mathcal{L}(X,Y)$ *is called* $\mathrm{ri}-$***subgradient*** *of the set-valued function* F *at* $\overline{x}$ *if* $\langle x^*, \overline{x}\rangle - F(\overline{x}) \in F^*_{\mathrm{ri}\,K}(x^*)$*. The set of all* $\mathrm{ri}-$*subgradients of* F *at* $\overline{x}$ *is denoted by* $\partial_{\mathrm{ri}\,K}F(\overline{x})$*. By convention, if* $\overline{x} \notin \mathrm{dom}\, F$ *then* $\partial_{\mathrm{ri}\,K}F(\overline{x}) = \emptyset$*.*

The condition (1) from Theorem 2 can be equivalently rewritten as $x^* \in \partial_{\mathrm{ri}\,K}F(x_1)$.

Proposition 3 *Let* $\overline{x} \in \mathrm{dom}\, F$*. Then*

$$F(\overline{x}) \in l-\underset{\mathrm{ri}}{\mathrm{Min}}\{F(x): \ x \in X\} \textit{ if and only if } 0 \in \partial_{\mathrm{ri}\,K}F(\overline{x}).$$

For proving this proposition one can see the idea from [2].

3 Perturbation approach via relative interior

In this section we consider an unconstrained, a constrained and a Lagrange set-valued optimization problems and we formulate duality statements.

3.1 Unconstrained set-valued optimization

Let us consider the unconstrained set-valued optimization problem

$$(P_{\mathrm{ri}}) \qquad l - \operatorname*{Min}_{x\in X} F(x)$$

and W a topological vector space.

An element $\overline{x} \in \operatorname{dom} F$ is called ri −*efficient solution* to (P_{ri}) if $F(\overline{x}) \in l - \mathrm{Min}_{\mathrm{ri}}\{F(x) : x \in \operatorname{dom} F\}$.

The perturbation function $\Phi : X\times W \to \mathcal{P}(Y)$ is defined by $\Phi(x,0) = F(x)$ for all $x \in X$ and the ri −conjugate function of Φ is $\Phi^*_{\mathrm{ri}\,K} : \mathcal{L}(X,Y)\times\mathcal{L}(W,Y) \to \mathcal{P}(\mathcal{P}(Y))$ defined by $\Phi^*_{\mathrm{ri}\,K}(y^*,x^*) = u - \mathrm{Max}_{\mathrm{ri}}\{\langle y^*,x\rangle + \langle x^*,w\rangle - \Phi(x,w) :\ (x,w) \in X\times W\}$ for all $(y^*,x^*) \in \mathcal{L}(X,Y)\times\mathcal{L}(W,Y)$. The set-valued dual problem associated to (P_{ri}) is

$$(D_{\mathrm{ri}}) \qquad l - \operatorname*{Max}_{x^*\in\mathcal{L}(W,Y)} [-\Phi^*_{\mathrm{ri}\,K}(0,x^*)].$$

We denote by $\mathcal{A}_{D_{\mathrm{ri}}} = \{(x^*,x,w) \in \mathcal{L}(W,Y)\times \operatorname{dom}\Phi : -\langle x^*,w\rangle + \Phi(x,w) \in -\Phi^*_{\mathrm{ri}\,K}\ (0,x^*)\} = \{(x^*,x,w) \in \mathcal{L}(W,Y)\times X\times W : (0,x^*) \in \partial_{\mathrm{ri}\,K}\Phi(x,w)\}$ the set of the feasible solutions.

An element $\widetilde{x^*} \in \mathcal{L}(W,Y)$ is called ri −*efficient solution* to (D_{ri}) if there exists an $(\widetilde{x},\widetilde{w}) \in \operatorname{dom}\Phi$ such that $(\widetilde{x^*},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{D_{\mathrm{ri}}}$ and $-\langle\widetilde{x^*},\widetilde{w}\rangle + \Phi(\widetilde{x},\widetilde{w}) \in l - \mathrm{Max}_{\mathrm{ri}}\{-\langle x^*,w\rangle + \Phi(x,w) : (x^*,x,w) \in \mathcal{A}_{D_{\mathrm{ri}}}\}$.

The set-valued weak duality for problems (P_{ri}) and (D_{ri}) holds in the same manner as in [2], but with respect to the relative interior.

Theorem 3 *Let $x_0 \in \operatorname{dom} F$ and $(x^*,x,w) \in \mathcal{A}_{D_{\mathrm{ri}}}$. The following statements are true*

(*i*) *If $F(x_0) \trianglelefteq^l_{\mathrm{ri}\,K} -\langle x^*,w\rangle + \Phi(x,w)$, then $-\langle x^*,w\rangle + \Phi(x,w) \trianglelefteq^l_{\mathrm{ri}\,K} F(x_0)$.*

(*ii*) *If $F(x_0) \trianglelefteq^l_{\mathrm{ri}\,K} -\langle x^*,w\rangle + \Phi(x,w)$, then $-\langle x^*,w\rangle + \Phi(x,w) \sim^l F(x_0)$.*

Proof. (*i*) We have that $F(x_0) = \Phi(x_0,0) = -\langle x^*,0\rangle + \Phi(x_0,0)$ and from the hypothesis that $-\langle x^*,0\rangle + \Phi(x_0,0) \trianglelefteq^l_{\mathrm{ri}\,K} -\langle x^*,w\rangle + \Phi(x,w)$, for all $w \in W$. This last relation is equivalent with $-(0,x^*)(x_0,0) + \Phi(x_0,0) \trianglelefteq^l_{\mathrm{ri}\,K} -(0,x^*)(x,w) + \Phi(x,w)$. From $(x^*,x,w) \in \mathcal{A}_{D_{\mathrm{ri}}}$ follows that $(0,x^*) \in \partial_{\mathrm{ri}\,K}\Phi(x,w)$. For the function Φ and the linear continuous operator $(0,x^*) \in \mathcal{L}(X,Y)\times\mathcal{L}(W,Y)$ follows from Theorem 2 (*i*) that $-(0,x^*)(x,w) + \Phi(x,w) \trianglelefteq^l_{\mathrm{ri}\,K} -(0,x^*)(x_0,0) + \Phi(x_0,0)$. This means that $-\langle x^*,w\rangle + \Phi(x,w) \trianglelefteq^l_{\mathrm{ri}\,K} F(x_0)$.

(*ii*) Follows in a similar manner by using Theorem 2 (*ii*).

Some optimality conditions for the set-valued optimization problems (P_{ri}) and (D_{ri}) are given below and the proof follows the lines of the coresponding ones from [2].

Theorem 4 *Let* $\overline{x} \in \operatorname{dom} F$ *and* $(\widetilde{x^*}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D_{\text{ri}}}$ *such that*

$$F(\overline{x}) \trianglelefteq^l_{\text{ri}\,K} -\langle \widetilde{x^*}, \widetilde{w} \rangle + \Phi(\widetilde{x}, \widetilde{w}).$$

Then the following statements are fulfilled

(i) $\overline{x}$ *is a* ri *−efficient solution to* (P_{ri}).

(ii) $\widetilde{x^*}$ *is a* ri *−efficient solution to* (D_{ri}).

Theorem 5 *Let* $\overline{x} \in \operatorname{dom} F$. *If there exists an element* $\overline{x} \in \mathcal{L}(W,Y)$ *such that* $(\overline{x^*}, \overline{x}, 0) \in \mathcal{A}_{D_{\text{ri}}}$, *then* $\overline{x^*}$ *is a* ri *−efficient solution to* (D_{ri}).

Remark 3 *Let* $\overline{x} \in \operatorname{dom} F$. *Each element* $\widetilde{x^*} \in \mathcal{L}(W,Y)$ *for which there exists an* $(\widetilde{x}, \widetilde{w}) \in \operatorname{dom} \Phi$ *such that* $(\widetilde{x^*}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D_{\text{ri}}}$ *and* $F(\overline{x}) \trianglelefteq^l_{\text{ri}\,K} -\langle \widetilde{x^*}, \widetilde{w} \rangle + \Phi(\widetilde{x}, \widetilde{w})$ *is a* ri *−efficient solution to* (D_{ri}).

3.2 Constrained set-valued optimization

We consider the general set-valued optimization problem with cone constraints

$$(PC_{\text{ri}}) \qquad l - \operatorname*{Min}_{G(x)\cap(-C)\neq\emptyset} F(x),$$

where X and W are topological vector spaces, Y and Z are separated locally convex spaces, $K \subset Y$ is a pointed, convex cone with $\text{ri}\,K \neq \emptyset$, $C \subset Z$ is a nonempty, pointed and convex cone, $F : X \to \mathcal{P}(Y)$ and $G : X \to \mathcal{P}(Z)$ are proper set-valued functions and $\{x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) : G(x) \cap (-C) \neq \emptyset\} \neq \emptyset$. The set of the feasible solutions is given by $\mathcal{A}_{PC_{\text{ri}}} = \{x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) : G(x) \cap (-C) \neq \emptyset\}$.

Let D, E vector spaces and $M \subseteq D$. The *indicator set-valued function* $\Delta^E_M(x) : D \to \mathcal{P}(E)$ associated to the set M with respect to the space E is defined by $\{0\}$ if $x \in M$ or $\emptyset$ if $x \notin M$. Then problem (PC_{ri}) becomes

$$l - \operatorname*{Min}_{x\in X}[F(x) + \Delta^Y_{\mathcal{A}_{PC_{\text{ri}}}}(x)].$$

The perturbation function $\Phi_C : X \times W \to \mathcal{P}(Y)$ is defined by $\Phi_C(x,0) = F(x) + \Delta^Y_{\mathcal{A}_{PC_{\text{ri}}}}(x)$ for all $x \in X$ and the set-valued dual problem associated to (PC_{ri}) is

$$(DC_{\text{ri}}) \qquad l - \operatorname*{Max}_{x^*\in\mathcal{L}(W,Y)} [-(\Phi_C)^*_{\text{ri}\,K}(0, x^*)].$$

The set of the feasible solutions is given by $\mathcal{A}_{DC_{\text{ri}}} = \{(x^*, x, w) \in \mathcal{L}(W,Y) \times \operatorname{dom} \Phi_C : -\langle x^*, w \rangle + \Phi(x,w) \in -(\Phi_C)^*_{\text{ri}\,K}(0,x^*)\} = \{(x^*, x, w) \in \mathcal{L}(W,Y) \times X \times W : (0, x^*) \in \partial_{\text{ri}\,K} \Phi_C(x,w)\}$.

An element $\widetilde{x^*} \in \mathcal{L}(W,Y)$ is called ri $-$*efficient solution* to (DC_{ri}) if there exists an $(\widetilde{x},\widetilde{w}) \in \operatorname{dom}\Phi_C$ such that $(\widetilde{x^*},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{DC_{\text{ri}}}$ and $-\langle\widetilde{x^*},\widetilde{w}\rangle + \Phi_C(\widetilde{x},\widetilde{w}) \in l-\text{Max}_{\text{ri}}\{-\langle x^*,w\rangle + \Phi_C(x,w) : (x^*,x,w) \in \mathcal{A}_{DC_{\text{ri}}}\}$.

By particularizing the results from the previous subsection we obtain the weak duality and some optimality conditions.

Theorem 6 *Let $x_0 \in \mathcal{A}_{PC_{\text{ri}}}$ and $(x^*,x,w) \in \mathcal{A}_{DC_{\text{ri}}}$. The following statements are true*

(i) If $F(x_0) \trianglelefteq^l_{\text{ri}\,K} -\langle x^*,w\rangle + \Phi_C(x,w)$, then $-\langle x^*,w\rangle + \Phi_C(x,w) \trianglelefteq^l_{\text{ri}\,K} F(x_0)$.

(ii) If $F(x_0) \trianglelefteq^l_{\text{ri}\,K} -\langle x^*,w\rangle + \Phi_C(x,w)$, then $-\langle x^*,w\rangle + \Phi_C(x,w) \sim^l F(x_0)$.

Theorem 7 *Let $\overline{x} \in \mathcal{A}_{PC_{\text{ri}}}$ and $(\widetilde{x^*},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{DC_{\text{ri}}}$ such that $F(\overline{x}) \trianglelefteq^l_{\text{ri}\,K} -\langle\widetilde{x^*},\widetilde{w}\rangle + \Phi_C(\widetilde{x},\widetilde{w})$. Then the following statements are fulfilled*

(i) $\overline{x}$ is a ri $-$*efficient solution to (PC_{ri}).*

(ii) $\widetilde{x^*}$ is a ri $-$*efficient solution to (DC_{ri}).*

Theorem 8 *Let $\overline{x} \in \mathcal{A}_{PC_{\text{ri}}}$. If there exists an element $\overline{x} \in \mathcal{L}(W,Y)$ such that $(\overline{x^*},\overline{x},0) \in \mathcal{A}_{DC_{\text{ri}}}$, then $\overline{x^*}$ is a* ri $-$*efficient solution to (DC_{ri}).*

Remark 4 *Let $\overline{x} \in \mathcal{A}_{PC_{\text{ri}}}$. Each element $\widetilde{x^*} \in \mathcal{L}(W,Y)$ for which there exists an $(\widetilde{x},\widetilde{w}) \in \operatorname{dom}\Phi_C$ such that $(\widetilde{x^*},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{DC_{\text{ri}}}$ and $F(\overline{x}) \trianglelefteq^l_{\text{ri}\,K} -\langle\widetilde{x^*},\widetilde{w}\rangle + \Phi_C(\widetilde{x},\widetilde{w})$ is a* ri $-$*efficient solution to (DC_{ri}).*

Lagrange set-valued optimization

Here we attach to the general set-valued optimization problem with cone constraints (PC_{ri}) a dual problem obtained by particularizing the perturbation function to be the classical Lagrange approach from the scalar case. Let consider the Lagrange-type perturbation function $\Phi_L(x,z) : X \times Z \to \mathcal{P}(Y)$ associated to problem (PC_{ri}) defined by

$$\Phi_L(x,z) = \begin{cases} F(x) & \text{if } x \in X \text{ and } (G(x)-z) \cap (-C) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $x^* \in \mathcal{L}(Z,Y)$. Then the ri $-$conjugate function of Φ_L is $(\Phi_L)^*_{\text{ri}\,K}(0,x^*) = u-\text{Max}_{\text{ri}}\{\langle x^*,z\rangle - F(x) : x \in Z, z \in G(x)+C\}$. The set-valued Lagrange-type dual problem associated to (PC_{ri}) is

$$(DCL_{\text{ri}}) \qquad\qquad l-\operatorname*{Max}_{x^*\in\mathcal{L}(Z,Y)}[-(\Phi_L)^*_{\text{ri}\,K}(0,x^*)].$$

The set of the feasible solutions is given by $\mathcal{A}_{DCL_{\text{ri}}} = \{(x^*,x,z) \in \mathcal{L}(Z,Y), x \in X, z \in G(x)+C, -\langle x^*,z\rangle + F(x) \in -(\Phi_L)^*_{\text{ri}\,K}(0,x^*)\} = \{(x^*,x,z) \in \mathcal{L}(Z,Y), x \in X, z \in G(x)+C, (0,x^*) \in \partial_{\text{ri}\,K}\Phi_L(x,z)\}$.

An element $\widetilde{x^*} \in \mathcal{L}(Z,Y)$ is called ri $-$*efficient solution* to (DCL_{ri}) if there exists an $(\widetilde{x},\widetilde{z}) \in \operatorname{dom}\Phi_L$ such that $(\widetilde{x^*},\widetilde{x},\widetilde{z}) \in \mathcal{A}_{DCL_{\text{ri}}}$ and $-\langle\widetilde{x^*},\widetilde{z}\rangle + F(\widetilde{x}) \in l-\text{Max}_{\text{ri}}\{-\langle x^*,z\rangle + F(x) : (x^*,x,z) \in \mathcal{A}_{DCL_{\text{ri}}}\}$.

The set-valued strong duality for problems (PC_{ri}) and (DCL_{ri}) holds.

Theorem 9 *Let (F,G) be $K\times C$ convex function, $(F,G)(X)+K\times C$ be a closed set and K closed cone. Moreover let $0\notin \operatorname{ri}K\cap K$, $\operatorname{ri}[(F,G)(X)+K\times C]\neq\emptyset$, $(0,0)\notin \operatorname{ri}[(F,G)(X)+K\times C]$, $0\in \operatorname{qi}[G(X)+C]$ and $\overline{x}\in\mathcal{A}_{PC_{\rm ri}}$ such that $0\in F(\overline{x})$. Then there exists an element $\overline{x^*}\in\mathcal{L}(Z,Y)$ such that $\overline{x^*}$ is a $\operatorname{ri}-$efficient solution to the dual problem $(DCL_{\rm ri})$.*

Proof. Since $\overline{x}\in\mathcal{A}_{PC_{\rm ri}}$, there exists at least one $\overline{z}\in G(\overline{x})\cap(-C)$. As (F,G) is a $K\times C-$convex function, it follows that the set $(F,G)(X)+K\times C$ is nonempty and convex and from the hypotheses it is also closed. From the hypotheses $\operatorname{ri}[(F,G)(X)+K\times C]\neq\emptyset$ and $(0,0)\notin\operatorname{ri}[(F,G)(X)+K\times C]$ and by applying Lemma 2 we obtain that there exists an $(y^*,z^*)\in Y^*\times Z^*\setminus\{(0,0)\}$ such that

$$(2)\qquad \langle y^*,y\rangle+\langle z^*,z\rangle\geq 0,\ \text{ for each } (y,z)\in(F,G)(X)+K\times C.$$

Next we prove that $y^*\in K^*$, where by definition $K^*=\{x^*\in X^*:\langle x^*,x\rangle\geq 0 \text{ for all } x\in X\}$. Since $\overline{z}\in G(\overline{x})\cap(-C)$ follows that $0\in G(\overline{x})+C$. We let $x=\overline{x}$ and $z=0$ in (2) and obtain that $\langle y^*,y\rangle\geq 0,$ for each $y\in F(\overline{x})+K$. Since $0\in F(\overline{x})$ we can particularize the inequality above to $\langle y^*,0+k\rangle\geq 0,$ for each $k\in K$ and consequently $\langle y^*,k\rangle\geq 0,$ for each $k\in K$, i.e. $y^*\in K^*$.

Now we prove that $z^*\in C^*$. Let $x=\overline{x}$ and $y=0$ and in (2) we obtain $\langle z^*,z\rangle\geq 0,$ for each $z\in G(\overline{x})+C$. Let us consider $c\in C$ an arbitrary point. From $\overline{z}\in G(\overline{x})\cap(-C)$ we obtain that $c=c+\overline{z}-\overline{z}\in C+G(\overline{x})-(-C)=G(\overline{x})+C+C\subseteq G(\overline{x})+C$. So, the previous inequality becomes $\langle z^*,c\rangle\geq 0,$ for each $c\in C$, i.e. $z^*\in C^*$.

Next we prove that $y^*\neq 0$. Assume that $y^*=0$ and (2) becomes $\langle z^*,z\rangle\geq 0$ for each $z\in G(X)+C$. Hence $-z^*\in N_{G(X)+C}(0)$. From the hypothesis $0\in\operatorname{qi}[G(X)+C]$ and from Theorem 1 relation (2) implies $N_{G(X)+C}(0)=\{0\}$ and therefore $-z^*=0$. But $(y^*,z^*)=(0,0)$ contradicts the choose of $(y^*,z^*)\in Y^*\times Z^*\setminus\{(0,0)\}$. So, $y^*\neq 0$ and from $y^*\in K^*$ follows that $y^*\in K^*\setminus\{0\}$. This means that we can choose $\overline{k}\in K$ such that $\langle\overline{k},y^*\rangle=1$ and we define the element $\overline{x^*}:Z\to Y$ by $\overline{x^*}(z)=\langle z^*,z\rangle(-\overline{k})$ for all $z\in Z$ and we prove that $(\overline{x}^*,\overline{x},0)\in\mathcal{A}_{DCL_{\rm ri}}$. We proceed by contradiction. We assume that there exists $x\in(\operatorname{dom}F)\cap(\operatorname{dom}G)$ and $z\in G(x)+C$ such that $-\langle\overline{x^*},z\rangle+F(x)\trianglelefteq^l_{\operatorname{ri}K}-\langle\overline{x^*},0\rangle+F(\overline{x})$. This means that $F(\overline{x})\subseteq F(x)-\langle\overline{x^*},z\rangle+\operatorname{ri}K$. Thus, from the hypothesis $0\in F(\overline{x})$ follows that there exist $y\in F(x)$ and $k\in\operatorname{ri}K$ such that $0=y-\langle\overline{x}^*,z\rangle+k\in Y$. Since y^* is a linear operator and $\langle y^*,\overline{k}\rangle=1$ we get $\langle y^*,0\rangle=\langle y^*,y\rangle-\langle y^*,\langle z^*,z\rangle(-\overline{k})\rangle+\langle y^*,k\rangle=\langle y^*,y\rangle+\langle z^*,z\rangle\langle y^*,\overline{k}\rangle+\langle y^*,k\rangle=\langle y^*,y\rangle+\langle z^*,z\rangle+\langle y^*,k\rangle$. From $k\in\operatorname{ri}K$ follows that $\operatorname{ri}K\neq\emptyset$. Also we have the hypothesis $0\notin\operatorname{ri}K\cap K$, so $0\notin\operatorname{ri}K$ and K closed convex cone. Using these for $y^*\in K^*\setminus\{0\}$ we can apply Lemma 2 and we obtain that $\langle y^*,k\rangle\geq 0$ for each $k\in K$. But $y^*\in K^*\setminus\{0\}$ and from $0\notin\operatorname{ri}K\cap K$ follows that $0\notin K$ and the previous relation becomes equivalently with $\langle y^*,k\rangle>0$. Therefore we have the conclusion that

$$(3)\qquad 0>\langle y^*,y\rangle+\langle z^*,z\rangle.$$

Considering the fact that $x\in(\operatorname{dom}F)\cap(\operatorname{dom}G)$, $y\in F(x)\subseteq F(x)+K$ and $z\in G(x)+C$, it follows that (3) is in contradiction to (2). This leads to $(\overline{x^*},\overline{x},0)\in$

$\mathcal{A}_{DCL_{\text{ri}}}$. From here by applying Theorem 8 follows that $\overline{x^*}$ is a ri $-$efficient solution to (DCL_{ri}).

Remark 5 *One can also consider the case of Fenchel approach or Fenchel-Lagrange approach and construct the corresponding dual problem associated to (PC_{ri}) and deliver the set-valued strong duality.*

4 Conclusions

We introduced here some new set relations with respect to the relative interior. For an unconstrained and then constrained set-valued optimization problem we formulated duality results and optimality conditions.

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Emilia-Loredana Pop
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
1 Mihail Kogălniceanu, 400084 Cluj-Napoca, România
e-mail: pop_emilia_loredana@yahoo.com

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An extension of a Flett's mean value theorem [1]

Emil C. Popa

Abstract

We present a new extension of Flett's mean value theorem with a Trahan-type condition.

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Key words and phrases: Flett's mean value theorem, real function, Cauchy theorem.

1 Introduction

Mean value theorems play an important role in mathematical analysis.

Theorem 1 *(Flett's mean value theorem, 1958)*
If $f : [a,b] \to \mathbb{R}$ is a differentiable function on $[a,b]$ and $f'(a) = f'(b)$, there exist a number $c \in (a,b)$, such that

$$f'(c) = \frac{f(c) - f(a)}{c - a}. \tag{1}$$

Flett's original proof (see[4]) uses the Rolle mean value theorem and the geometric interpretation states that if the curve $y = f(x)$ has a tangent at each point in $[a,b]$ and if the tangents at $(a, f(a))$ and $(b, f(b))$ are parallel, then there exists a point $c \in (a,b)$ such that the tangent at $(c, f(c))$ passes through the point $(a, f(a))$.

An interesting case is in the following

Theorem 2 *(T. Riedel-P.K. Sahoo, 2001) If $f : [a,b] \to \mathbb{R}$ is a differentiable function on $[a,b]$, then there exists a number $c \in (a,b)$ such that*

$$f'(c) = \frac{f(c) - f(a)}{c - a} + \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a) \tag{2}$$

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In recent years some extensions and generalizations of Flett's theorem were obtained (see[1],[2],[3],[5],[6],[7],[8]). For example

Theorem 3 *(I. Pawlikowska,1999) Let f be n-times differentiable on $[a,b]$ and $f^{(n)}(a) = f^{(n)}(b)$. Then there exists $c \in (a,b)$ such that*

$$f(c) - f(a) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}(c-a)^k f^{(k)}(c) \tag{3}$$

For $n = 1$ we obtain the Flett's relation (1).

Theorem 4 *(U. Abel, M. Ivan, T. Riedel, 2004) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and possesses derivatives of order n at a and b such that $f^{(n)}(a) = f^{(n)}(b)$, then there exists $c \in (a,b)$ such that in any neighborhood of the point c there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with*

$$[a, c_0, \ldots, c_n; f] = 0 \tag{4}$$

where $[a, c_0, \ldots, c_n; f]$ is the divided difference of f on the knots $a, c_0, \ldots, c_n$.

In the special case $n = 1$, we obtain that if $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable at a and b with $f'(a) = f'(b)$, then there exist two distinct points $a_1, b_1 \in (a,b)$ such that

$$[a, a_1, b_1; f] = 0 \tag{5}$$

Theorem 5 *(J. Molnarova, 2011) Let f, g be n-times differentiable on $[a,b]$ and $g^{(n)} \neq g^{(n)}(b)$. Then there exists $c \in (a,b)$ such that*

$$f(a) - T_n(f,c)a = \frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)} \left[g(a) - T_n(g,c)(a)\right] \tag{6}$$

where

$$T_n(f,x_0)(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is the n-th Taylor polynomial of f.

For $n = 1$ we obtain

$$f'(c) - \frac{f(c)-f(a)}{c-a} = \frac{f'(b)-f'(a)}{g'(b)-g'(a)} \left[g'(c) - \frac{g(c)-g(a)}{c-a}\right] \tag{7}$$

In 1966 D.H. Trahan replace the condition $f'(a) = f'(b)$ of the Flett's theorem with the following

$$\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] \cdot \left[f'(b) - \frac{f(b)-f(a)}{b-a}\right] \geq 0 \tag{8}$$

In this paper we present an extension (in the Cauchy sense) for the Flett's theorem with the Trahan-type condition.

In a particular case we obtain a new proof for the Flett's theorem.

2 Main result

Theorem 6 *If $f, g : [a, b] \to \mathbb{R}$ are differentiable on $[a, b]$, $g'(x) > 0$ on $[a, b]$ and*

$$\left[\frac{f'(a)}{g'(a)} - \frac{f(b) - f(a)}{g(b) - g(a)}\right] \cdot \left[\frac{f'(b)}{g'(b)} - \frac{f(b) - f(a)}{g(b) - g(a)}\right] \geq 0 \tag{9}$$

then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(c) - f(a)}{g(c) - g(a)} \quad or \quad \frac{f'(c)}{g'(c)} = \frac{f(c) - f(b)}{g(c) - g(b)}. \tag{10}$$

For $g(x) = x$ we obtain the Flett's theorem with the Trahan condition.
Proof. Case 1. If there is $c_1 \in (a, b)$ such that

$$\frac{f(c_1) - f(a)}{g(c_1) - g(a)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

we consider the function

$$\varphi(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, \ \varphi : [c_1, b] \to \mathbb{R}.$$

Using the Rolle theorem we have $c \in (c_1, b) \subset (a, b)$ such that $\varphi'(c) = 0$, or

$$\frac{f'(c)}{g'(c)} = \frac{f(c) - f(a)}{g(c) - g(a)}.$$

Case 2. We consider now that for any $x \in (a, b)$ we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} \neq \frac{f(b) - f(a)}{g(b) - g(a)}.$$

If there are $x_1, x_2 \in (a, b)$ with

$$\frac{f(x_1) - f(a)}{g(x_1) - g(a)} > \frac{f(b) - f(a)}{g(b) - g(a)} > \frac{f(x_2) - f(a)}{g(x_2) - g(a)}$$

then there exists c_2 between x_1 and x_2 such that

$$\frac{f(c_2) - f(a)}{g(c_2) - g(a)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$

and we are in Case 1.

So we can consider that for example

$$\frac{f(x) - f(a)}{g(x) - g(a)} > \frac{f(b) - f(a)}{g(b) - g(a)}, \tag{11}$$

where $x \in (a, b)$.

If there is $x_0 \in (a,b)$ with

$$\frac{f(x_0)-f(b)}{g(x_0)-g(b)} \geq \frac{f(b)-f(a)}{g(b)-g(a)}$$

we have

$$f(x_0)-f(a) > \frac{f(b)-f(a)}{g(b)-g(a)} \cdot [g(x_0)-g(a)]\,,$$
$$f(b)-f(x_0) \geq \frac{f(b)-f(a)}{g(b)-g(a)} \cdot [g(b)-g(x_0)]$$

and by adding

$$f(b)-f(a) > \frac{f(b)-f(a)}{g(b)-g(a)} \cdot [g(b)-g(a)]$$

what is false. Hence we have

$$\frac{f(x)-f(b)}{g(x)-g(b)} < \frac{f(b)-f(a)}{g(b)-g(a)} \tag{12}$$

for $x \in (a,b)$.
From (11) and (12) we get

$$\frac{f'(a)}{g'(a)} \geq \frac{f(b)-f(a)}{g(b)-g(a)} \geq \frac{f'(b)}{g'(b)}. \tag{13}$$

From (13) and (9) we have

$$\frac{f'(a)}{g'(a)} = \frac{f(b)-f(a)}{g(b)-g(a)} \text{ or } \frac{f'(b)}{g'(b)} = \frac{f(b)-f(a)}{g(b)-g(a)}. \tag{14}$$

Next using the functions

$$\Psi_1(x) = \begin{cases} \dfrac{f(x)-f(a)}{g(x)-g(a)}, & x \in (a,b], \\ \dfrac{f'(a)}{g'(a)}, & x = a, \end{cases}$$

$$\Psi_2(x) = \begin{cases} \dfrac{f(x)-f(b)}{g(x)-g(b)}, & x \in [a,b), \\ \dfrac{f'(b)}{g'(b)}, & x = b, \end{cases}$$

$\Psi_1, \Psi_2 : [a,b] \to \mathbb{R}$, and the conditions (14) we obtain $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(c)-f(a)}{g(c)-g(a)} \text{ or } \frac{f'(c)}{g'(c)} = \frac{f(c)-f(b)}{g(c)-g(b)}.$$

Similarly considering the case

$$\frac{f(x)-f(a)}{g(x)-g(a)} < \frac{f(b)-f(a)}{g(b)-g(a)}$$

for any $x \in (a,b)$.

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Emil C. Popa
Lucian Blaga University of Sibiu
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7
RO-550012 Sibiu, Romania
e-mail: ecpopa2002@yahoo.com

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First order approximated semi-infinite optimization problems [1]

Augusta Raţiu, Dorel I. Duca

Abstract

In this paper, we consider the semi-infinite optimization problem which I attach the first order approximate problem. Relationships are established between the feasible sets of the two issues and then between their optimal solutions.

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Key words and phrases: semi-infinite optimization, approximated problem, incave function, optimality conditions.

1 Introduction

We consider the optimization problem

$$(\mathrm{P}) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X \\ & g_t(x) \leqq 0, \ t \in T, \end{array}$$

where X is a subset of $\mathbb{R}^n$, T is a nonempty set, and $f : X \to \mathbb{R}$ and $g_t : X \to \mathbb{R}$, $t \in T$ are functions.

Let

$$\mathcal{F}(P) := \{x \in X : g_t(x) \leqq 0, \ t \in T\}$$

denote the set of all feasible solutions of Problem (P) and

$$v(P) := \inf\{f(x) : x \in \mathcal{F}(P)\}$$

is the optimal value of Problem (P).

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For $x^0 \in \mathcal{F}(P)$ we consider the active index set

$$T(x^0) := \{t \in T : g_t(x^0) = 0\}.$$

The Problem (P) is semi-infinite if it has an infinite number of constraints, i.e. the set T is infinite.

The term of semi-infinite optimization (SIO) was invented in 1962 by Cooper, Charnes and Kortanek [3], [4]. We will find applications in various field such as: optimal control, transportation problems, robotics,fuzzy sets, engineering design, Chebyshev approximation, economics (portfolio problem) [10], statistics (risk theory), but the first applications of these optimization problem were in game theory, mecanics, economics and being developed by Kortanek. In 1970, he suggested with Gustafson the first numerical methods for (SIO) models. Kanzi and Nobakhtian established some alternative theorems and necessary optimality conditions of Fritz-John and Karush-Kuhn-Tucker. Few authors who have treated (SIO) problems would be: Krabs [9], Anderson and Nash [1], Guddat [8], Bonnans and Shapiro [2], Polak [11] and more recently Goberna and Lopez [6].

If the set T is finite, then the Problem (P) is a classic optimization problem. The transition at T, infinite set, involved an adequate mathematical apparatus. For semi-infinite optimization problems, theorems of John and Karush-Kuhn-Tucker have a statement of the following form:

Theorem 1 (*Dual necessary optimality conditions*): *Let X be a subset of $\mathbb{R}^n$, T is a nonempty set, x^0 be an interior point of X and f, $g_t : X \to \mathbb{R}$, $t \in T$ are functions.*

If x^0 is a local solution of Problem (P) and the functions f, g_t, $t \in T(x^0)$ are differentiable at x^0 then:

(a) There exist multipliers $\mu_0, \mu_1, ..., \mu_k \geq 0$ and indices $t_1, ..., t_k \in T(x^0)$, $k \leq n+1$, such that:

$$\sum_{j=0}^{k} \mu_j = 1$$

and

$$\mu_0 \nabla f(x^0) - \sum_{j=1}^{k} \mu_j \nabla g_{t_j}(x^0) = 0, \text{ (Fritz-John condition)}$$

(b) If $\left[\nabla g_t(x^0)\right](d) > 0$, $\forall t \in T(x^0)$, then there exist multipliers $\mu_1, ..., \mu_k \geq 0$ and indices $t_1, ..., t_k \in T(x^0)$, $k \leq n$, such that

$$\nabla f(x^0) - \sum_{j=1}^{k} \mu_j \nabla g_{t_j}(x^0) = 0, \text{ (Karush-Kuhn-Tucker condition)}.$$

For semi-infinite optimization problems, a sufficient condition for a point to be solution at a problem is:

Theorem 2 (*First order sufficient condition*): *Let X be a subset of $\mathbb{R}^n$, T is a nonempty set, x^0 be an interior point of X and f, $g_t : X \to \mathbb{R}$, $t \in T$ are functions.*

If x^0 is a feasible solution of Problem (P) and the functions f, g_t, $t \in T(x^0)$ are differentiable at x^0 and there is no a direction $d \in \mathbb{R}^n \backslash \{0\}$ satisfying

$$\left[\nabla f(x^0)\right](d) \leq 0 \text{ and } \left[\nabla g_t(x^0)\right](d) \geq 0, \forall t \in T(x^0),$$

then x^0 is a local minimizer of Problem (P).

For a proof of these two results, we refer to [10].

2 Definitions and preliminary results

For solving optimization Problem (P), there are various manners to approach. One of these is to attach to Problem (P) another optimization problem whose solutions give us the (information about) optimal solutions of the initial Problem (P).

Let $\eta : X \times X \to X$ be a function, x^0 be an interior point of X. Assume that the functions $f : X \to \mathbb{R}$ and $g_t : X \to \mathbb{R}$, $t \in T$ are differentiable at x^0.

In this paper, we attach to Problem (P), the problem:

$$\text{(A(P))} \qquad \begin{array}{ll} \min & f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \\ \text{such that:} & \\ & x \in X \\ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T \end{array}$$

called the η-approximated optimization problem.

Let

$$\mathcal{F}(A(P)) := \{x \in X : g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T\}$$

denote the set of all feasible solutions of Problem $(A(P))$.

Example 1 *For the problem*

$$(P_1) \qquad \begin{array}{ll} \min & f(x) := (x_1 - 1)^2 + x_2 \\ \textit{such that} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := -x_1 \leq 0 \\ & g_2(x) := -x_2 \leq 0 \\ & g_k(x) := x_1 x_2 - \frac{1}{k} \leq 0,\ k \in \mathbb{N},\ k \geq 3, \end{array}$$

we have

$$\mathcal{F}(P) = (\{0\} \times [0, +\infty[) \cup ([0, +\infty[\times \{0\}),$$

and hence $x^0 = (1, 0)$ is a solution of Problem (P_1). Let's $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x, x^0) = x - x^0, \textit{ for all } x \in \mathbb{R}^2.$$

Then

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) = x_2$$

$$g_1(x^0) + \left[\nabla g_1(x^0)\right]\left(\eta(x,x^0)\right) = -x_1$$

$$g_2(x^0) + [\nabla g_2(x^0)]\left(\eta(x,x^0)\right) = -x_2$$

$$g_k(x^0) + \left[\nabla g_k(x^0)\right]\left(\eta(x,x^0)\right) = x_2 - \frac{1}{k},\ k = 3, 4, ...$$

and hence the η-approximated optimization problem is

$$(A\,(P_1))\qquad \begin{array}{ll} \min & x_2 \\ \textit{such that:} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & x_2 - \frac{1}{k} \leq 0,\ k \in \mathbb{N},\ k \geq 3, \end{array}$$

with

$$\mathcal{F}(A\,(P_1)) = [0, +\infty[\times\{0\}.$$

We remark that

$$\mathcal{F}(A\,(P_1)) \subseteq \mathcal{F}(P_1)$$

Moreover,

$$\mathcal{F}(A\,(P_1)) \neq \mathcal{F}(P_1).$$

Definition 1 *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$ be a differentiable function at x^0 and $\eta : X \times X \to X$ be a function. We say that the function f is incave at x^0 with respect to (w.r.t.) η if*

$$f(x) - f(x^0) \leq \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right),\ \textit{for all } x \in X.$$

3 Main results

We establish relationships between the set of all feasible solution of Problem (P) and $(A(P))$.

Theorem 3 *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t : X \to \mathbb{R}$, $t \in T$. If for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η, then every feasible solution of Problem $(A(P))$ is a feasible solution of Problem (P), i.e.*

$$\mathcal{F}(A(P)) \subseteq \mathcal{F}(P)$$

Proof. Let $x \in \mathcal{F}(A(P))$. It follows that

$$\left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) + g_t(x^0) \leq 0,\ t \in T. \tag{1}$$

The functions g_t, $t \in T$, are incave at x^0 w.r.t. η and then

$$g_t(x) - g_t(x^0) \leq \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ t \in T$$

Now, from (1) we obtain

$$g_t(x) \leq 0,\ t \in T$$

Hence,

$$x \in \mathcal{F}(P).$$

Example 2 *For the problem*

$$(P_2)\qquad \begin{array}{ll} \min & f(x) := (x_1 - 1)^2 + x_2 \\ \textit{such that} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := -x_2 \leq 0 \\ & g_2(x) := -(x_1 - 2)^2 - (x_2 - 1)^2 + 1 \leq 0 \\ & g_k(x) := -x_1 - x_2 + \frac{k+1}{k+2} \leq 0,\ k \in \mathbb{N},\ k \geq 3, \end{array}$$

the point $x^0 = (1, 0)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x, x^0) = x - x^0,\ \textit{for all } x \in \mathbb{R}^2,$$

we have

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) = x_2$$
$$g_1(x^0) + \left[\nabla g_1(x^0)\right]\left(\eta(x, x^0)\right) = -x_2$$
$$g_2(x^0) + \left[\nabla g_2(x^0)\right]\left(\eta(x, x^0)\right) = 2x_1 + 2x_2 - 3$$
$$g_k(x^0) + \left[\nabla g_k(x^0)\right]\left(\eta(x, x^0)\right) = -x_1 - x_2 + \frac{k+1}{k+2},\ k = 3, 4, ...$$

and hence the η-approximated optimization problem is

$$(A(P_2))\qquad \begin{array}{ll} \min & x_2 \\ \textit{such that} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -x_2 \leq 0 \\ & 2x_1 + 2x_2 - 3 \leq 0 \\ & -x_1 - x_2 + \frac{k+1}{k+2} \leq 0,\ k \in \mathbb{N},\ k \geq 3, \end{array}$$

Obviously, the functions g_k, $k \in \mathbb{N}$, are diferentiable at x^0 and incave at x^0 w.r.t. η. Consequently,

$$\mathcal{F}(A(P_2)) \subseteq \mathcal{F}(P_2).$$

Theorem 4 *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, $g_t : X \to \mathbb{R}$, $t \in T$. If*

(a) f is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

then

$$\inf(A(P)) \geq \inf(P).$$

Proof. By Theorem 3, we have

$$\mathcal{F}(A(P)) \subseteq \mathcal{F}(P),$$

hence

$$\inf\{f(x) : x \in \mathcal{F}(A(P))\} \geq \inf\{f(x) : x \in \mathcal{F}(P)\}. \tag{2}$$

From the hypothesis (a), for each

$$x \in \mathcal{F}(A(P))$$

we have

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq f(x)$$

It follows that

$$\inf\{f(x) : x \in \mathcal{F}(A(P))\} \leq \inf\{f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) : x \in \mathcal{F}(A(P))\}. \tag{3}$$

From (2) and (3), we have:

$$\inf\{f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) : x \in \mathcal{F}(A(P))\} \geq \inf\{f(x) : x \in \mathcal{F}(P)\}.$$

Example 3 *For the problem*

$$(P_3) \quad \begin{array}{ll} \min & f(x) := -(x_1 - 9)^2 - (x_2 - 10)^2 \\ \textit{such that} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := -x_1 \leq 0 \\ & g_2(x) := -x_2 \leq 0 \\ & g_3(x) := -(x_1 - 2)^2 - (x_2 - 1)^2 + 1 \leq 0 \\ & g_4(x) := x_1 + x_2 - 5 \leq 0 \\ & g_k(x) := -x_1 - x_2 + \frac{k+1}{k+2} \leq 0,\ k \in \mathbb{N},\ k \geq 5, \end{array}$$

the point $x^0 = (1, 0)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x, x^0) = x - x^0, \ \textit{for all } x \in \mathbb{R}^2,$$

we have

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) = 16x_1 + 20x_2 - 180$$

$$g_1(x^0) + \left[\nabla g_1(x^0)\right]\left(\eta(x, x^0)\right) = -x_1$$

$$g_2(x^0) + \left[\nabla g_2(x^0)\right]\left(\eta(x, x^0)\right) = -x_2$$

$$g_3(x^0) + \left[\nabla g_3(x^0)\right]\left(\eta(x, x^0)\right) = 2x_1 + 2x_2 - 3$$

$$g_4\left(x^0\right) + \left[\nabla g_4(x^0)\right]\left(\eta(x, x^0)\right) = x_1 + x_2 - 5$$

$$g_k(x^0) + \left[\nabla g_k(x^0)\right]\left(\eta(x, x^0)\right) = -x_1 - x_2 + \frac{k+1}{k+2},\ k = 5, 6, ...$$

and hence the η-approximated optimization problem is

$$(A\,(P_3)) \quad \begin{array}{ll} \min & 16x_1 + 20x_2 - 180 \\ \textit{such that} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & 2x_1 + 2x_2 - 3 \leq 0 \\ & x_1 + x_2 - 5 \leq 0 \\ & -x_1 - x_2 + \frac{k+1}{k+2} \leq 0,\ k \in \mathbb{N},\ k \geq 5, \end{array}$$

Obviously, the functions f, g_k, $k \in \mathbb{N}$, are diferentiable at x^0 and incave at x^0 w.r.t. η. Consequently,

$$\mathcal{F}(A\,(P_3)) \subseteq \mathcal{F}(P_3).$$

Moreover,

$$v\,(P_3) = v\,(A\,(P_3))\,.$$

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Augusta Raţiu
Babes-Bolyai University
Faculty of Mathematics and Computer Science
1 Mihail Kogalniceanu Street, 400084, Cluj-Napoca, Romania
e-mail: augu2003@yahoo.com, augusta.ratiu@ubbcluj.ro

Dorel I. Duca
Babes-Bolyai University
Faculty of Mathematics and Computer Science
1 Mihail Kogalniceanu Street, 400084, Cluj-Napoca, Romania
e-mail: dorelduca@yahoo.com, dduca@math.ubbcluj.ro

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Grüss-type inequalities for the BLaC operator [1]

Maria-Daniela Rusu

Abstract

The classical form of Grüss' inequality, first published by G. Grüss in [4], gives an estimate of the difference between the integral of the product and the product of the integrals of two functions in $C[a,b]$. The BLaC-wavelets ("Blending of Linear and Constant wavelets") were introduced by G.P. Bonneau, S. Hahmann and G. Nielson in 1996 (see [1]) and describe a compromise between the sharpness of the Haar wavelets and the smoothness of the linear ones. The aim of this paper is to discuss Grüss-type inequalities for the BLaC operator, both in the univariate and the bivariate case. Using a result of Gonska et al. from [3], quantitative Chebyshev-Grüss-type inequalities are also obtained in terms of the least concave majorant of the classical modulus of continuity. Interesting is how the blending parameter influences our results.

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1 Preliminaries

1.1 Auxiliary results

We introduce some classical results which we will need in the sequel.

Theorem 1. *(Grüss, 1935, see [4]) Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, such that $m \le f(x) \le M$, $p \le g(x) \le P$, for all $x \in [a,b]$, where $m, M, p, P \in \mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\cdot\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\right|$$
$$\le \frac{1}{4}(M-m)(P-p)$$

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Let $C(X) = C_{\mathbb{R}}((X,d))$ represent the Banach lattice of real-valued continuous functions defined on the compact metric space (X,d). We consider the positive linear operator $H : C(X) \to C(X)$ reproducing constant functions and we are interested in its degree of non-multiplicativity. Consider two functions $f,g \in C(X)$ and define the positive bilinear functional

$$D(f,g) := H(f \cdot g; x) - H(f;x) \cdot H(g;x).$$

A useful result was given in [6] (see Theorem 3.1.) in a general compact metric space.

Theorem 2. *If $f,g \in C(X)$, where (X,d) a compact metric space, and $x \in X$ fixed, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{H(d^2(\cdot,x);x)}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{H(d^2(\cdot,x);x)}\right)$$

holds.

For $X = [a,b]$, we have a slightly better result (see Theorem 4.1. in [6]):

Theorem 3. *If $f,g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right)$$

holds, where e_1 denotes the first monomial given by $e_1(t) = t$, $t \in [a,b]$.

We also need the following result (see Corollary 5.1. in [3]).

Proposition 1. *If $H : C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, then for $f,g \in C[a,b]$ and $x \in [a,b]$ fixed we have the inequalities:*

$$\begin{aligned}
&|D(f,g)| = |D(f \cdot g; x) - D(f;x) \cdot D(g;x)| \\
&\le \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2 \cdot \sqrt{H(e_2;x) - H(e_1;x)^2}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{H(e_2;x) - H(e_1;x)^2}\right) \\
&\le \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2 \cdot \sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{H((e_1-x)^2;x)}\right).
\end{aligned}$$

Remark 1. *If H reproduces linear functions, we have no improvement of Theorem 3. If on the other hand $H(e_1,x) \neq x$, then the inequality in the Proposition 1 is a better estimate.*

1.2 The BLaC operator: Definitions and Properties

The idea to examine BLaC operators comes from the BLaC-wavelets (*Blending of Linear and Constant wavelets*), introduced around 1996 by G.P. Bonneau, S. Hahmann and G.Nielson (see [1]). They present a multiresolution analysis that implies a function representation at multiple levels of detail. This is a tool for handling large sets of data. The wavelet coefficients are the ones who store the loss of detail in each level of representation. The wavelets are basis functions encoding the difference between two succesive levels. Throughout their work, they discriminate among *Haar and linear wavelets*. The Haar wavelets are not continuous, but have perfect locality, while the linear ones are continuous, but the regularity they possess can be a drawback. A compromise between the locality of the analysis and the regularity of the approximation is desired.

This compromise is obtained by using a blending parameter $0 < \Delta \leq 1$. We now introduce the operator.

Definition 1. *(see [2]) For $f \in C[0,1]$ and $x \in [0,1]$ the* BLaC operator *is given by*

$$BL_n(f;x) := \sum_{i=-1}^{2^n-1} f(\eta_i^n) \cdot \varphi_i^n(x).$$

Now we explain the definition.

For the real blending parameter, the *scaling functions* $\varphi_\Delta : \mathbb{R} \to [0,1]$ are given by

$$\varphi_\Delta := \begin{cases} \frac{x}{\Delta}, & \text{for } 0 \leq x < \Delta, \\ 1, & \text{for } \Delta \leq x < 1, \\ -\frac{1}{\Delta} \cdot (x - 1 - \Delta), & \text{for } 1 \leq x < 1 + \Delta, \\ 0, & \text{else }. \end{cases}$$

Remark 2. *For $\Delta = 1$, φ_Δ reduces to B-Spline functions of first order (or* hat-functions*), while for the case $\Delta \to 0$ the piecewise constant functions are obtained. That's why we choose $\Delta \in (0,1]$.*

For the index $i = -1, \ldots, 2^n - 1,\ n \in \mathbb{N}$, by dilatation and translation of φ_Δ we obtain the *family of fundamental functions*:

$$\varphi_i^n(x) := \varphi_\Delta(2^n x - i),\ x \in [0,1]. \tag{1}$$

The *midpoints* η_i^n of the support line of each fundamental function φ_i^n are given by

$$\eta_i^n := \frac{i}{2^n} + \frac{1}{2} \cdot \frac{1+\Delta}{2^n},\ \text{for } i = 0, \ldots, 2^n - 2.$$

For $i \in \{-1, 2^n - 1\}$ let $\eta_{-1}^n := 0$ and $\eta_{2^n-1}^n := 1$.

Proposition 2. *(Properties of the BLaC operator, see [2])*

i) $BL_n : C[0,1] \to C[0,1]$ *is positive and linear;*

ii) BL_n *is a modification of the piecewise linear interpolation operator* $S_{\Delta_n,1}$*;*

iii) BL_n *interpolates* f *at* η_i^n*,* $i = -1, \ldots, 2^n - 1$ *(also at the endpoints* 0 *and* 1*);*

iv) BL_n *reproduces constant functions;*

v) The first absolute moment of the BLaC operator is:

$$BL_n(|e_1 - x|\,; x) := \sum_{i=-1}^{2^n-1} |\eta_i^n - x| \cdot \varphi_i^n(x) \leq \frac{1}{2^n},$$

for all $x \in [0,1]$.

vi) The second moment of the BLaC operator is given by:

$$BL_n((e_1 - x)^2; x) := \sum_{i=-1}^{2^n-1} (\eta_i^n - x)^2 \cdot \varphi_i^n(x) \leq \frac{1}{2^{2n}},$$

for all $x \in [0,1]$;

In the next sections we will apply the above results to the BLaC operator. We will also generalize them in the compact metric space setting and give a bivariate inequality.

2 (Pre-)Grüss-type inequalities for the BLaC operator

If we take $H = BL_n$ in Theorem 2, then we obtain the following result.

Theorem 4. *For $f, g \in C[0,1]$ and $x \in [0,1]$ fixed, the following inequality*

$$\begin{aligned} &|D(f,g)| \\ &\leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{BL_n((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{BL_n((e_1-x)^2;x)}\right) \\ &\leq \frac{1}{4}\widetilde{\omega}\left(f; \frac{1}{2^{n-1}}\right) \cdot \widetilde{\omega}\left(g; \frac{1}{2^{n-1}}\right) \end{aligned}$$

holds.

We also have a pre-Grüss-type inequality, using the same method as in [6] (see Theorem 6.1).

Theorem 5. *Let $f, g \in C[0,1]$. Then the inequality*

$$\begin{aligned} &|D(f,g)| \\ &\leq \frac{1}{2}\min\{\|f\|_\infty \widetilde{\omega}\left(g; 4 \cdot BL_n(|e_1 - x|\,; x)\right), \|g\|_\infty \widetilde{\omega}\left(f; 4 \cdot BL_n(|e_1 - x|\,; x)\right)\} \\ &\leq \frac{1}{2}\min\left\{\|f\|_\infty \widetilde{\omega}\left(g; \frac{1}{2^{n-2}}\right), \|g\|_\infty \widetilde{\omega}\left(f; \frac{1}{2^{n-2}}\right)\right\} \end{aligned}$$

holds.

3 Chebyshev-Grüss-type inequalities for the BLaC operator

In a recent paper [3] we proved that replacing the second moments $H((e_1-x)^2;x)$ in Theorem 4 by the smaller quantity $H(e_2;x)-H(e_1;x)^2$ is sometimes a better choice. For the BLaC operator the situation is as follows:

We want to estimate the quantity

$$\begin{aligned} D(e_1,e_1) &:= BL_n((e_1-x)^2;x) - [BL_n(e_1-x;x)]^2 \\ &\leq \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2, \end{aligned}$$

for all $x \in [0,1]$.

Suppose $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, for $k \in \{0,\dots,2^n-1\}$. This only leaves out the case $x=1$, when we get $BL_n(e_1-1;1)=0$. We distinguish between two cases:

Case 1: $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$, for $k \in 0,\dots,2^n-1$.

1. First we treat the case $\mathbf{k}=\mathbf{0}$. We have:

$$\begin{aligned} BL_n(e_1-x;x) &= (\eta^n_{-1}-x)\cdot\varphi^n_{-1}(x) + (\eta^n_0-x)\cdot\varphi^n_0(x) \\ &= \frac{x(1-\Delta)}{2\Delta} \\ BL_n((e_1-x)^2;x) &= (\eta^n_{-1}-x)^2\cdot\varphi^n_{-1} + (\eta^n_0-x)^2\cdot\varphi^n_0(x) \\ &\leq \frac{1}{2^{2n}}, \end{aligned}$$

so we get

$$\begin{aligned} D(e_1,e_1) &\leq \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2 \\ &\leq \frac{1}{2^{2n}} - \left[\frac{x(1-\Delta)}{2\cdot\Delta}\right]^2 \\ &= \frac{1}{2^{2n}}\left[1-\left(\frac{2^n\cdot x\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right] \\ &= \frac{1}{2^{2n}}\left[1-\left(\underbrace{\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}}_{(*)}\right)^2\right], \end{aligned}$$

where we denote $A(x) := 2^{n+1}x - 2k - \Delta$. We need the quantity $(*)$ to be positive. This is the case when $0<\Delta<1$ and $x \in \left(0, \frac{\Delta}{2^n}\right)$, because

$$\begin{aligned} &A(x)+\Delta+2k \neq 0 \\ \Leftrightarrow\ &A(x) \neq -\Delta-2k \\ \Leftrightarrow\ &x \neq 0 \end{aligned}$$

In the sequel we apply Proposition 1 to our BLaC operator and we obtain the following result.

Theorem 6. *For $f, g \in C[0,1]$ and $x \in \left[0, \frac{\Delta}{2^n}\right)$, we have the inequality*

$$|D(f,g)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2 \cdot \sqrt{\frac{1}{2^{2n}}\left[1 - \left(\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}\right)^2\right]}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{\frac{1}{2^{2n}}\left[1 - \left(\frac{(A(x)+\Delta+2k)(1-\Delta)}{4\cdot\Delta}\right)^2\right]}\right).$$

This is an estimate better than the one in Theorem 4 for $0 < \Delta < 1$ and $x \neq 0$.

2. For $\mathbf{1 \leq k \leq 2^n - 1}$, we have:

$$\begin{aligned} BL_n(e_1 - x; x) &= \frac{1}{2^{n+1}} \cdot \frac{(1-\Delta)}{\Delta}[2(2^n x - k) - \Delta] \\ BL_n((e_1 - x)^2; x) &= (\eta_{k-1}^n - x)^2 \cdot \varphi_{k-1}^n + (\eta_k^n - x)^2 \cdot \varphi_k^n(x) \\ &\leq \frac{1}{2^{2n}}, \end{aligned}$$

so we get

$$\begin{aligned} D(e_1, e_1) &\leq \frac{1}{2^{2n}} - [BL_n(e_1 - x; x)]^2 \\ &\leq \frac{1}{2^{2n}} - \frac{1}{2^{2n+2}}\left[\frac{(1-\Delta)\,[2(2^n \cdot x - k) - \Delta]}{\Delta}\right]^2 \\ &= \frac{1}{2^{2n}}\left[1 - \left(\underbrace{\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}}_{(**)}\right)^2\right] \end{aligned}$$

where we denote $A(x) := 2^{n+1}x - 2k - \Delta$. We need the quantity $(**)$ to be positive. This is the case when $0 < \Delta < 1$ and $x \neq \frac{k}{2^n} + \frac{\frac{1}{2}\cdot\Delta}{2^n}$, because

$$\begin{aligned} & A(x) \neq 0 \\ \Leftrightarrow\ & 2^{n+1}x - 2k - \Delta \neq 0 \\ \Leftrightarrow\ & x \neq \frac{2k+\Delta}{2^{n+1}}. \end{aligned}$$

We apply Proposition 1 to our BLaC operator to obtain the following

Theorem 7. *For $f, g \in C[0,1]$ and $x \in \left[\frac{k}{2^n}, \frac{k+\Delta}{2^n}\right)$ with $k \in \{1, \ldots, 2^n - 1\}$, we have* *the inequality*

$$|D(f,g)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right]}\right)$$
$$\cdot\,\widetilde{\omega}\left(g; 2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)\cdot(1-\Delta)}{2\cdot\Delta}\right)^2\right]}\right),$$

This is an estimate better than the one in Theorem 4 for $x \neq \frac{k}{2^n}+\frac{\frac{1}{2}\cdot\Delta}{2^n}$ *and* $0<\Delta<1$.

Case 2: $x \in \left[\frac{k+\Delta}{2^n}, \frac{k+1}{2^n}\right)$, for $k \in 0,\ldots,2^n-1$. We have:

$$\begin{aligned} BL_n(e_1-x;x) &= (x-\eta_k^n)\cdot\varphi_k^n(x)\\ &= \frac{2(2^n x-k)-\Delta-1}{2n+1}\\ BL_n((e_1-x)^2;x) &= (x-\eta_k^n)^2\cdot\varphi_k^n(x)\\ &\leq \frac{1}{2^{2n}}, \end{aligned}$$

so we get

$$\begin{aligned} D(e_1,e_1) &\leq \frac{1}{2^{2n}} - [BL_n(e_1-x;x)]^2\\ &\leq \frac{1}{2^{2n}} - \frac{1}{2^{2n+2}}\cdot[A(x)-1]^2\\ &= \frac{1}{2^{2n}}\left[1-\left(\underbrace{\frac{A(x)-1}{2}}_{(***)}\right)^2\right] \end{aligned}$$

where we denote $A(x) := 2^{n+1}x-2k-\Delta$. We need the quantity $(***)$ to be positive. This is the case when $x \neq \frac{k}{2^n}+\frac{\frac{1}{2}\cdot(1+\Delta)}{2^n}$, because

$$\begin{aligned} &A(x)\neq 1\\ \Leftrightarrow\ &2^{n+1}x-2k-\Delta\neq 1\\ &\Leftrightarrow x\neq\frac{2k+\Delta+1}{2^{n+1}}. \end{aligned}$$

Applying Proposition 1 to the BLaC operator gives

Theorem 8. *For* $f,g\in C[0,1]$ *and* $x\in\left[\frac{k+\Delta}{2^n},\frac{k+1}{2^n}\right)$ *with* $k\in\{0,\ldots,2^n-1\}$, *we have the inequality*

$$|D(f,g)|\leq\frac{1}{4}\widetilde{\omega}\left(f;2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)-1}{2}\right)^2\right]}\right)\cdot\widetilde{\omega}\left(g;2\cdot\sqrt{\frac{1}{2^{2n}}\left[1-\left(\frac{A(x)-1}{2}\right)^2\right]}\right).$$

This is an estimate better than the one in Theorem 4 for $x \neq \frac{k}{2^n}+\frac{\frac{1}{2}\cdot(1+\Delta)}{2^n}$.

4 Bivariate approximation

4.1 Tensor product BLaC operator

We define the bivariate BLaC operator and then derive a Grüss-type inequality.

Let $X = [0,1]^2$ be the compact metric space equipped with the Euclidian metric

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

The two-dimensional scaling functions $\varphi_\Delta(x,y)$ are given by

$$\varphi_\Delta(x,y) = \varphi_\Delta(x)\cdot\varphi_\Delta(y) := \begin{cases} \varphi_\Delta(x), & \text{for } \Delta \le y < 1, \\ \varphi_\Delta(y), & \text{for } \Delta \le x < 1, \\ \frac{xy}{\Delta^2}, & \text{for } 0 \le x, y < \Delta, \\ -\frac{x}{\Delta^2}\cdot(y-1-\Delta), & \text{for } 0 \le x < \Delta,\ 1 \le y < 1+\Delta \\ -\frac{y}{\Delta^2}\cdot(x-1-\Delta), & \text{for } 0 \le y < \Delta,\ 1 \le x < 1+\Delta \\ \frac{1}{\Delta^2}\cdot(x-1-\Delta)(y-1-\Delta), & \text{for } 1 \le x, y < 1+\Delta \\ 0, & \text{else }, \end{cases}$$

and the bivariate fundamental functions $\varphi^n_{i,j}(x,y)$ are defined by:

$$\varphi^n_{i,j}(x,y) = \varphi_\Delta(2^n(x - i2^{-n}))\cdot\varphi_\Delta(2^n(y - j2^{-n})).$$

The interpolation points $\eta^n_{i,j} \in \mathbb{R}^2$ look as follows:

$$\begin{aligned} &\eta^n_{i,j} = \left(\frac{2i+1+\Delta}{2^{n+1}}, \frac{2j+1+\Delta}{2^{n+1}}\right), \text{ for } i,j = 0,\dots,2^n-2, \\ &\eta^n_{-1,-1} = (0,0),\ \eta^n_{2^n-1,2^n-1} = (1,1), \\ &\eta^n_{-1,j} = \left(0, \frac{2j+1+\Delta}{2^{n+1}}\right),\ \eta^n_{i,-1} = \left(\frac{2i+1+\Delta}{2^{n+1}}, 0\right), \\ &\eta^n_{2^n-1,j} = \left(1, \frac{2j+1+\Delta}{2^{n+1}}\right),\ \eta^n_{i,2^n-1} = \left(\frac{2i+1+\Delta}{2^{n+1}}, 1\right). \end{aligned}$$

Definition 2. *(see [5]) For $f \in C([0,1]^2)$ and $x,y \in [0,1]$ the* bivariate BLaC operator *is given by*

$$BL_n f(x,y) = BL_n f(x,y,n) := \sum_{i=-1}^{2^n-1}\sum_{j=-1}^{2^n-1} f(\eta^n_{i,j})\cdot\varphi^n_{i,j}(x,y).$$

Remark 3. *The properties of the BLaC operator in the univariate case also apply here.*

4.2 A Grüss-type inequality for the bivariate BLaC operator on $[0,1]^2$

We apply Theorem 2 to the bivariate BLaC operator and obtain:

Theorem 9. *If $f_0, f_1 \in C(X)$, where $X = [0,1]^2$ is a compact metric space, then the inequality*

$$\begin{aligned} |D(f_0,f_1)| &\leq \frac{1}{4}\prod_{j=0}^{1}\widetilde{\omega_d}(f_j; 4\sqrt{BL_n(d^2(\cdot,(x,y));(x,y))}) \\ &\leq \frac{1}{4}\widetilde{\omega_d}\left(f_0; \frac{\sqrt{2}}{2^{n-4}}\right)\cdot \widetilde{\omega_d}\left(f_1; \frac{\sqrt{2}}{2^{n-4}}\right) \end{aligned}$$

holds.

Proof. The second moment of the bivariate BLaC operator is of interest here.

Let $x \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $y \in [\frac{l}{2^n}, \frac{l+1}{2^n}]$, $k, l \in \{0, \ldots, 2^n - 1\}$. For $x = 1$ and $y = 1$, take $k = 2^n - 1$ and $l = 2^n - 1$, respectively. The second moment of the bidimensional BLaC operator is:

$$\begin{aligned} BL_n(d^2(\cdot,(x,y));(x,y)) &= \underbrace{\sum_{i=-1}^{2^n-1}\sum_{j=-1}^{2^n-1} d^2(\eta^n_{i,j},(x,y))\cdot\varphi^n_{i,j}(x,y)}_{(1)} \\ &= (\underbrace{d^2(\eta^n_{k-1,l-1},(x,y))}_{\leq\frac{1}{2^{2n-3}}}\underbrace{\varphi^n_{k-1,l-1}(x,y)}_{\leq 1} + \underbrace{d^2(\eta^n_{k,l-1},(x,y))\varphi^n_{k,l-1}(x,y)}_{\leq\frac{1}{2^{2n-3}}\cdot 1} \\ &+ \underbrace{d^2(\eta^n_{k-1,l},(x,y))\varphi^n_{k-1,l}(x,y)}_{\leq\frac{1}{2^{2n-3}}\cdot 1} + \underbrace{d^2(\eta^n_{k,l},(x,y))\varphi^n_{k,l}(x,y))}_{\leq\frac{1}{2^{2n-3}}\cdot 1} \\ &\leq \frac{4}{2^{2^{2n-3}}} = \frac{1}{2^{2n-5}}, \end{aligned}$$

so the sum (1) has at most 4 terms. The idea of the above calculations is that the maximum distance in each component is smaller than or equal to $\frac{1}{2^{n-1}}$, so we have:

$$d(\eta^n_{k-1,l-1},(x,y))^2 \leq \left(\frac{1}{2^{n-1}}\right)^2 + \left(\frac{1}{2^{n-1}}\right)^2 = \frac{2}{2^{2n-2}} = \frac{1}{2^{2n-3}}.$$

Then we get

$$\begin{aligned} BL_n(d^2(\cdot,(x,y));(x,y))^{\frac{1}{2}} &= \sqrt{BL_n(d^2(\cdot,(x,y));(x,y))} \\ &= \sqrt{\sum_{i=-1}^{2^n-1}\sum_{j=-1}^{2^n-1} d^2(\eta^n_{i,j},(x,y))\cdot\varphi^n_{i,j}(x,y)} \\ &\leq \sqrt{4\left(\frac{1}{2^{2n-3}}\right)}. \end{aligned}$$

The Grüss-type inequality becomes:

$$
\begin{aligned}
|D(f_0,f_1)| &\le \frac{1}{4}\prod_{j=0}^{1}\widetilde{\omega_d}(f_j;4\sqrt{BL_n(d^2(\cdot,(x,y));(x,y))})\\
&\le \frac{1}{4}\widetilde{\omega_d}\left(f_0;4\sqrt{\frac{4}{2^{2n-3}}}\right)\cdot\widetilde{\omega_d}\left(f_1;4\sqrt{\frac{4}{2^{2n-3}}}\right)\\
&= \frac{1}{4}\widetilde{\omega_d}\left(f_0;4\sqrt{\frac{1}{2^{2n-5}}}\right)\cdot\widetilde{\omega_d}\left(f_1;4\sqrt{\frac{1}{2^{2n-5}}}\right)\\
&= \frac{1}{4}\widetilde{\omega_d}\left(f_0;\frac{\sqrt{2}}{2^{n-4}}\right)\cdot\widetilde{\omega_d}\left(f_1;\frac{\sqrt{2}}{2^{n-4}}\right),
\end{aligned}
$$

for $f_0, f_1 \in C(X)$, where $X=[0,1]^2$ is the compact metric space equipped with the Euclidian metric. Our proof is completed.

The bivariate case is of particular interest because it can be applied in the image compression process.

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Maria-Daniela Rusu
University of Duisburg-Essen
Faculty of Mathematics
Forsthausweg 2
47057 Duisburg, Germany
e-mail: maria.rusu@uni-due.de

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Evaluations of the remainder using devided differences [1]

Daniel Florin Sofonea

Abstract

A representation of the rest in a approximation formula is given using properties of divided differences. Also, the rest is of simple form is proved.

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Key words and phrases: divided differences, Cebîşev system, positive linear operator, functional.

1 Introduction

The starting point of research given below it is motivated by the result of O. Aramă [2] according to which the rest of approximation formula of nonconcave function on $[0, 1]$ by Bernstein polynomials always keeps a constant sign. Furthermore, the rest is of simple form.

As is know, the transformation which for each bounded function f gives the restriction on $[0, 1]$ of Bernstein polynomial $B_n f$ is a positive linear operator and preserves linear functions. Therefore, it arises the question if the O. Aramă's results are valid for each linear and positive transformation which preserves linear functions. In this paper we will give an affirmative answer to this problem.

The system of functions $\varphi_i \in C[a, b], i = \overline{1, m}$ is called a Cebîşev system on $[a, b]$ if

$$V\begin{pmatrix} \varphi_1, & \varphi_2, & \dots, & \varphi_m \\ x_1, & x_2, & \dots, & x_m \end{pmatrix} \neq 0$$

for every m distinct points $x_i \in [a, b]$, $i = \overline{1, m}$, where

$$V\begin{pmatrix} \varphi_1, & \varphi_2, & \dots, & \varphi_m \\ x_1, & x_2, & \dots, & x_m \end{pmatrix} := |\varphi_i(x_j)|_{i,j=\overline{1,m}}$$

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is the determinant of the values of the functions $\varphi_1, \ldots, \varphi_m$ on the points $x_i, i = \overline{1, m}$. Let

$$(1) \qquad f_0, f_1, ..., f_n, f_{n+1}, \quad n \geq 1,$$

be a Cebîşev system, where $f_i \in C[a, b]$, $i = \overline{0, n+1}$.

It is called a *divided difference* of the function $f \in C[a, b]$ on the knots x_i, $i = \overline{0, n+1}$ with respect to the sequence of functions (1), the quotient

$$V\begin{pmatrix} f_0, & f_1, & \ldots, & f_n, & f \\ x_0 & x_1, & \ldots & x_n, & x_{n+1} \end{pmatrix} : V\begin{pmatrix} f_0, & f_1, & \ldots, & f_n, & f_{n+1} \\ x_0 & x_1, & \ldots & x_n, & x_{n+1} \end{pmatrix}$$

which is denote by

$$\left[\begin{matrix} f_0, f_1, \ldots, f_{n+1} \\ x_0, x_1, \ldots, x_{n+1} \end{matrix} ; f\right].$$

For the following system of functions

$$(2) \qquad e_i(x) = x^i, \ i = \overline{0, n+1}$$

is obtained the case of usual divided difference.

The function $f \in C[a, b]$ is called convex, nonconcave, concave, respectively nonconvex of order n with respect to the sequence of functions (1), if

$$\left[\begin{matrix} f_0, f_1, \ldots, f_{n+1} \\ x_0, x_1, \ldots, x_{n+1} \end{matrix} ; f\right] > 0, \geq 0, < 0, \text{ respectively } \leq 0,$$

where $x_i \in [a, b]$, $i = \overline{0, n+1}$ are distinct points.

In [8] T. Popoviciu introduced the concept of functional of simple form. A linear functional $F : C[a, b] \to \mathbb{R}$ is of *simple form* if for any $f \in C[a, b]$ there are the distinct points $\theta_i = \theta_i(f) \in [a, b]$, $i = \overline{0, n+1}$ such that

$$F(f) = K\left[\begin{matrix} f_0, f_1, \ldots, f_{n+1} \\ \theta_0, \theta_1, \ldots, \theta_{n+1} \end{matrix} ; f\right],$$

where $K \neq 0$ is a constant independent of function f.

Theorem 1 *([8]) A necessary and sufficient condition for a linear functional $F : C[a, b] \to R$ to be of simple form with respect to the sequence of functions (1), is that*

$$F(h) \neq 0$$

for any convex function $h \in C[a, b]$.

A linear functional $F : C[a, b] \to \mathbb{R}$ has degree of exactness n if it vanishes in the first $n + 1$ functions of (1).

Theorem 2 *([8]). Let n be a fixed natural number and $F : C[a,b] \to R$ be a linear and bounded functional which has degree of exactness n. A necessary and sufficient condition for the functional F to be of simple form with respect to the sequence of functions (2) is that*

$$F(e_{n+1})F(\varphi_{n+1,\lambda}) \geq 0, \quad \forall \lambda \in [a,b],$$

where

(3) $$\varphi_{n+1,\lambda}(t) = (t-\lambda)_+^n = \left(\frac{t-\lambda+|t-\lambda|}{2}\right)^n, \quad t \in [a,b].$$

Theorem 3 *([8]). Let $F : C[a,b] \to R$ be a linear and bounded functional of simple form with respect to the sequence of functions (2), with degree of exactness equal to m, $m \geq 0$. If $F_1 : C^{(m+1)}[a,b] \to R$ is defined by:*

$$F_1(f) = F(f) - F(e_{m+1})\frac{f^{(m+1)}(c)}{(m+1)!}\ , \ f \in C^{(m+1)}[a,b],$$

where $c \in (a,b)$ is given by the following identity:

$$F(e_{m+2}) - (m+2)F(e_{m+1})c = 0,$$

then F_1 is a functional of simple form with degree of exactness equal to $m+2$.

Note $\varepsilon_y : C[a,b] \to R$ the functional defined by $\varepsilon_y(f) = f(y)$.

Theorem 4 *([9]). Let $f_j \in C[a,b]$, $j = 0,1,2$ be a Cebîşev system on $[a,b]$ and x_0 is a fixed point from $[a,b]$. If $F : C[a,b] \to R$ is a linear and positive functional which verifies:*

$$F(f_j) = \epsilon_{x_0}(f_j), j = 0,1,2,$$

then

$$F = \epsilon_{x_0} \in C[a,b].$$

2 Divided differences in evaluation of the remainder

In this section we propose a new approximation formula of a linear and positive functional. The remainder is given using divided differences.

Theorem 5 *([4]) If $F : C[a,b] \to R$ is a linear and positive functional which verifies*

$$F(e_0) = 1, F(e_j) = a_j,\ j = 1,2,$$

then, for any $f \in C[a,b]$, there exist the distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1,2$, such that

(4) $$F(f) = f(a_1) + (a_2 - a_1^2)[\theta_1, \frac{\theta_1+\theta_2}{2}, \theta_2; f]$$

In [7] (see also [6], pp. 176) T. Popoviciu established the following result:

If $f \in C[a,b]$ and $x_0, x_1, ..., x_n$ are distinct points which belong to $[a,b]$, then there exist two distinct points y_1, $y_2 \in [a,b]$ such that

$$(5) \qquad [x_0, x_1, x_2, \ldots, x_n; f] = [y_1, y_1 + \frac{1}{n}(y_2 - y_1), y_1 + \frac{2}{n}(y_2 - y_1), \ldots, y_2; f].$$

In applications this result can be used in the following way:

Let $x_0 \in [a,b]$, $f \in C[a,b]$ and $F : C[a,b] \to R$ a linear and positive functional which verifies $F(e_0) = 1$, $F(e_k) = a_k$, $k = 1,2$. If it is considered the following approximation formula

$$F(f) \approx f(x_0),$$

then

a) if $a_1 = x_0$, then there is a system of distinct points r_1, r_2, r_3 from $[a,b]$ such that:

$$F(f) - f(x_0) = (a_2 - a_1^2)[r_1, r_2, r_3; f],$$

b) if $a_1 \neq x_0$, then there are two system θ_1, θ_2;r_1, r_2, r_3 of distinct points from $[a,b]$ such that

$$F(f) - f(x_0) = (a_1 - x_0)[\theta_1, \theta_2; f] + (a_2 - a_1^2)[r_1, r_2, r_3; f].$$

The last identity follows from (4) using the properties of divided differences ([8]).

Denote by

$$R(f) = F(f) - f(a_1) = (a_2 - a_1^2)[\theta_1, \frac{\theta_1 + \theta_2}{2}, \theta_2; f].$$

Let $R_1 : C^{(2)}[a,b] \to R$ be a functional defined by

$$R_1(f) = R(f) - R(e_2)\frac{f''(c)}{2}, \;\; f \in C^{(2)}[a,b],$$

where c verifies

$$3cR(e_2) = R(e_3), \quad R(e_2) \neq 0.$$

Since R has degree of exactness 1 and R is of simple form, using Theorem 3 (m=1), we obtain that R_1 has degree of exactness equal 3 and R_1 is of simple form. Therefore, it is proved the following result

Theorem 6 *([3]) If $F : C^{(2)}[a,b] \to R$ is a linear and positive functional which verifies*

$$F(e_0) = 1, \quad F(e_i) = a_i, \quad i = 1,2,3,4, \quad a_2 \neq a_1^2,$$

then for any $f \in C^{(2)}[a,b]$ there are two distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1,2$, such that

$$(6) \qquad \begin{aligned} F(f) = f(a_1) + \frac{a_2 - a_1^2}{2} f''\left(\frac{a_3 - a_1^3}{3(a_2 - a_1^2)}\right) \\ + \frac{a_1^6 - 3a_1^4 a_2 + 4a_1^3 a_3 - 3a_1^2 a_4 + 3a_2 a_4 - 2a_3^2}{3(a_2 - a_1^2)}[\theta_1, \theta_{31}, \theta_{22}, \theta_{13}, \theta_2; f], \end{aligned}$$

where

$$\theta_{ij} = \frac{i}{4}\theta_1 + \frac{j}{4}\theta_2.$$

Let F be a functional linear and positive which verifies

$$F(e_0) = 1, F(e_i) = a_i, \ , i = \overline{1,6}. \quad (7)$$

Denote by $w_i = a_i - a_1^i$, $i = \overline{1,6}$.

Theorem 7 *If $F : C^{(4)}[a,b] \to \mathbb{R}$ is a linear and positive functional which verifies conditions (7), $w_2 \neq 0$ and $3w_2w_4 - 2w_3^2 \neq 0$, then for any $f \in C^{(4)}[a,b]$ there are two distinct points $\theta_i = \theta_i(f) \in [a,b]$, $i = 1,2$, such that*

$$\begin{aligned} F(f) &= f(a_1) + \frac{w_2}{2} f''(z_1) + \frac{1}{24}\left(w_4 - \frac{2}{3}\frac{w_3^2}{w_2}\right) f^{(4)}(z_2) \\ &+ K\left[\theta_1, \theta_{51}, \theta_{42}, \theta_{33}, \theta_{24}, \theta_{15}, \theta_2; f\right], \end{aligned}$$

were

$$z_1 = \frac{w_3}{3w_2}, \ z_2 = \frac{27w_2^2w_5 - 10w_3^3}{45w_2(3w_2w_4 - 2w_3^2)}, \ \theta_{ij} = \frac{i\theta_1 + j\theta_2}{6},$$

$$K = w_6 - \frac{5w_3^4}{27w_2^3} - \frac{9w_2}{5(3w_2w_4 - 2w_3^2)} \cdot \left(w_5 - \frac{10}{27}\frac{w_3^3}{w_2^2}\right)^2.$$

Proof. Denote by

$$R(f) = F(f) - f(a_1) - \frac{w_2}{2} f''(z_1) - \frac{1}{24}\left(w_4 - \frac{2}{3}\frac{w_3^2}{w_2}\right) f^{(4)}(z_2).$$

From Theorem 3 it follows that R has degree of exactness 5 and R is of simple form, therefore, there are the distinct points $y_i = y_i(f), i = \overline{1,7}$ such that

$$R(f) = R(e_6)[y_1, y_2, y_3, y_4, y_5, y_6, y_7; f].$$

Using Popoviciu's result ([7]) follows that there exists two distinct points $\theta_1, \theta_2 \in [a,b]$ such that

$$[y_1, y_2, y_3, y_4, y_5, y_6, y_7; f] = [\theta_1, \theta_{51}, \theta_{42}, \theta_{33}, \theta_{24}, \theta_{15}, \theta_2; f].$$

3 Numerical Examples

In this section we will give some applications of the result obtained in previous section. We will apply the approximation formula given in Theorem 7 for the following functional

$$F : C[a,b] \to \mathbb{R}, \ F(f) = \frac{1}{b-a}\int_a^b f(t)dt.$$

Since $a_k = F(e_k) = \dfrac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$, from Theorem 7 we obtain

$$\frac{1}{b-a}\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)^2 f''\left(\frac{a+b}{2}\right) + \frac{1}{1920}(b-a)^4 f^{(4)}\left(\frac{a+b}{2}\right)$$
$$+ \frac{1}{448}(b-a)^6 \left[\theta_1, \frac{5\theta_1+\theta_2}{6}, \frac{2\theta_1+\theta_2}{3}, \frac{\theta_1+\theta_2}{2}, \frac{\theta_1+2\theta_2}{3}, \frac{\theta_1+5\theta_2}{6}, \theta_2; f\right],$$

where $f \in C^{(4)}[a,b]$ and $\theta_i = \theta_i(f) \in [a,b]$.

In the next part of this paper we will apply the approximation formula given in Theorem 7 for a sequence of linear and positive operators.

Theorem 8 *If $(L_n)_{n=1}^{\infty}$ is a sequence of linear and positive operators defined on $C[a,b]$ which verifies*

$$L_n(e_0) = e_0, L_n(e_k) = a_{kn},\ k = \overline{1,6}$$

then for any $x \in K = \left\{x \in [a,b]\,\middle|\, w_2 \neq 0,\ 3w_2w_4 - 2w_3^2 \neq 0\right\}$ and $f \in C^{(4)}[a,b]$ there are two distinct points $\theta_{in} = \theta_{in}(f;x) \in [a,b]$, $i = 1,2$, such that

$$L_n(f)(x) = f(a_{1n}(x)) + \frac{w_{2n}(x)}{2} f''(z_{1n}(x)) + \frac{1}{24}\left(w_{4n}(x) - \frac{2}{3}\frac{w_{3n}^2(x)}{w_{2n}(x)}\right) f^{(4)}(z_{2n}(x))$$
$$+ K_n(x)\left[\theta_{1n}, \frac{5\theta_{1n}+\theta_{2n}}{6}, \frac{2\theta_{1n}+\theta_{2n}}{3}, \frac{\theta_{1n}+\theta_{2n}}{2}, \frac{\theta_{1n}+2\theta_{2n}}{3}, \frac{\theta_{1n}+5\theta_{2n}}{6}, \theta_{2n}; f\right],$$

were

$$w_{in} = a_{in} - a_{1n}^i,\ i = \overline{1,6},\ z_{1n} = \frac{w_{3n}}{3w_{2n}},\ z_{2n} = \frac{27w_{2n}^2 w_{5n} - 10w_{3n}^3}{45w_{2n}(3w_{2n}w_{4n} - 2w_{3n}^2)},$$
$$K_n = w_{6n} - \frac{5w_{3n}^4}{27w_{2n}^3} - \frac{9w_{2n}}{5(3w_{2n}w_{4n} - 2w_{3n}^2)} \cdot \left(w_{5n} - \frac{10}{27}\frac{w_{3n}^3}{w_{2n}^2}\right)^2.$$

We will consider Bernstein operators $B_n : C[0,1] \to C[0,1],\ n = 1,2,\cdots$, defined by

$$B_n(f)(x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k}{n}\right),\ \text{ where } p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Let $a_{kn}(x) = B_n(e_k)(x),\ k = \overline{1,6}$. Then we obtain

$$a_{0n}(x) = 1,\ a_{1n}(x) = x,\ a_{2n}(x) = x^2 + \frac{x(1-x)}{n},$$

$$a_{3n}(x) = x^3 + \frac{x(1-x)}{n^2}\left[(3n-2)x + 1\right],$$

$$a_{4n}(x) = x^4 + \frac{x(1-x)}{n^3}\left[(6n^2 - 11n + 6)x^2 + (7n-6)x + 1\right],$$

$$a_{5n}(x)=x^5+\frac{x(1-x)}{4}\left[(10n^3-35n^2+50n-24)x^3+(25n^2-60n+36)x^2+(15n-14)x+1\right],$$

$$a_{6n}(x)=x^6+\frac{x(1-x)}{n^5}\left[(15n^4-85n^3+225n^2-274n+120)x^4\right.$$
$$\left.+(65n^3-300n^2+476n-240)x^3+(90n^2-239n+150)x^2+(31n-30)x+1\right].$$

For any $x\in(0,1)$ and $f\in C^{(4)}[0,1]$ there are two distinct points $\theta_{in}=\theta_{in}(f;x)\in[0,1]$, $i=\overline{1,2}$, such that

$$B_n(f)(x)=f(x)+c_{1n}(x)f''(z_{1n}(x))+c_{2n}(x)f^{(4)}(z_{2n}(x))$$
$$+K_n(x)\left[\theta_{1n},\frac{5\theta_{1n}+\theta_{2n}}{6},\frac{2\theta_{1n}+\theta_{2n}}{3},\frac{\theta_{1n}+\theta_{2n}}{2},\frac{\theta_{1n}+2\theta_{2n}}{3},\frac{\theta_{1n}+5\theta_{2n}}{6},\theta_{2n};f\right],$$

where

$$c_{1n}(x)=\frac{x(1-x)}{2n},\ c_{2n}(x)=-\frac{x(1-x)}{72n^3}\left[(9n-10)x^2-(9n-10)x-1\right],$$

$$z_{1n}(x)=\frac{(3n-2)x+1}{3n},$$

$$z_{2n}(x)=\frac{(405n^2-990n+568)x^3-(405n^2-1260n+852)x^2-(315n-318)x-17}{45n\left[(9n-10)x^2-(9n-10)x-1\right]},$$

$$K_n(x)=\frac{x(1-x)}{405n^5\left[(9n-10)x^2-(9n-10)x-1\right]}\cdot$$

$$\left[(54675n^3-243000n^2+339660n-151376)x^6-(164025n^3-729000n^2+1018980n-454128)x^5\right.$$
$$+(164025n^3-753300n^2+1086750n-497748)x^4-(54675n^3-291600n^2+475200n-238616)x^3$$
$$\left.-(24300n^2-71685n+47658)x^2-(3915n-4038)x-41\right].$$

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Daniel Florin Sofonea
Lucian Blaga University of Sibiu
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7
RO-550012 Sibiu, Romania
e-mail: sofoneaflorin@yahoo.com

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Remarks on Bernstein-Euler-Jacobi (BEJ) type operators [1]

Elena-Dorina Stănilă

Abstract

We introduce and study a class of positive linear operators defined as a composition (two Bernstein and an Euler-Jacobi Beta operator in the middle) that reduces in special cases to many known operators. Using the method presented by Z. Finta we prove direct and converse inequalities of type A in terms of a K-functional. This is done for two special cases, that is, for the composition of two different Bernstein operators and for a particular case of the general composition that reproduces linear function.

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1 Introduction

We introduce and study a class of positive linear operators that are given by

Definition 1 *For $r > 0, a, b \geq -1, n, m > 1$ we define $R_{m,n} : C[0,1] \to C[0,1]$ by*

$$R_{m,n} = R_{m,n}^{(r,a,b)} = B_m \circ \mathcal{B}_r^{a,b} \circ B_n. \tag{1}$$

Here $\mathcal{B}_r^{a,b}$ is the Euler-Jacobi Beta operator introduced in [14] *and B_n, B_m are the n-th and m-th Bernstein operators.*

We use the convention $B_\infty = \mathcal{B}_\infty^{a,b} = Id$. The purpose of introducing such an operator is to explain known operators using decomposition.

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In the following table we show that for different values of the indices we find many known operators.

m	**r**	**a**	**b**	**n**	**Other Notations**
n	n	-1	-1	∞	U_n – "genuine" Bernstein-Durrmeyer, Goodman-Sharma [9]
n	n	0	0	∞	M_n – Classical Durrmeyer, Durrmeyer [4] and Lupaş [11]
n	n	> -1	> -1	∞	$M_n^{a,b}$ – Durrmeyer with Jacobi weights, Păltănea [16]
n	$n \cdot \varrho$	-1	-1	∞	U_n^{ϱ} – Gonska - Păltănea [8]
n	$n \cdot \varrho$	> -1	> -1	∞	$P_n^{\varrho,a,b}$ – Mache - Zhou [14]
∞	n	0	0	n	$V_n^{0,0} = \mathbb{B}_n \circ B_n$ – Lupaş [12]
∞	n	-1	-1	n	S_n – Lupaş - Lupaş [13]
∞	n	> -1	> -1	n	$V_n^{a,b}$ – Raşa [17]
∞	$\alpha(n)$	-1	-1	n	S_n^{α} – Stancu [18]
∞	$n \cdot \varrho$	> -1	> -1	n	$Q_n^{\varrho,c,d}$ – Gonska [7]
m	∞	> -1	> -1	n	$R_{m,n}^{\infty}$
m	$n \cdot \varrho$	-1	-1	n	$R_{m,n}^{\varrho}$

In what follows let $\omega_{\varphi}^2(f,t)_{C[0,1]} = \sup_{0<h\le t} ||\Delta^2_{h\varphi(x)} f(x)||_{C[0,1]}$ be the Ditzian-Totik modulus of smoothness, where $\varphi(x) = \sqrt{x(1-x)}, x \in [0,1]$ and $f \in C[0,1]$ with the usual supremum norm $||f|| = ||f||_{C[0,1]}$. We denote by $K_{2,\varphi}(f,t)_{C[0,1]}$ the K-functional of the pair of spaces $C[0,1]$ and a corresponding Sobolev space with weight function φ^2 given by

$$K_{2,\varphi}(f,t)_{C[0,1]} = \inf\{||f-g|| + t||\varphi^2 g''|| : g \in W^2_{\infty}(\varphi)\},$$

where $W^2_{\infty}(\varphi)$ consists of all functions $g \in C[0,1]$ such that g' is absolutely continuous on $[0,1]$ and $||\varphi^2 g''||$ is finite.

2 Direct and converse results

Direct and strong converse inequality of type A, in the terminology of [2], exist for the Bernstein operators in [10], for "genuine" Bernstein-Durrmeyer operators in [15], and for a special selection of Stancu operators in [5].

Using the method presented by Finta in [5] and [6] we can give such results for two more cases, that is, for the composition of two different Bernstein operators and for a particular case of the general composition that reproduces linear functions.

We will need the following auxiliary results. Note that Lemma 2 is of crucial importance in our proofs.

Lemma 1 *Let* $R_2(f,u,x) = \int\limits_x^u (u-v)f''(v)dv$ *be the integral remainder of* f *in Taylor expansion. Then for* $x, y \in [0,1]$ *we have:*

$$(2) \qquad |R_2(f,u,x)| \le \frac{|u-x|}{\varphi^2(x)} \left| \int\limits_u^x \varphi^2(x) f''(v)dv \right|.$$

Proof. Lemma 1 results from Lemma 9.6.1 ([3], p.140).

Lemma 2 *Let* $f \in C[0,1]$. *Then*

$$(3) \qquad \frac{1}{n}||\varphi^2 B_n'' f|| \le C_0 ||B_n f - f||$$

where $C_0 > 0$ *is an absolute constant.*

Proof. See the inequality (2.1) in ([10], p.317)

2.1 Case I - a composition of two Bernstein operators

We define

$$R_{m,n}^{\infty} f = (B_m \circ B_n)(f,x) = \sum_{k=0}^{m} p_{m,k}(x) B_n f\left(\frac{k}{m}\right),$$

a positive linear operator that reproduces linear functions, where $p_{m,k}$ are the Bernstein basis polynomials with $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, x \in [0,1]$.
For the first case we need the following result:

Lemma 3 *Let* $f \in C[0,1]$. *Then*

$$(4) \qquad ||R_{m,n}^{\infty} f - B_n f|| \le \frac{1}{m} ||\varphi^2 B_n'' f||.$$

Proof. $R_{m,n}^{\infty} f = B_m(B_n f) = \sum\limits_{k=0}^{m} p_{m,k}(x) B_n f(\frac{k}{m})$ where $p_{m,k} = \binom{m}{k} x^k (1-x)^{m-k}$. Hence

$$(R_{m,n}^{\infty} f - B_n f)(x) = \sum_{k=0}^{m} p_{m,k}(x) \left(B_n f\left(\frac{k}{m}\right) - B_n f(x) \right)$$

and by Taylor expansion with integral remainder, namely

$$B_n f\left(\frac{k}{m}\right) = B_n f(x) + \left(\frac{k}{m} - x\right) B_n' f(x) + \int\limits_x^{\frac{k}{m}} \left(\frac{k}{m} - u\right) B_n'' f(u) du,$$

we can write

$$R_{m,n}^{\infty} f - B_n f = \sum_{k=0}^{m} p_{m,k}(x) \left[\left(\frac{k}{m} - x\right) B_n' f(x) + \int\limits_x^{\frac{k}{m}} \left(\frac{k}{m} - u\right) B_n'' f(u) du \right].$$

By simple computation we obtain:

$$\sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k}{m}-x\right)=0, \tag{5}$$

and

$$\sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k}{m}-x\right)^2=\frac{\varphi^2(x)}{m}. \tag{6}$$

Then in view of (5), (6) and Lemma 1, we obtain

$$\begin{aligned} ||R^{\infty}_{m,n}f-B_nf|| &\le \sum_{k=0}^{m} p_{m,k}(x)\left|\int_x^{\frac{k}{m}}\left(\tfrac{k}{m}-u\right)B''_nf(u)du\right| \\ &\le \frac{||\varphi^2 B''_nf||}{\varphi(x)^2}\sum_{k=0}^{m} p_{m,k}(x)\left(\tfrac{k}{m}-x\right)^2 \\ &= \frac{1}{m}||\varphi^2 B''_nf||, \end{aligned} \tag{7}$$

which completes the proof.

Theorem 1 *Let $f\in C[0,1]$. Then there exists a constant $C>0$ such that*

$$||R^{\infty}_{m,n}f-f||\le C\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}. \tag{8}$$

Proof. We have

$$||R^{\infty}_{m,n}f-f||\le ||R^{\infty}_{m,n}f-B_nf||+||B_nf-f||. \tag{9}$$

Let $g\in W^2_{\infty}(\varphi)$. In view of Lemma 3 and Lemma 7.4 ([1], p.234) we obtain

$$\begin{aligned} ||R^{\infty}_{m,n}f-B_nf|| &\le \frac{1}{m}||\varphi^2 B''_nf|| \\ &\le \frac{1}{m}\{||\varphi^2 B''_n(f-g)||+||\varphi^2 B''_ng||\} \\ &\le \frac{1}{m}\{2n||f-g||+12||\varphi^2 g''||\} \\ &\le 12\frac{n}{m}\{||f-g||+\frac{1}{n}||\varphi^2 g''||\}. \end{aligned}$$

So

$$\begin{aligned} ||R^{\infty}_{m,n}f-B_nf|| &\le 12\frac{n}{m}\inf\{||f-g||+\frac{1}{n}||\varphi^2 g''||: g\in W^2_{\infty}(\varphi)\} \\ &= 12\frac{n}{m}K_{2,\varphi}(f,n^{-1})_{C[0,1]}. \end{aligned}$$

Because $K_{2,\varphi}(f, n^{-1})_{C[0,1]}$ is equivalent with $\omega_\varphi^2(f, n^{-1/2})_{C[0,1]}$ in view of Theorem 2.1.1 ([3], p.11) we obtain the existence of a constant $C_1 \neq C_1(f, n, m) > 0$ such that

$$||R_{m,n}^\infty f - B_n f|| \leq 12\frac{n}{m}C_1\omega_\varphi^2(f, n^{-1/2})_{C[0,1]}. \tag{10}$$

On the other hand it has been shown in [3] that for some constant $C_2 \neq C_2(f, n, m) > 0$

$$||B_n f - f|| \leq C_2\omega_\varphi^2(f, n^{-1/2})_{C[0,1]}, \tag{11}$$

for every $f \in C[0,1]$. Thus, by (9), (10) and (11) we obtain for a constant $C = 12\frac{n}{m}C_1 + C_2$ the estimate (8).

Corollary 1 *Under the assumption of Theorem 1 we have*

$$||R_{m,n}^\infty f - f|| \leq C||B_n f - f|| \tag{12}$$

where $C > 0$ is constant.

Proof. In view of [10] we have for some absolute constant $M > 0$

$$M\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \leq ||B_n f - f||. \tag{13}$$

Thus by Theorem 1 we get (12).

Theorem 2 *Let $\alpha_1 = C_0\frac{n}{m} < 1$, where C_0 denotes the absolute constant in Lemma 2 and the pair (n, m) is chosen accordingly. Then there exists a constant $C > 0$ such that for all $f \in C[0,1]$ we have*

$$C^{-1}||B_n f - f|| \leq ||R_{m,n}^\infty f - f|| \leq C||B_n f - f|| \tag{14}$$

and

$$C^{-1}\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \leq ||R_{m,n}^\infty f - f|| \leq C\omega_\varphi^2(f, n^{-1/2})_{C[0,1]}. \tag{15}$$

Proof. We have

$$\begin{aligned} ||B_n f - f|| &\leq ||R_{m,n}^\infty f - f|| + ||R_{m,n}^\infty f - B_n f|| \\ &\leq ||R_{m,n}^\infty f - f|| + C_0\frac{n}{m}||B_n f - f|| \end{aligned}$$

in view of Lemma 2 and (4). But $\alpha_1 = C_0\frac{n}{m} < 1$, by assumption, and therefore

$$||B_n f - f|| \leq ||R_{m,n}^\infty f - f|| + \alpha_1||B_n f - f||.$$

So

$$(1 - \alpha_1)||B_n f - f|| \leq ||R_{m,n}^\infty f - f||.$$

Hence by Corollary 1 we obtain (14) for some $C > 0$. The inequalities in (15) are direct consequences of (13) and (14). Thus the theorem is proved.

2.2 Case II - general composition

We define

$$\begin{aligned} R^{\varrho}_{m,n}f(x) &= (B_m \circ \mathcal{B}^{-1,-1}_{n\varrho} \circ B_n)(f,x) \\ &= \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1} B_n f(t)dt}{B(k\varrho,(m-k)\varrho)}, \end{aligned}$$

a positive linear operator that reproduces linear functions with $\varrho > 0$.

Lemma 4 *Let $f \in C[0,1]$ and $\varrho > 0$. Then*

$$||R^{\varrho}_{m,n}f - B_n f|| \leq \frac{\varrho+1}{m\varrho+1}||\varphi^2 B''_n f||. \tag{16}$$

Proof.

$$R^{\varrho}_{m,n}f(x) = \sum_{k=0}^{m} p_{m,k}(x) \frac{1}{B(k\varrho,(m-k)\varrho)} \int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1} B_n f(t)dt, \quad x \in (0,1).$$

Hence

$$R^{\varrho}_{m,n}f(x) - B_n f(x) = \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}[B_n f(t) - B_n f(x)]dt}{B(k\varrho,(m-k)\varrho)}$$

and by Taylor expansion with integral remainder, that is,

$$B_n f(t) = B_n f(x) + (t-x)B'f(x) + \int_x^t (t-u)B''_n f(u)du,$$

we can write

$$R^{\varrho}_{m,n}f(x) - B_n f(x) =$$

$$= \sum_{k=0}^{m} p_{m,k}(x) \frac{\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}\left[(t-x)B'f(x) + \int_x^t (t-u)B''_n f(u)du\right]dt}{B(k\varrho,(m-k)\varrho)}.$$

By simple computations we obtain

$$\sum_{k=0}^{m} p_{m,k}(x) \frac{1}{B(k\varrho,(m-k)\varrho)} \int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}(t-x)dt = 0 \tag{17}$$

and

$$(18)\quad \sum_{k=0}^{m} p_{m,k}(x)\frac{1}{B(k\varrho,(m-k)\varrho)}\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}(t-x)^2dt = \frac{\varrho+1}{m\varrho+1}\varphi^2(x),$$

respectively. Then in view of (17), Lemma 1 and (18), we get

$$\begin{aligned} |R^{\varrho}_{m,n}f(x)-B_nf(x)| &\leq \sum_{k=0}^{m} p_{m,k}(x)\frac{\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}\left|\int_x^t (t-u)B''_nf(u)du\right| dt}{B(k\varrho,(m-k)\varrho)} \\ &\leq \frac{\|\varphi^2 B''_nf\|}{\varphi^2(x)}\sum_{k=0}^{m} p_{m,k}(x)\frac{\int_0^1 t^{k\varrho-1}(1-t)^{(m-k)\varrho-1}(t-x)^2dt}{B(k\varrho,(m-k)\varrho)} \\ &= \frac{\varrho+1}{m\varrho+1}\|\varphi^2B''_nf\| \end{aligned}$$

which completes the proof.

Theorem 3 *Let $f\in C[0,1]$. Then there exists a constant $C>0$ such that*

$$(19)\qquad \|R^{\varrho}_{m,n}f-f\|\leq C\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}.$$

Proof. We have

$$(20)\qquad \|R^{\varrho}_{m,n}f-f\|\leq \|R^{\varrho}_{m,n}f-B_nf\|+\|B_nf-f\|.$$

Let $g\in W^2_{\infty}(\varphi)$. In view of Lemma 4 and Lemma 7.4 ([1], p.234) we obtain

$$\begin{aligned} \|R^{\infty}_{m,n}f-B_nf\| &\leq \frac{1+\varrho}{m\varrho+1}\|\varphi^2B''_nf\| \\ &\leq \frac{1+\varrho}{m\varrho+1}\{\|\varphi^2B''_n(f-g)\|+\|\varphi^2B''_ng\|\} \\ &\leq \frac{1+\varrho}{m\varrho+1}\{2n\|f-g\|+12\|\varphi^2g''\|\} \\ &\leq 12\frac{n(1+\varrho)}{m\varrho+1}\{\|f-g\|+\frac{1}{n}\|\varphi^2g''\|\}. \end{aligned}$$

So

$$\begin{aligned} \|R^{\infty}_{m,n}f-B_nf\| &\leq 12\frac{n(1+\varrho)}{m\varrho+1}\inf\{\|f-g\|+\frac{1}{n}\|\varphi^2g''\| : g\in W^2_{\infty}(\varphi)\} \\ &= 12\frac{n(1+\varrho)}{m\varrho+1}K_{2,\varphi}(f,n^{-1})_{C[0,1]}. \end{aligned}$$

Because $K_{2,\varphi}(f,n^{-1})_{C[0,1]}$ is equivalent with $\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}$ in view of Theorem 2.1.1 ([3], p.11) we obtain the existence of a constant $C_1\neq C_1(f,n,m,\varrho)>0$ such that

$$(21)\qquad \|R^{\varrho}_{m,n}f-B_nf\|\leq 12\frac{n(1+\varrho)}{m\varrho+1}C_1\omega^2_{\varphi}(f,n^{-1/2})_{C[0,1]}.$$

On the other hand it has been shown in [3] that for some constant $C_2 \neq C_2(f, n, m, \varrho) > 0$

$$||B_n f - f|| \leq C_2 \omega_\varphi^2(f, n^{-1/2})_{C[0,1]}, \tag{22}$$

for every $f \in C[0,1]$. Thus, by (20), (21) and (22) for a constant $C = 12\dfrac{n(1+\varrho)}{m\varrho+1}C_1 + C_2$ we obtain (19).

Corollary 2 *Under the assumption of* Theorem 3 *we have*

$$||R_{m,n}^{\varrho} f - f|| \leq C ||B_n f - f|| \tag{23}$$

where $C > 0$ is constant.

Proof. In view of [10] we have for some absolute constant $M > 0$

$$M \omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \leq ||B_n f - f|| \tag{24}$$

Thus by Theorem 3 we get (23).

Theorem 4 *Let $\alpha_2 = C_0 \dfrac{n(\varrho+1)}{m\varrho+1} < 1$, where C_0 denotes the absolute constant in Lemma 2 and the triplet (n, m, ϱ) is chosen accordingly. Then there exists a constant $C > 0$ such that for all $f \in C[0,1]$ we have*

$$C^{-1}||B_n f - f|| \leq ||R_{m,n}^{\varrho} f - f|| \leq C ||B_n f - f|| \tag{25}$$

and

$$C^{-1}\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \leq ||R_{m,n}^{\varrho} f - f|| \leq C \omega_\varphi^2(f, n^{-1/2})_{C[0,1]}. \tag{26}$$

Proof. We have

$$\begin{aligned} ||B_n f - f|| &\leq ||R_{m,n}^{\varrho} f - f|| + ||R_{m,n}^{\varrho} f - B_n f|| \\ &\leq ||R_{m,n}^{\varrho} f - f|| + C_0 \frac{n(1+\varrho)}{m\varrho+1} ||B_n f - f|| \end{aligned}$$

in view of Lemma 2 and (16). But $\alpha_2 = C_0 \dfrac{n(1+\varrho)}{m\varrho+1} < 1$ by assumption, and therefore

$$||B_n f - f|| \leq ||R_{m,n}^{\varrho} f - f|| + \alpha_2 ||B_n f - f||.$$

So

$$(1 - \alpha_2)||B_n f - f|| \leq ||R_{m,n}^{\varrho} f - f||.$$

Hence by Corollary 2 we obtain (25) for some $C > 0$. The inequalities in (26) are direct consequences of (24) and (25). Thus the theorem is proved.

3 Concluding remarks

This method can be applied for compositions of linear positive operators that reproduce linear functions under the assumption that a strong converse inequality as the one given in Lemma 2 exists for the operator on the right hand side.
This method does not cover all the cases of the composition as a result of the restrictions applied to the constants.

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Elena-Dorina Stănilă
University of Duisburg-Essen
Faculty of Mathematics
Forsthausweg 2
47057 Duisburg, Germany
e-mail: elena.stanila@stud.uni-due.de

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A new kind of a generalized bottleneck assignment problem [1]

Oana Ruxandra Tuns (Bode), Luciana Neamţiu

Abstract

The present paper points out a new kind of a generalized bottleneck assignment problem. The problem to be considered here represents the mathematical modeling of assigning unemployed persons to professional training programs, based on some given restrictions. Necessary and sufficient optimality conditions are given. A polynomial algorithm for solving this problem is given.

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1 Introduction

Classic assignment problems deal with the question how to assign n tasks to n machines (or workers) in the best possible way. The classic assignment problem is to find a one-to-one matching between n tasks and n workers, while the objective being to minimize the total cost of the assignments. While the objective of a classic assignment problem is to minimize or maximize the sum of the costs of the assignments of tasks to workers, the objective of the bottleneck assignment problem is to minimize the maximum of the costs of the assignments (or to maximize the minimum of the efficiency of the assignments).

For the first time the bottleneck assignment problem was mentioned in literature in 1953, by Fulkerson, Glicksberg and Gross [4]. In Ford and Fulkerson [3] the authors give an example that involves assigning workers to jobs in such a way that the minimum efficiency of such assignments will be maximized. The bottleneck assignment problem has been also studied by Gross [6], Garfinkel [5], Ravindran and

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Ramaswami [9]. During the time, the assignment problems knew different generalizations. A very useful overview regarding the variety of models of the assignment problems can be find in Pentico [8]. Some of the variety of models of the assignment problems that can be find here are the lexicographic bottleneck problem, the assignment problem with side constraints and r-lexicographic multi-objective problem. The lexicographic bottleneck problem have been studied for example in Burkard and Rendl [2] or in Sokkalingam and Aneja [10]. Another useful work for researchers and practitioners is the one authored by Burkard et al. [1]. It provides a comprehensive treatment of assignment problems from their conceptual beginnings through present-day.

The problem to be discussed in the present paper represents a new kind of a generalized bottleneck assignment problem. It images the modeling of a concrete economic problem based on some given restrictions and involves assigning unemployed persons to professional training programs (PTPs). Assume that in the same period of time, there are organized different PTPs for the unemployed persons. For each PTP there is known its efficiency, defined from the point of view of finding a place to work by the unemployed persons after graduating it. All the registered unemployed persons need to participate to these programs. For each PTP there is known the maximum number of the persons that can attend it and the score that each unemployed person has if attends it (this score was calculated based on historical data about each unemployed person taking into account his/her education or professional experience). The problem that arises is how to assign all the registered unemployed persons to the PTPs, based on each person's score, such that the following restrictions to be fulfilled:

i) all unemployed persons to attend the PTPs;

ii) each unemployed person to attend exactly one PTP;

iii) the assignment of the unemployed persons to a PTP to be done such that to maximize the minimum score of the assignments;

iv) the efficiency of the PTP for which the minimum score is reached, to be as small as possible and to be reached as few times as possible.

Furthermore, the present paper is organized as follows. In Section 2 we give the mathematical model of our concrete economic problem and we study some properties of its optimal solutions. Based on these properties, in Section 3 a polynomial algorithm for solving our problem is presented and an example is given. Section 4 contains the conclusions of this paper.

2 Mathematical modeling of our economic problem

Let us denote by:

-m the number of the total PTPs identified by the variable i, $i \in \{1, \ldots, m\}$. Let $I = \{1, \ldots, m\}$;

-e_i, $i \in I$ the efficiency of the PTP i;

-n the total number of the unemployed persons that need to attend the PTPs.

Let $J = \{1, \ldots, n\}$;

-a_i, $i \in I$ the maximum number of the persons that can participate to the PTP i, $i \in I$;

-r_{ij}, $i \in I$, $j \in J$ the score corresponding to each unemployed person j if attends the PTP i, $i \in I$. Let $R \in \mathcal{M}_{m\times n}\{\mathbb{R}_+^*\}$ be the matrix which contains the scores r_{ij};

-y_{ij}, $i \in I$, $j \in J$ the binary variable having the significance $y_{ij} = 1$, if the unemployed person j will participate to the course i, and $y_{ij} = 0$, otherwise.

In the formulation of the restrictions of our practical problem the values of the efficiencies of the PTPs doesn't interfere. It interfere just the arrangement of the efficiency of one PTP in relation to the other PTPs. Therefore, we assume that the arrangement of the PTPs was done in a descending order of their efficiency, i.e. $e_i \geq e_{i+1}$, $\forall\, i \in I$.

Let $\mathcal{Y}$ be the set of the matrices $Y = [y_{ij}] \in M_{m\times n}(\mathbb{R})$ which fulfill the following conditions:

$$y_{ij} \in \{0, 1\},\ \forall\, i \in I, \forall\, j \in J; \tag{1}$$

$$\sum_{i\in I} y_{ij} = 1,\ \forall\, j \in J; \tag{2}$$

$$\sum_{j\in J} y_{ij} \leq a_i,\ \forall\, i \in I. \tag{3}$$

Let $f = (f_1, f_2, f_3) : \mathcal{Y} \to \mathbb{R}^3$ be the function given by

$$f_1(Y) = \min\Big\{r_{ij} \,|\, i \in I,\ j \in J,\ y_{ij} = 1\Big\} = \min\Big\{r_{ij}y_{ij} \,|\, i \in I,\ j \in J\Big\},\ \forall\, Y \in \mathcal{Y}, \tag{4}$$

$$f_2(Y) = \min\Big\{i \in I \,|\, \exists\, j \in J \quad \text{such that} \quad r_{ij}y_{ij} = f_1(Y)\Big\},\ \forall\, Y \in \mathcal{Y}, \tag{5}$$

(6)
$$f_3(Y) = \sum_{(i,j)\in I\times J;\, r_{ij}y_{ij}=f_1(Y)} y_{ij} = card\Big\{y_{ij} \,|\, i \in I,\ j \in J,\ r_{ij}y_{ij} = f_1(Y)\Big\},\ \forall\, Y \in \mathcal{Y}.$$

We suppose that

$$\sum_{i\in I} a_i \geq n, \tag{7}$$

i.e. the total number of the persons that can attend the PTPs is greater that the total number of the registered unemployed persons. Condition (7) assures that $\mathcal{Y} \neq \emptyset$.

Based on (7) and using the above notations, the mathematical model attached to our economic problem is a problem of lexicographic optimization type. In order to give this mathematical model we will introduce in $\mathbb{R}^3$ the following order relation of lexicographic type, denoted by $<_{\max-\max-\min}$.

Definition 1 *Let* $x = (x_1, x_2, x_3)$ *and* $y = (y_1, y_2, y_3)$ *be two elements from* $\mathbb{R}^3$.

We say that $x <_{\max-\max-\min} y$ *if and only if one of the followings conditions occurs:*

i) $x_1 < y_1$;

ii) $x_1 = y_1$ *and* $x_2 < y_2$;

iii) $x_1 = y_1$, $x_2 = y_2$ *and* $x_3 > y_3$.

Therefore, our problem denoted by (PS) can be graphically given by

$$(PS) \quad \begin{cases} f(Y) \to lex - \max - \max - \min \\ Y \in \mathcal{Y}. \end{cases} \tag{8}$$

A point $Y^0 \in \mathcal{Y}$ *is said to be an optimal solution of the problem* (PS) *if there is no other point* $Y \in \mathcal{Y}$ *such as to have* $f(Y^0) <_{\max-\max-\min} f(Y)$, *i.e. neither one of the following restrictions to occur:*

i) $f_1(Y) > f_1(Y^0)$;

ii) $f_1(Y) = f_1(Y^0)$ *and* $f_2(Y) > f_2(Y^0)$;

iii) $f_1(Y) = f_1(Y^0)$, $f_2(Y) = f_2(Y^0)$ *and* $f_3(Y) < f_3(Y^0)$.

Based on the restrictions (2) and (3), the problem (PS) can be view as a particular type of an unbalanced transportation problem of time type, having the property that all its variables have a boolean value. On the other hand, based on restriction (1), the problem (PS) can be view as a generalization of the bottleneck assignment problem. Also, our problem can be seen as a resources assignment problem [7]. Whatever we consider this problem, as far as we know a such type of problem have not been studied yet.

In what follows, we will study some properties of the optimal solutions of the problem (PS). Let

$$\lambda = \min\{r_{ij} \mid i \in I,\, j \in J\}, \tag{9}$$

$$h = \min\{i \in I \mid \exists\, k \in J \text{ such that } r_{hk} = \lambda\}, \tag{10}$$

$$J_{h,\lambda} = \{j \in J \mid r_{hj} = \lambda\}, \tag{11}$$

and

(12) $q = card(J_{h,\lambda})$, i.e. q represents the number of the elements of the set $J_{h,\lambda}$.

Proposition 1 *If*

$$\sum_{i \in I} a_i < n + card(J_{h,\lambda}), \tag{13}$$

then for each $Y \in \mathcal{Y}$ *the following conditions occur:*

$$f_1(Y) = \lambda \tag{14}$$

and

$$f_2(Y) = h. \tag{15}$$

Proof. Let $Y^0 = [y^0_{ij}] \in \mathcal{Y}$. For each $i \in I \backslash \{h\}$ the restriction (3) is fulfilled and we get

$$\sum_{i \in I \backslash \{h\}} \sum_{j \in J} y^0_{ij} \leq \sum_{i \in I \backslash \{h\}} a_i. \tag{16}$$

Based on restriction (2) we obtain that

$$\sum_{i \in I} \sum_{j \in J} y^0_{ij} = \sum_{j \in J} \sum_{i \in I} y^0_{ij} = \sum_{j \in J} 1 = n. \tag{17}$$

On the other hand, we have that

$$\sum_{i \in I} \sum_{j \in J} y^0_{ij} = \sum_{i \in I \backslash \{h\}} \sum_{j \in J} y^0_{ij} + \sum_{j \in J} y^0_{hj} \leq \sum_{i \in I \backslash \{h\}} a_i + \sum_{j \in J} y^0_{hj}. \tag{18}$$

Based on (17) and (18) we deduce that

$$\sum_{j \in J} y^0_{hj} \geq n - \sum_{i \in I \backslash \{h\}} a_i. \tag{19}$$

From (13) we deduce that

$$a_h - card(J_{h,\lambda}) < n - \sum_{i \in I \backslash \{h\}} a_i. \tag{20}$$

Therefore, (19) and (20) imply the existence of

$$card\{j \in J_{h,\lambda} \,|\, y^0_{hj} = 1\} \geq 1. \tag{21}$$

Based on (21) and (9) it results that $f_1(Y^0) = \lambda$. Then, from (10) we deduce that $f_2(Y^0) = h$.

As $Y^0 \in \mathcal{Y}$ was chosen arbitrary, it results that (14) and (15) are true, for each $Y \in \mathcal{Y}$. $\diamond$

Proposition 2 *If*

$$\sum_{i \in I} a_i \geq n + card(J_{h,\lambda}), \tag{22}$$

then for each $Y^0 \in \mathcal{Y}$ an optimal solution of the problem (PS) we have

$$y^0_{hj} = 0, \, \forall \, j \in J_{h,\lambda}. \tag{23}$$

Proof. Let us suppose that there exists an optimal solution $Y^0 = [y^0_{ij}] \in \mathcal{Y}$ of the problem (PS) and there exists $k \in J_{h,\lambda}$ such that $y^0_{hk} = 1$. Then, based on (9) and (10) we get that

$$f_1(Y^0) = \lambda \tag{24}$$

and

$$f_2(Y^0) = h. \tag{25}$$

Now, we will build a matrix $Y^* = [y^*_{ij}] \in \mathcal{Y}$ such that $f_1(Y^*) \geq \lambda$ or $f_1(Y^*) = \lambda$, but $f_2(Y^*) > h$.

For this, without restricting the generality, we can suppose that $h = 1$ and $J_{h,\lambda} = \{1, \ldots, q\}$. Indeed, if $h > 1$, then we can interchange in the scores matrix R the line h with the first line, and then interchange the columns between them, such that the first q elements from the first line to be equal to λ. Based on the above, (12) implies

$$r_{hj} > \lambda, \forall\, j \in J \backslash \{1, \ldots, q\}. \tag{26}$$

We build the matrix $Y^* = [y^*_{ij}]$ such that

$$y^*_{ij} = \begin{cases} 1, & \text{if} \quad i = 1 \quad \text{and} \quad j \in J \backslash \{1, \ldots, \sum_{s=2}^{m} a_s\}, \\ 0, & \text{if} \quad i = 1 \quad \text{and} \quad j \in J \cap \{1, \ldots, \sum_{s=2}^{m} a_s\}, \\ 1, & \text{if} \quad i = 2 \quad \text{and} \quad j \in J \cap \{1, \ldots, a_1\}, \\ 0, & \text{if} \quad i = 2 \quad \text{and} \quad j \in J \backslash \{1, \ldots, a_1\}, \\ 1, & \text{if} \quad 3 \leq i \leq m \quad \text{and} \quad j \in J \cap \{\sum_{s=2}^{i-1} a_s + 1, \ldots, \sum_{s=2}^{i} a_s\}, \\ 0, & \text{if} \quad 3 \leq i \leq m \quad \text{and} \quad j \in J \backslash \{\sum_{s=2}^{i-1} a_s + 1, \ldots, \sum_{s=2}^{i} a_s\}. \end{cases} \tag{27}$$

It is not difficult to see that $Y^* \in \mathcal{Y}$. From the way we have build the matrix Y^* and based on (9) and (26) we obtain that $f_1(Y^*) \geq \lambda$.

If $f_1(Y^*) > \lambda$, then (24) contradicts the optimality of Y^0.

If $f_1(Y^*) = \lambda$, then there exists $(i, j) \in I \times J$ such that $r_{ij} y^*_{ij} = \lambda$. Let $u = \min\{i \in I \,|\, \exists\, j \in J \,\text{such that}\, r_{ij} y^*_{ij} = \lambda\}$. As $y^*_{ij} = 0, \forall\, j \in \{1, \ldots, q\} \cap J$, from (27) it results that $u \geq 2$. Therefore, $f_2(Y^*) = u > 1 = f_2(Y^0)$. Again, the optimality of Y^0 is contradicted.

Hence, (23) must be true. $\diamond$

From *Proposition 2*, it results that if $q = n$, then to determine an optimal solution of the problem (PS) is equivalent to determine an optimal solution of a problem of the same type as the problem (PS), but in which in the scores matrix the line h does not appear. In what follows, we suppose that $q < n$.

Let us now consider the following lexicographic optimization problem

$$(PM) \quad \begin{cases} \varphi(Y) = \begin{pmatrix} f_1(Y) \\ f_2(Y) \end{pmatrix} \to lex - \max - \max \\ Y \in \mathcal{Y}. \end{cases} \tag{28}$$

Let $\tilde{Y}$ be an optimal solution of the problem (PM), $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$. We build the matrix $C_{\lambda,h} = [c_{ij}]$ given by

$$c_{ij} = \begin{cases} 0, & \text{if} \quad r_{ij} > \lambda, \\ 1, & \text{if} \quad r_{ij} = \lambda \quad \text{and} \quad i \geq h, \\ n+1, & \text{if} \quad (r_{ij} < \lambda) \quad \text{or} \quad (r_{ij} = \lambda \quad \text{and} \quad i < h). \end{cases} \tag{29}$$

Let $g_{\lambda,h} : \mathcal{Y} \to \mathbb{R}$ be the function given by

$$g_{\lambda,h}(Y) = \sum_{i\in I}\sum_{j\in J} c_{ij}y_{ij},\ \forall\, Y \in \mathcal{Y}.$$

Let us consider the following optimization problem

$$(30)\qquad (AC_{\lambda,h}) \quad \begin{cases} g_{\lambda,h}(Y) \to \min \\ Y \in \mathcal{Y}. \end{cases}$$

The problem $(AC_{\lambda,h})$ represents a transportation problem which can be solved using the potential plan algorithm.

Let

$$E_1 = \Big\{(i,j) \in I \times J \,|\, r_{ij} = \lambda,\, i \geq h\Big\}$$

and

$$E_2 = \Big\{(i,j) \in I \times J \,|\, (r_{ij} < \lambda) \quad \text{or} \quad (r_{ij} = \lambda \quad \text{and} \quad i < h)\Big\}.$$

We remark that

$$(31)\qquad g_{\lambda,h}(Y) = \sum_{(i,j)\in E_1} y_{ij} + (n+1) \sum_{(i,j)\in E_2} y_{ij}.$$

Remark 1 *$\tilde{Y}$ is an optimal solution of the problem (PS) if and only if $\tilde{Y}$ is an optimal solution of the problem (PM) and is also an optimal solution of the problem $(AC_{\lambda,h})$.*

From *Remark 1* it results that the solving of the problem (PS) can be reduced by solving two problems: on one hand, we have to solve the problem (PM) and, on the other hand, we have to verify the optimality of the solution $\tilde{Y}$ of the problem (PM) for the problem $(AC_{\lambda,h})$. In order to do this, we have to apply the potential plan theorem. Then, we obtain the following result.

Remark 2 *$\tilde{Y}$ is an optimal solution for the problem $(AC_{\lambda,h})$ if and only if*

$$\sum_{s=1}^{m} c_{sj}\tilde{y}_{sj} \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J.$$

If $\tilde{Y}$ is not an optimal solution of the problem $(AC_{\lambda,h})$, then applying the potential plan algorithm to the problem $(AC_{\lambda,h})$ we obtain an optimal solution of the problem (PS).

3 An algorithm for solving the problem (PM)

Based on *Proposition 1* and *Proposition 2* we give a polynomial algorithm for solving the problem (PM).

Let R^k be the work matrix that we will build at each iteration. Initially, $R^1 = R$. The optimal solution of (PM) will be memorized by the matrix $Y = [y_{ij}] \in \mathcal{M}_{m\times n}\{0,1\}$, $i \in I$, $j \in J$, and the optimal value of the function φ by the vector $F = (F_1, F_2)$. Initially, $Y = O_{m\times n}$ (null matrix).

The idea of our algorithm is the following:

- at each iteration k it is determined the value $\lambda_k := \min\{r^k_{ij} \,|\, i \in I,\, j \in J\}$. If $\lambda_k = +\infty$, we set $F = \varphi(Y)$ and we stop;

- at each iteration we pass through the work matrix R^k from left to right and from up to down. When we find the first element r^k_{ij} equal to λ_k we verify if the column and the line that contains it have more than two finite elements each one:

  - if *Yes*, i.e. $\exists\, p, r \in I$, $\exists\, q, s \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{rj} \neq +\infty, r^k_{iq} \neq +\infty, r^k_{ir} \neq +\infty$, then we set $r^{k+1}_{ij} := +\infty$;

  - if *No*, we have the following three possible cases:

  (i) if the number of the finite elements from the column that contains it, is exactly two and the number of the finite elements from the line that contains it, is greater than two, i.e. $\exists\, p \in I, \exists\, q, s \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{hj} = +\infty$, $\forall h \in I \setminus \{i, p\}$, $r^k_{iq} \neq +\infty$, $r^k_{is} \neq +\infty$, then we set $r^{k+1}_{hj} := +\infty$, $\forall h \in I$, $y_{pj} := 1$,$a_p := a_p - 1$;

  (ii) if the number of the finite elements from the column that contains it, is exactly two and the number of the finite elements from the line that contains it, is exactly two, also, i.e. $\exists\, p \in I, \exists\, q \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{hj} = +\infty, \forall h \in I \setminus \{i, p\}$ and $r^k_{iq} \neq +\infty$, $r^k_{it} = +\infty, \forall t \in J \setminus \{j, q\}$, then we set $r^{k+1}_{it} := +\infty, \forall t \in J$, $r^{k+1}_{hj} := +\infty, \forall h \in I$, $y_{iq} := 1$, $y_{pj} := 1$, $a_i := a_i - 1$;

  (iii) if the number of the finite elements from the column that contains it, is greater than two and the number of the finite elements from the line that contains it, is exactly two, i.e. $\exists\, p, s \in I, \exists\, q \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{sj} \neq +\infty, r^k_{iq} \neq +\infty$,
  $r^k_{ih} = +\infty, \forall h \in J \setminus \{j, q\}$, then we set $r^{k+1}_{ih} := +\infty, \forall h \in I$, $y_{iq} := 1$, $a_i := a_i - 1$;

The efficiency of our algorithm results from the fact that we pass through the scores matrix R from up to down. The algorithm described above is presented below:

Input

the natural numbers m, n;
the elements of natural vector $a = (a_1, \ldots, a_m)$;

the elements of natural matrix $R = [r_{ij}],\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}$;

Output

ok — *true* if a solution exists,
$Y = [y_{ij}],\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}$ and $F = (F_1, F_2)$ — the solution

Algorithm

```
ok := false; sw := 0;
for j = 1 to n do
    s_j := 0;
    for i = 1 to m do
        y_ij := −1;
    end for
end for
I := {1, ..., m}; J := {1, ..., n};
while J ≠ ∅ do
    r := min{r_ij | i ∈ I, j ∈ J};
    for i = 1 to m do
        if i ∈ I then
          for j = 1 to n do
            if j ∈ J then
              if r_ij = r then
                s_j := s_j + 1; r_ij := +∞;
                if s_j = m then
                  y_ij := 1;
                  if sw = 0 then
                    F_1 := r_ij; F_2 := i; sw := 1;
                  end if
                  a_i := a_i − y_ij;
                  if a_i < 1 then I := I \ {i};
                  end if
                  J := J \ {j};
                else
                  y_ij := 0;
                  if s_j = m − 1 then
                    s_j := m;
                    for k = 1 to m do
                      if k ∈ I then
                        if y_kj = −1 then
                          y_kj := 1; r_kj := +∞; a_k := a_k − y_kj;
                            if a_k = 0 then I := I \ {k};
                            end if
                        end if
                      end if
```

end for
end if
end if
end if
end if
end for
end if
end for
end while
if $I = \emptyset$ then $OK := false$;
end if
if $OK = false$ then
output: there is an error in the algorithm;
else
output: $Y = [y_{ij}]$ is the optimal solution of the problem (PM) and $F = (F_1, F_2)$ is the optimal value of the function φ;
end if
End Algorithm

Now, an easiest example to point out how our algorithm works is given.

Example: Let $m = 3$, $n = 6$, $a_1 = 2$, $a_2 = 3$, $a_3 = 3$ and let the matrices

$$R = R^1 = \begin{bmatrix} 2 & 7 & 3 & 4 & 1 & 5 \\ 3 & 1 & 2 & 3 & 2 & 5 \\ 7 & 1 & 8 & 3 & 7 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iteration 1: We get $a_1 = 1$, $a_2 = 3$, $a_3 = 3$,

$$R^2 = \begin{bmatrix} 2 & +\infty & 3 & 4 & +\infty & 5 \\ 3 & +\infty & 2 & 3 & 2 & 5 \\ 7 & +\infty & 8 & 3 & 7 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iteration 2: We get $a_1 = 1$, $a_2 = 3$, $a_3 = 2$,

$$R^3 = \begin{bmatrix} +\infty & +\infty & 3 & 4 & +\infty & 5 \\ 3 & +\infty & +\infty & 3 & +\infty & 5 \\ 7 & +\infty & 8 & 3 & +\infty & +\infty \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Iteration 3: We get $a_1 = 0$, $a_2 = 2$, $a_3 = 0$,

$$R^4 = \begin{bmatrix} +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, Y is the optimal solution of the problem (PM) and $F = (4, 1)$ is the optimal value of the function φ.

4 Conclusions

The problem considered in the present paper points out a new kind of a generalized bottleneck assignment problem. On one hand, it can be seen as a generalization of the following three types of bottleneck assignment problems: lexicographic, r-lexicographic multi-objective or with side constraints.

Although in order to determine the optimal solution of our problem we use a type of lexicographic ordering introduced by us, our problem is different from the *lexicographic bottleneck problem* ([2], [8] or [10]). The first objective of our problem (to find an assignment of the persons to PTPs such that to maximize the minimum score of the assignments) is analogously to the first objective of a lexicographic bottleneck problem (to minimize the largest cost coefficient). The difference appears at the second objective of the studied problem: for our problem the second objective is that the minimum score to be reached at a PTP with an efficiency as small as possible (so the objective function is given by (5)), while in the lexicographic bottleneck problem is to minimize the second largest cost coefficient.

For this new type of problem, we give necessary and sufficient optimality conditions. The solving of our economic problem (PS) can be reduced by solving two problems: (PM) and $(AC_{\lambda,h})$.

First, we have to solve the lexicographic optimization problem (PM). A polynomial algorithm for solving it is given. This algorithm can be used for solving the classic bottleneck assignment problem.

If Y^0 is an optimal solution of (PM) and (λ, h) the optimal value, then we have to verify the optimality of Y^0 for the problem $(AC_{\lambda,h})$. For this, we use the *Remark 2*. If Y^0 is not an optimal solution of the problem $(AC_{\lambda,h})$, then we can obtain an optimal solution of the problem (PS) applying the potential plan algorithm to solve the problem $(AC_{\lambda,h})$, using Y^0 as the initial plan.

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Oana Ruxandra Tuns (Bode)
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
M. Kogălniceanu Street, no. 1, 400084, Cluj-Napoca, România
e-mail: oanabode@yahoo.com

Luciana Neamţiu
Oncology Institute Prof. Dr. I. Chiricuţă
Department
Republicii Street, no. 34-36, 400015, Cluj-Napoca, România
e-mail: luciana@iocn.ro

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Characterization theorem's of Jacobi polynomials [1]

Ioan Ţincu

Abstract

The aim of this paper is to prove of Turan's inequality and on equalities concerning the roots of the Jacobi polynomial.

2010 Mathematics Subject Classification: 33C45, 33C52.
Key words and phrases: interpolation, orthogonal polynomials, roots.

1 Introduction

For $\alpha > -1, \beta > -1$, let $R_n^{(\alpha,\beta)} ={}_2 F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \dfrac{1-x}{2}\right)$ be the Jacobi polynomials of degree n normalized by $R_n^{(\alpha,\beta)}(1) = 1$.

In 1950, P. Turán showed that for $n \geq 1$

$$\begin{vmatrix} P_n(x) & P_{n-1}(x) \\ P_{n+1}(x) & P_n(x) \end{vmatrix} > 0, \quad \forall x \in (-1,1).$$

Later, G. Gosper [1] studied an inequality of this type for Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ using a complicated demonstration.

A. Lupaş has raised the problem of finding a much simpler proofs of Turan's inequality, using interpolation theory. By using interpolation theory, Ţincu proved Turan inequality for the case $\alpha = \beta$ ([3]).

The following formulas are known:

$$(1)\qquad (1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) + \\ + n(n+\alpha+\beta+1)y(x) = 0, \quad y(x) = R_n^{(\alpha,\beta)}(x),$$

$$(2)\qquad \left[R_n^{(\alpha,\beta)}(x)\right]' = \frac{n(n+\alpha+\beta+1)}{2(\alpha+1)} R_{n-1}^{(\alpha-1,\beta-1)}(x).$$

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2 Main result

From $R_n^{(\alpha,\beta)}(x) ={}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right)$ we have
$R_n^{(\alpha,\beta)}(x) = \frac{\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)} \cdot \frac{x^n}{2^n} + ...,$
$R_{n+1}^{(\alpha-1,\beta-1)}(x) = \frac{\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha)}{\Gamma(n+\alpha+\beta)\Gamma(n+\alpha+1)} \cdot \frac{x^{n+1}}{2^{n+1}} + ...,$
$R_{n-1}^{(\alpha+1,\beta+1)}(x) = \frac{\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+1)} \cdot \frac{x^{n-1}}{2^{n-1}} + ...$

We consider

$$\Delta_{2n}(x) = \left[R_n^{(\alpha,\beta)}(x)\right]^2 - R_{n-1}^{(\alpha+1,\beta-1)}(x) \cdot R_{n+1}^{(\alpha-1,\beta-1)}(x) = a_{0,n}x^{2n} + ...$$

where

$$a_{o,n} = -\frac{n+\beta}{4^n\alpha(n+\alpha+\beta+1)}\left[\frac{\Gamma(2n+\alpha+\beta+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}\right]^2 \neq 0$$

and observe that $\Delta_{2n} \in \Pi_{2n}$.

Theorem 1 *If $\alpha > 0$, $\beta > -1$ and $n \geq 1$, then*

$$\left|\begin{array}{cc} R_n^{(\alpha,\beta)}(x) & R_{n+1}^{(\alpha-1,\beta-1)}(x) \\ R_{n-1}^{(\alpha+1,\beta+1)}(x) & R_n^{(\alpha,\beta)}(x) \end{array}\right| > 0, \quad \forall x \in (-1,1). \tag{3}$$

Proof. From (2) we obtain

$$R_{n-1}^{(\alpha+1,\beta+1)}(x) = \frac{2}{n+\alpha+\beta+1} \cdot \frac{\alpha+1}{n}\left[R_n^{(\alpha,\beta)}(x)\right]', \tag{4}$$

$$\left[R_{n+1}^{(\alpha-1,\beta-1)}(x)\right]' = \frac{n+\alpha+\beta}{2} \cdot \frac{n+1}{\alpha} R_n^{(\alpha,\beta)}(x), \tag{5}$$

$$\left[R_{n+1}^{(\alpha-1,\beta-1)}(x)\right]'' = \frac{n+\alpha+\beta}{2} \cdot \frac{n+1}{\alpha}\left[R_n^{(\alpha,\beta)}(x)\right]'. \tag{6}$$

Using (1), (5) and (6) one finds

$$\frac{n+1}{\alpha} \cdot \frac{n+\alpha+\beta}{2}(1-x^2)\left[R_n^{(\alpha,\beta)}(x)\right]' + [\beta-\alpha-(\alpha+\beta)x] \cdot \frac{n+\alpha+\beta}{2} \cdot n+1\alpha R_n^{(\alpha,\beta)}(x) +$$

$$+(n+1)(n+\alpha+\beta)R_{n+1}^{(\alpha-1,\beta-1)}(x) = 0,$$

$$R_{n+1}^{(\alpha-1,\beta-1)}(x) = -\frac{1-x^2}{2\alpha}\left[R_n^{(\alpha,\beta)}(x)\right]' - \frac{[\beta-\alpha-(\alpha+\beta)x]}{2\alpha} R_n^{(\alpha,\beta)}(x). \tag{7}$$

From (4) and (7) we have

$$\Delta_{2n}(x) = \left[R_n^{(\alpha,\beta)}(x)\right]^2 + \frac{2(\alpha+1)}{n(n+\alpha+\beta+1)} \left[R_n^{(\alpha,\beta)}(x)\right]' \cdot \left\{ \frac{1-x^2}{2\alpha} \left[R_n^{(\alpha,\beta)}(x)\right]' + \frac{\beta-\alpha-(\alpha+\beta)x}{2\alpha} R_n^{(\alpha,\beta)}(x) \right\}. \tag{8}$$

Let us define the polynomial $f(x) = \dfrac{\Delta_{2n}(x)}{1-x}$, $x \in (-1,1)$. According to Hermite interpolation formula

$$f(x) = H_{2n-1}(x_1, x_1, x_2, x_2, ..., x_n, x_n; f|x) = \sum_{k=0}^{n} \varphi_k(x) A_k(f;x), \tag{9}$$

where $x_1, x_2, , x_n \in (-1,1)$ are the roots of $R_n^{(\alpha,\beta)}(x)$ and

$$\varphi_k(x) = \left\{ \frac{R_n^{(\alpha,\beta)}(x)}{(x-x_0)\left[R_n^{(\alpha,\beta)}(x_k)\right]'} \right\}^2,$$

$$A_k(f;x) = f(x_k) + (x-x_0)\left[f'(x_k) - \frac{\alpha-\beta+(\alpha+\beta+2)x_k}{1-x_k^2} f(x_k)\right].$$

If we prove that $A_k(f;x) > 0, \ \ \forall\, x \in (-1,1)$, then it follows

$$H_{2n-1}(x_1, x_1, x_2, x_2, , x_n, x_n; f|x) > 0,$$

that is $\Delta_{2n}(x) > 0, \ x \in (-1,1)$.

Further, we investigate $A_k(f;x)$. From (2), (4) and (7) we have

$$f(x_k) = \frac{\Delta_{2n}(x_k)}{1-x_k} = \frac{(\alpha+1)(1+x_k)}{\alpha n(n+\alpha+\beta+1)} \left\{ \left[R_n^{(\alpha,\beta)}(x)\right]' \right\}^2, \tag{10}$$

$$\left[R_{n-1}^{(\alpha+1,\beta+1)}(x)\right]' = \frac{2(\alpha+1)}{n(n+\alpha+\beta+1)} \left[R_n^{(\alpha,\beta)}(x)\right]'', \tag{11}$$

$$\left[R_{n+11}^{(\alpha-1,\beta-1)}(x)\right]' = \frac{x}{\alpha}\left[R_n^{(\alpha,\beta)}(x)\right]' - \frac{1-x^2}{2\alpha}\left[R_n^{(\alpha,\beta)}(x)\right]'' + \frac{\alpha+\beta}{2\alpha} R_n^{(\alpha,\beta)}(x) - \frac{\beta-\alpha-(\alpha+\beta)x}{2\alpha}\left[R_n^{(\alpha,\beta)}(x)\right]'. \tag{12}$$

Using (1), (10), (11) and (12) one finds

$$\Delta'_{2n}(x_k) = \frac{\alpha+1}{\alpha n(n+\alpha+\beta+1)} \left[\alpha-\beta+(\alpha+\beta+2)x_k\right] \cdot \left\{ \left[R_n^{(\alpha,\beta)}(x)\right]' \right\}^2,$$

$$f'(x_k)=\frac{(\alpha+1)\left[R_n^{(\alpha,\beta)}(x)\right]^2}{\alpha n(n+\alpha+\beta+1)(1-x_k)}\left[\alpha-\beta+1+(\alpha+\beta+2)x_k\right],$$

$$\frac{f'(x_k)}{f(x_k)}=\frac{\alpha-\beta+1+(\alpha+\beta+3)x_k}{1-x_k^2},$$

$$A_k(f;x)=f(x_k)\left\{1-(x-x_k)\left[\frac{f'(x_k)}{f(x_k)}-\frac{\alpha-\beta+(\alpha+\beta+2)x_k}{1-x_k^2}\right]\right\}$$

$$=f(x_k)\left[1-\frac{(x-x_k)(1+x_k)}{1-x_k^2}\right]=\frac{1-x}{1-x_k}f(x_k)>0,\quad \forall x\in(-1,1).$$

In conclusion

$$f(x)>0,\ \Delta_{2n}(x)>0,\quad \forall\ x\in(-1,1).$$

Corollary 1 *For $\alpha>0,\ \beta>-1,\ n\in\mathbb{N}^*$ and $x\in(-1,1)$, we have*

$$\begin{vmatrix} R_n^{(\alpha,\beta)}(x) & R_{n+1}^{(\alpha-1,\beta-1)}(x) \\ R_{n-1}^{(\alpha+1,\beta+1)}(x) & R_n^{(\alpha,\beta)}(x) \end{vmatrix} \tag{13}$$

$$=\frac{(\alpha+1)(1-x)^2}{\alpha n(n+\alpha+\beta+1)}\sum_{k=0}^{n}\frac{1+x_k}{1-x_k}\cdot\left[\frac{R_n^{(\alpha,\beta)}(x)}{x-x_k}\right]^2.$$

Theorem 2 *If $\{x_1,x_2,...,x_n\}\subset\mathbb{R}$, with $x_i\neq x_j$ for $i\neq j,\ i,j\in\{1,2,...,n\}$ verifies*

$$\frac{\alpha-\beta+(\alpha+\beta+2)x_j}{2(a-x_j^2)}=\sum_{k=1}^{n}\frac{1}{x_j-x_k}, \tag{14}$$

for $\alpha>-1,\ \beta>-1$ then $x_j\in(-1,1),\quad\forall\ j=1,2,...,n.$

Proof. Let $P(x)=\prod_{k=1}^{n}(x-x_k)$. We obtain

$$\frac{P'(x)}{P(x)}=\sum_{k=1}^{n}\frac{1}{x-x_k},\ \ \frac{P'(x)}{P(x)}-\frac{1}{x-x_j}=\sum_{\substack{k\neq j\\k=1}}^{n}\frac{1}{x-x_k},$$

$$\frac{(x-x_j)P'(x)-P(x)}{(x-x_j)P(x)}=\sum_{\substack{k\neq j\\k=1}}^{n}\frac{1}{x-x_k},$$

$$\lim_{x\to x_j}\frac{(x-x_j)P'(x)-P(x)}{(x-x_j)P(x)}=\lim_{x\to x_j}\sum_{\substack{k\neq j\\k=1}}^{n}\frac{1}{x-x_j},$$

$$\sum_{\substack{k\neq j\\k=1}}^{n}\frac{1}{x_j-x_k}=\lim_{x\to x_j}\frac{P''(x)+(x-x_j)P''(x_j)-P'(x)}{P(x)+(x-xj)P'^{(x)}},$$

(15)
$$\sum_{\substack{k\neq j\\k=1}}^{n}\frac{1}{x_j-x_k}=\frac{P''(x_j)}{2P'(x_j)}.$$

From (14) and (15) we have

$$\frac{P''(x_j)}{P'(x_j)}=\frac{\alpha-\beta+(\alpha+\beta+2)x_j}{1-x_j^2},$$

$$(1-x_j^2)P''(x_j)+[\beta-\alpha-(\alpha+\beta+2)x_j]P'(x_j)=0.$$

Let

$$h(x)=(1-x^2)P''(x)+[\beta-\alpha-(\alpha+\beta+2)x]P'(x),\quad h\in\Pi_n.$$

We observe $h(x_j)=0,\quad j\in\{1,2,...,n\}$. In conclusion exists $c_n\in\mathbb{R}$ such that $h(x)=c_nP(x)$,

(16)
$$(1-x^2)P''(x)+[\beta-\alpha-(\alpha+\beta+2)x]P'(x)-c_nP_n(x)=0.$$

Because $\{R_0^{(\alpha,\beta)}(x),R_1^{(\alpha,\beta)}(x),...,R_n^{(\alpha,\beta)}(x)\}$ is base in Π_n, exists $a_k\in\mathbb{R},\quad k=\overline{0,n}$ such that

$$P(x)=\sum_{k=0}^{n}a_kR_k^{(\alpha,\beta)}(x).$$

From(1) and (16) we obtain

$$\sum_{k=0}^{n}k(k+\alpha+\beta+1)a_kR_k^{(\alpha,\beta)}(x)=c_n\sum_{k=0}^{n}a_kR_k^{(\alpha,\beta)}(x),$$

$$a_k=0,\quad\forall\ k\in\{0,1,...,n-1\},$$

$$c_n=-n(n+\alpha+\beta+1),\ a_n\neq0.$$

In conclusion, the polynomial P verifies (1), hence it exists $\lambda_n\in R$ such that $P(x)=\lambda_nR_n^{(\alpha,\beta)}(x)$. Therefore $x_j\in(-1,1),\quad\forall\ j\in\{1,2,...,n\}$.

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Ioan Ţincu
"Lucian Blaga" University of Sibiu
Faculty of Sciences
Departament of Mathematics and Informatics
Str. I. Raţiu, no. 5-7, 2400-Sibiu, Romania
e-mail: tincuioan@yahoo.com

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About four convergences to e and 1/e [1]

Andrei Vernescu

Abstract

We give the shortest proof of the classical two sided estimation of the standard convergence to e. We also give the similar results for three other convergences, we show the equivalence of all these and we give a mnemonic rule.

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Key words and phrases: The number e (or the constant of Napier, or the number of Euler), sequence, speed of convergence of a sequence, GM-AM inequality, logarithmical derivative.

1 Introduction

The number e, also called the constant of Napier, or the number of Euler, is one of the most important constants of all mathematics. The ("slow") speed of convergence of the sequence of general term $(1+1/n)^n$ to this number is described by the two-sided estimate

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1} \tag{1}$$

(see [4], p. 38, Entry 170).

Firstly, we intend to show a very short proof of (1) and, secondly, to emphasize the speed of convergence of three other basic sequences related to e and respectively to $1/e$. Our second aim is to propose two mnemonic rules which may be useful in order to keep these results: the first concerns their memorization, the second will "automatically" develop their proofs, in a certain sense.

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2 A proof of (1)

Let's begin, by isolating the constant e, for each part of (1). So, (1) is equivalent to

$$(2)\qquad \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n+1}\right)<e<\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right).$$

Passing to the positive real variable x, we will show that the functions $u, v : (0,\infty)\to(0,\infty)$ given by

$$u(x)=\left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{2x+1}\right)$$

and

$$v(x)=\left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{2x}\right)$$

are strictly increasing, respectively strictly decreasing, which, in combination with the relations $\lim_{x\to\infty}u(x)=\lim_{x\to\infty}v(x)=e$, gives the inequality (2). The monotony of the functions u and v is, respectively, the same as the one of their logarithms. Performing the usual calculations of logarithmical derivatives, we obtain

$$(3)\qquad (\log u(x))'=\log(x+1)-\log x-\frac{2}{2x+1},$$

$$(4)\qquad (\log v(x))'=\log(x+1)-\log x+\frac{2}{2x+1}-\left(\frac{1}{x}+\frac{1}{x+1}\right).$$

We will use now the inequality of Hermite-Hadamard, applied to the convex function $t\mapsto 1/t$, $t>0$, on the interval $[x,x+1]\subset(0,\infty)$, that is

$$(5)\qquad \frac{2}{2x+1}<\log(x+1)-\log x<\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+1}\right)$$

(see [3], p. 51, Entry 1. 9. 1.(i); also see [1], p. 273, Entry 3.6.18). Using the left part of (5), we minorize $(\log u(x))'$ as follows

$$(\log u(x))'>\frac{2}{2x+1}-\frac{2}{2x+1}=0;$$

using the right part of (5), we majorize $(\log v(x))'$

$$(\log v(x))'<\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+1}\right)+\frac{2}{2x+1}-\left(\frac{1}{x}+\frac{1}{x+1}\right)=-\frac{1}{2x(x+1)(2x+1)}<0.$$

These give the searched monotonies and complete the proof.

The reader may compare the length of this proof to the one of [4]. The proof given here was improving an idea of [7].

3 The speed of convergence of other three sequences and the mnemonic rule to keep it

Taking again (1), under the number (6), these speeds are described by the two-sided estimations (7) – (9) of the following ones:

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1} \tag{6}$$

$$\frac{e}{2n+1} < \left(1+\frac{1}{n}\right)^{n+1} - e < \frac{e}{2n} \tag{7}$$

$$\frac{1}{2ne} < \frac{1}{e} - \left(1-\frac{1}{n}\right)^n < \frac{1}{(2n-1)e} \tag{8}$$

$$\frac{1}{(2n-1)e} < \left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{e} < \frac{1}{(2n-2)n}. \tag{9}$$

(The inequality (7) was obtained in [6], multiplying the three parts of (6) by $1+1/n$ and by some elementary calculations; (8) and (9) were obtained in [2], in a different way).

Firstly note that all the median parts of these four inequalities are strictly positive, because of the monotonic convergence of the involved sequences to their limits. Indeed, denote $e_n = (1+1/n)^n$, $f_n = (1+1/n)^{n+1}$, $g_n = (1-1/n)^n$ and $h_n = (1-1/n)^{n-1}$; to see its monotonies in a unitary manner, let us enounce the following

Lemma 1 *Let $x \in \mathbb{R}$, $x \neq 0$. The sequence of general term $e_n(x) = (1+x/n)^n$ is strictly increasing beginning at $n > -x$.*

Proof. Apply the GM-AM inequality for $n+1$ numbers chosen as follows: $a_1 = a_2 = a_3 = \ldots = a_n = 1 + x/n$ and $a_{n+1} = 1$, where $n > -x$. We get

$$\sqrt[n+1]{\left(1+\frac{x}{n}\right)^n} < \frac{n\left(1+\frac{x}{n}\right)+1}{n+1},$$

i.e.

$$\sqrt[n+1]{\left(1+\frac{x}{n}\right)^n} < 1 + \frac{x}{n+1},$$

which gives $e_n(x) < e_{n+1}(x)$.

For $x = 1$, we find that the sequence $(e_n)_n$ is strictly increasing (directly treated in [5]); for $x = -1$, we find that the sequence $(g_n)_n$ is also strictly increasing. The obvious identities $f_n = 1/g_{n+1}$ and $h_n = 1/e_{n-1}$ give that the sequences $(f_n)_n$ and

$(h_n)_n$ are strictly decreasing. Therefore, $e_n < e$, $f_n > e$, $g_n < 1/e$, $h_n > 1/e_n$, and all the median parts of (6) – (9) are strictly positive.

Starting from (6), a first mnemonic rule helps us to retain all the two-sided estimates (6) – (9), as follows:

Reading all the coefficients of the denominators of the extreme parts of the four inequalities, from left to right, we remark that these give the following finite decreasing sequence of eight numbers:

$$2n+2 > 2n+1 = 2n+1 > 2n = 2n > 2n-1 = 2n-1 > 2n-2.$$

When we pass from the inequalities containing the two sequences which converge to e, at the other two, related to sequences which converge to $1/e$*, the constant e of the extreme parts pass at the denominators.*

All the inequalities (6) – (9) *contain at the denominator an even coefficient and an odd coefficient and when we "descend" in these relations, the coefficient of the denominator of the right part becomes the coefficient of the denominator of the left part for the next inequality.*

4 The proof of the inequalities (6) – (9)

The second mnemonic rule, that we establish in order to ease the proofs of these inequalities, coincides to the following remark which shows a surprising versatility of all the relations (6) – (9) to (2):

Isolating the number e, in each of the inequalities (6) – (9), *and performing some elementary calculations, we are conducted, in all the four cases, to a part or another of the "central" inequality* (2).

(This operation for (6), that is (1), was mentioned in the section 2. The going pasts from (7), respectively (8), to (2) contain a cancellation of a factor n or $n+1$; the going pasts from (8), respectively, (9) to (2) contain a substitution $n \mapsto n+1$.)

We also may conclude that all the two-sided estimates (7) – (8) are equivalent to (1).

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Andrei Vernescu
Valahia University of Târgovişte
Department of Science
Bd. Unirii 18, 130086 Târgovişte, Romania
e-mail: avernescu@gmail.com

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Generalized characterization theorem for the $K-$functional associated with the algebraic version of trigonometric Jackson Integrals [1]

Teodora D. Zapryanova

Abstract

The purpose of this paper is to present a characterization of a certain Peetre $K-$functional in $L_p[-1,1]$ norm, for $1 \leq p \leq 1/\lambda, \lambda \in (0,1]$ by means of modulus of smoothness. We generalize the earlier characterization of Ivanov [3] and extend the result to a more general setting.

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1 Introduction

1.1 Notations

Let X be a normed space. For a given "differential" operator D we set $X \cap D^{-1}(X) = \{g \in X : Dg \in X\}$. Let X be one of the spaces $L_p[-1,1], 1 \leq p < \infty$ or $C[-1,1]$. In this case we denote the norm in X by $\|.\|_p , 1 \leq p \leq \infty$, where $\|.\|_\infty$ means the uniform norm. Two examples of the operator D are

$$\begin{aligned} D_1 g(x) &:= \frac{1-x^2}{\varphi(x)} (\varphi(x) g'(x))' \text{ , where } \varphi(x) = \left(1-x^2\right)^\lambda, \lambda \in (0,1], \\ D_2 g(x) &:= \left(1-x^2\right) g''(x). \end{aligned}$$

We define for every $f \in X$ and $t > 0$ the K-functionals

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(1) $$K(f,t;X,Y,D_1) := \inf\left\{\|f-g\|_p + t\,\|D_1 g\|_p : g\in Y\right\},$$

(2) $$K(f,t;X,Y,D_2) := \inf\left\{\|f-g\|_p + t\,\|D_2 g\|_p : g\in Y\right\},$$

where Y is a given subspace of $X\cap D_1^{-1}(X)$ or $X\cap D_2^{-1}(X)$, respectively.

The K-functional (1) with $X = C[-1,1]$, $Y = C^2[-1,1]$ and $\lambda = 1/2$ is equivalent to the approximation error of Jackson type operator $G_{s,n}$ in uniform norm (such equivalence was established in [5]), while the K-functional (2) with $X = L_p[-1,1], Y = C^2[-1,1]$ is well-known and is equivalent to the approximation error of Bernstain polynomials in the interval $[0,1]$ ($p=\infty$) and characterizes the best polynomial approximations ($1\le p\le\infty$).

We recall that the operator $G_{s,n} : C[-1,1] \to \Pi_{sn-s}$ is defined by (see [4])

$$G_{s,n}(f,x) = \pi^{-1}\int_{-\pi}^{\pi} f(\cos(\arccos x + v))K_{s,n}(v)dv,$$

where

$$K_{s,n}(v) = c_{n,s}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s}, \qquad \pi^{-1}\int_{-\pi}^{\pi} K_{s,n}(v)dv = 1.$$

Π_r denotes the set of all algebraic polynomials of degree not exceeding r (r is natural number).

Notation $\Phi(f,t)\sim\Psi(f,t)$ means that there is a positive constant γ, independent of f and t, such that $\gamma^{-1}\Psi(f,t)\le\Phi(f,t)\le\gamma\Psi(f,t)$.

By c we denote positive constants, independent of f and t, that may differ at each occurrence.

For r - natural number we denote
$C^r[a,b] = \left\{f : f, f', ..., f^{(r)} \in C[a,b] \text{ (continuous function in } [a,b])\right\}.$

1.2 Known results

The idea for the equivalence of the approximation errors of a given sequence of operators and the values of proper K-functionals was studied systematically in [1].

Such equivalence was established for the algebraic version of trigonometric Jackson integrals $G_{s,n}$ and K-functionals (1) with $X = C[-1,1]$, $Y = C^2[-1,1], p=\infty$ and $\lambda = 1/2$ in [5] (see Theorem A).

Theorem A. *For $s \geqslant 2, \lambda = 1/2$ and every $f \in C[-1,1]$ we have*

$$\|f - G_{s,n}f\|_\infty \sim K(f, \frac{1}{n^2}; C[-1,1], C^2, D_1).$$

Using a linear transform of functions in [3] Ivanov compares the K-functional

$$K(f,t;X,Y,D_3) := \inf\left\{\|f-g\|_p + t\left\|(\psi g')'\right\|_p : g \in Y\right\}, \tag{3}$$

with the already characterized K-functional

$$K(f,t;X,Y,D_4) := \inf\left\{\|f-g\|_p + t\left\|\psi g''\right\|_p : g \in Y\right\}, \tag{4}$$

where $\psi(x) = x(1-x)$; X is one of the spaces $L_p[0,1]$, $1 \le p < \infty$ or $C[0,1]; Y$ is a given subspace of $X \cap D_3^{-1}(X)$ or $X \cap D_4^{-1}(X)$, respectively; $D_3 g := (\psi g')'$, $D_4 g := \psi g''$.

Ivanov proved in [3, Theorem 1] the following

Theorem B. *For every $t \in (0,1]$ and $f \in L_1[0,1]$ we have*

$$K(f,t;L_1[0,1],C^2,D_3) \sim K(Bf,t;L_1[0,1],C^2,D_4) + tE_0(f)_1,$$

where $\quad (Bf)(x) = f(x) + \int\limits_{1/2}^{x} \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2}\right) f(y)dy$

and $E_0(f)_1$ denotes the best approximation of f in $L_1[0,1]$ by constant.

1.3 New results

The aim of this paper is to define a modulus that is equivalent to the K-functional (1) for $1 \le p \le 1/\lambda$. We apply the method presented in [3].

First, let us note that the K-functionals $K(f,t;L_p[-1,1],C^2,D_1)$ for $\lambda = 1/2$ and $K(f,t;L_p[-1,1],C^2,D_2)$ are not equivalent. The inequality $K(f,t;L_p[-1,1],C^2,D_2) \le cK(f,t;L_p[-1,1],C^2,D_1)$ is not true for a fixed c, every f, every $t \in (0,1]$ and $1 \le p \le 1/\lambda$ because of functions like (with small positive ε)

$$f_\varepsilon(x) = \begin{cases} \arcsin x, & x \in [-1+\varepsilon, 1-\varepsilon]; \\ ax^3 + bx + d, & x \in [1-\varepsilon, 1]; \\ ax^3 + bx - d, & x \in [-1, -1+\varepsilon]; \end{cases}$$

where a, b, d are chosen such that $f_\varepsilon \in C^2$.

But these $K-$functionals can become equivalent if in the one of them instead f stays Af for appropriate operator A.

Let $f \in L_1[-1,1]$. For every $-1 < x < 1$ we define the value of the operator A by

$$(Af)(x) := f(x) + \int\limits_0^x f(y)\left[(y-x)\frac{\varphi'(y)}{\varphi(y)}\right]' dy, \tag{5}$$

where the derivative is on y, $\varphi(y) = (1-y^2)^\lambda$, $\lambda \in (0,1]$.

Using operator (5) we prove

Theorem 1 *a) For every $t \in (0,1]$ and $f \in L_p[-1,1]$, $\lambda \in (0,1)$, $1 \le p \le 1/\lambda$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],C^2,D_2).$$

b) For every $t \in (0,1]$, $f \in L_1[-1,1]$ and $\lambda = 1$ we have

$$K(f,t;L_1[-1,1],C^2,D_1) \sim K(Af,t;L_1[-1,1],C^2,D_2) + tE_0(f)_1.$$

We mention that in Theorem 1 a) there is no additional term $tE_0(f)_p$ in the equivalence relation, while in Theorem B there is. Moreover, the equivalence in Theorem B is valid only for $p = 1$, while Theorem 1 a) holds for $1 \le p \le 1/\lambda, \lambda \in (0,1)$. Although the operators D_1 for $\lambda = 1/2$ and D_3 are similar we cannot reduce one to another. We can write the operator $D_1 g(x)$ for $\lambda = 1/2$ of the form :

$$(D_1 g)(x) = (1-x^2)g''(x) - xg'(x).$$

On the other hand, the analogue of D_3 for the interval $[-1,1]$ is

$$\widetilde{D}_3 G(y) = (1-y^2)G''(y) - 2yG'(y),$$

i. e. $\widetilde{D}_3 G$ differs from $D_1 G$ by constant multiplier 2 in the term containing G'.

From Theorem 1 and characterizations of some weighted Peetre K-functionals in terms of weighted moduli established in [2, Ch. 2, Theorem 2.1.1] we get

Corollary 1 *a) For $f \in L_p[-1,1]$, $t \in (0,1]$ and $\lambda \in (0,1), 1 \le p \le 1/\lambda$, with $\phi = \sqrt{1-x^2}$ we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim \omega_\phi^2(Af,\sqrt{t})_p,$$

where ω_ϕ^2 is Ditzian-Totik modulus of smoothness, introduced in [2].

b) For $f \in L_1[-1,1]$, $t \in (0,1]$ and $\lambda = 1$ with $\phi = \sqrt{1-x^2}$ we have

$$K(f,t;L_1[-1,1],C^2,D_1) \sim \omega_\phi^2(Af,\sqrt{t})_1 + t\omega_1(f,1)_1.$$

The equivalence in Theorem 1 is no longer true for $1/\lambda < p < \infty$ as the following example shows. Let $F(x) = \arcsin x$, $\lambda = 1/2$. We have $E_0(F)_p \sim 1$ and thus $ct \le K(F,t;L_p[-1,1],C^2,D_1)$ for $2 < p < \infty$ (see Lemma 4). On the other hand $K(AF,t;L_p[-1,1],C^2,D_2) = 0$ for every p because $AF(x) = x$, i. e. $AF \in C^2[-1,1]$ and $D_2(AF) = \left(1-x^2\right)(AF)'' = 0$.

The connection between the K-functionals of f and Af with D_1 and D_2 as differential operators respectively, is not so satisfactory when $1/\lambda < p < \infty$. We have

Theorem 2 *For every $t \in (0,1]$ and $f \in L_p[-1,1]$, $\lambda \in (0,1], 1/\lambda < p < \infty$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \le c\left[K(Af,t^{\frac{1}{p}+(1-\lambda)};L_p[-1,1],C^2,D_2)+t^{\frac{1}{p}+(1-\lambda)}E_0(f)_p\right],$$

$$K(Af,t;L_p[-1,1],C^2,D_2)+tE_0(f)_p \le cK(f,t;L_p[-1,1],C^2,D_1).$$

The proof of Theorem 1 follows the scheme. First we establish in Lemma 2 the equivalence

$$K(f,t;L_p[-1,1],Z_1,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2) \text{ for } 1 \le p < \infty,$$

where Z_1 and Z_2 are suitable subspaces of C^2 (see Definition 2). On the other hand these variations of Y produce K- functionals equal to the K - functionals we compare in Theorem 1. In Lemma 1 and Lemma 3 b) respectively we prove that

$$\begin{aligned} K(f,t;L_p[-1,1],Z_1,D_1) &= K(f,t;L_p[-1,1],C^2,D_1) \text{ for } 1 \le p < \infty \text{ and} \\ K(F,t;L_p[-1,1],Z_2,D_2) &= K(F,t;L_p[-1,1],C^2,D_2) \text{ for } 1 \le p \le 1/\lambda, \lambda \in (0,1). \end{aligned}$$

The last two relations we obtain using Lemma 2 from [3, p.116]. We state this lemma, as we use it several times.

Definition 1 *For given $Y \subset X \cap D^{-1}(X)$ and a positive number γ we define $S_\gamma(Y)$ as the set of all $g \in X \cap D^{-1}(X)$ such that for every $\varepsilon > 0$ there is $h \in Y$ such that $\|g-h\| < \varepsilon$ and $\|Dh\| < \gamma\|Dg\| + \varepsilon$.*

Lemma B *Let $Y_1, Y_2 \subset X \cap D^{-1}(X)$ and $\rho > 0$. Then for a given positive γ the following statements are equivalent:*

i) $K(f,t;X,Y_1,D) \le K(f,\gamma t;X,Y_2,D)$ for every $f \in X$, $0 < t$.

ii) $K(f,t;X,Y_1,D) \le K(f,\gamma t;X,Y_2,D)$ for every $f \in X$, $0 < t \le \rho$.

iii) $Y_2 \subset S_\gamma(Y_1)$.

In particular, i) with $\gamma = 1$ holds when $Y_2 \subset Y_1$.

Theorems 1 and 2 are proved in Section 3.

2 Properties of the operators

In the next statement we collect some properties of operator A.

Theorem 3 *a) A is a linear operator, satisfying $\|Af\|_p \le c\|f\|_p$ for every $1 \le p \le \infty$.*

b) $Af = f$ for every $f \in \Pi_0$.

c) If $f, f' \in AC_{loc}(-1,1)$, then $(Af)(0) = f(0)$, $(Af)'(0) = f'(0)$ and

$$\begin{aligned} \left(1-x^2\right)(Af)''(x) &= \frac{1-x^2}{\varphi(x)}(\varphi(x)f'(x))', \qquad -1 < x < 1, \\ \textit{i.e. } D_2Af &= D_1f \qquad \textit{for} \quad -1 < x < 1. \end{aligned}$$

Proof. We write (5) as

$$(Af)(x) = f(x) + \int_{-1}^{1} R(x,y) f(y) dy,$$

where the kernel $R : (-1,1) \times (-1,1) \to \mathbb{R}$ is given as follows: $R(0,y) = 0$; for $x \in (-1,0)$ we have $R(x,y) = -\left[(y-x)\frac{\varphi'(y)}{\varphi(y)}\right]'_y = -\frac{\varphi'(y)}{\varphi(y)} - (y-x)\left(\frac{\varphi'(y)}{\varphi(y)}\right)'$ if $y \in (x,0)$ and $R(x,y) = 0$ if $y \notin (x,0)$; for $x \in (0,1)$ we have $R(x,y) = \left[(y-x)\frac{\varphi'(y)}{\varphi(y)}\right]'_y = \frac{\varphi'(y)}{\varphi(y)} + (y-x)\left(\frac{\varphi'(y)}{\varphi(y)}\right)'$ if $y \in (0,x)$ and $R(x,y) = 0$ if $y \notin (0,x)$.

We set $\varphi(x) = (1-x)^{\lambda}\varphi_1(x)$, where $\varphi_1(x) = (1+x)^{\lambda}$. Then $\frac{\varphi'(x)}{\varphi(x)} = \frac{-\lambda}{1-x} + g_1(x)$, $g_1(x) := \frac{\varphi_1'(x)}{\varphi_1(x)}$. As $\varphi_1(x) \geq c > 0$ for $x \in [0,1]$ we have $g_1(x) \in C^1[0,1]$, $|g_1(x)| \leq c$ and $|g_1'(x)| \leq c$.Thus for $x > 0$ and $y \in (0,x)$ we have

$$\begin{aligned} R(x,y) &= \frac{-\lambda}{1-y} + g_1(y) + (y-x)\left[\frac{-\lambda}{(1-y)^2} + g_1'(y)\right] \\ &= \frac{-\lambda}{(1-y)^2}\left[(1-y) + (y-x)\right] + g_1(y) + (y-x)g_1'(y) \\ &= \frac{-\lambda(1-x)}{(1-y)^2} + g_1(y) + (y-x)g_1'(y). \end{aligned}$$

Similarly, if we set $\varphi(x) = (1+x)^{\lambda}\varphi_2(x)$, where $\varphi_2(x) = (1-x)^{\lambda}$ we have $\frac{\varphi'(x)}{\varphi(x)} = \frac{\lambda}{1+x} + g_2(x)$, $g_2(x) := \frac{\varphi_2'(x)}{\varphi_2(x)}$ and $g_2(x) \in C^1[-1,0]$.Thus for $x < 0$ and $y \in (x,0)$ we have

$$R(x,y) = -\left[\frac{\lambda(1+x)}{(1+y)^2} + g_2(y) + (y-x)g_2'(y)\right].$$

Then for $x \in (0,1)$ we get

$$\int_{-1}^{1} |R(x,y)|\, dy = \int_{0}^{x} |R(x,y)|\, dy \leq \int_{0}^{x} \frac{\lambda(1-x)}{(1-y)^2} dy + \int_{0}^{x} \left|g_1(y) + (y-x)g_1'(y)\right| dy.$$

The last integral is bounded as $|g_1(y) + (y-x)g_1'(y)|$ is bounded. The first integral on the right side

$$\lambda(1-x)\int_{0}^{x} \frac{dy}{(1-y)^2} = \lambda - \lambda(1-x) \leq \lambda.$$

Therefore

$$\int_{-1}^{1} |R(x,y)|\, dy \leq c, \text{ for } x \in (0,1).$$

By analogy

$$\int_{-1}^{1} |R(x,y)|\, dy \le c, \text{ for } x \in (-1,0).$$

Hence $\|Af\|_\infty \le c\,\|f\|_\infty$.

The function $R(x,y)$ is bounded in every fixed, closed interval $[a,b] \subset (-1,1)$. We investigate $R(x,y)$ when y is close to 1 and -1. Let first y be close to 1. As $y \in (0,x)$ we have $0 < y < x < 1$ and $1-y > 1-x > 0$. We notice that as $\frac{\varphi'(x)}{\varphi(x)} = \frac{-\lambda}{1-x} + g_1(x)$ and $g_1(x) \in C^1[0,1]$ we have $\left|\frac{\varphi'(x)}{\varphi(x)}\right| \le \frac{c}{1-x}$, $\left|\left(\frac{\varphi'(x)}{\varphi(x)}\right)'\right| \le \frac{c}{(1-x)^2}$ for $x \in [0,1)$. Then $\left|(y-x)\left(\frac{\varphi'(y)}{\varphi(y)}\right)'\right| \le c\left|(y-x)\frac{1}{(1-y)^2}\right| = c\left|\frac{(y-1)+(1-x)}{(1-y)^2}\right| \le \frac{c}{1-y} + \frac{c(1-x)}{(1-y)^2} < \frac{c}{1-y} + \frac{c(1-y)}{(1-y)^2} \le \frac{c}{1-y}$ and

$$|R(x,y)| \le \frac{c}{1-y} \quad \text{for } y > 0.$$

By analogy

$$|R(x,y)| \le \frac{c}{1+y} \quad \text{for } y < 0.$$

Therefore

$$|R(x,y)| \le \frac{c}{1-y^2} \quad \text{for } y \in (-1,1),\ x \in (-1,1)$$

and

$$\int_{-1}^{1} |R(x,y)|\, dx = \int_{-1}^{-|y|} |R(x,y)|\, dx + \int_{|y|}^{1} |R(x,y)|\, dx \le \frac{c}{1-y^2}(1-|y|) \le c.$$

Hence $\|Af\|_1 \le c\,\|f\|_1$. Now the Riesz-Thorin theorem proves a).

Part b) follows from $\int_{-1}^{1} R(x,y)dy = 0$. Indeed, for $x > 0$

$$\int_{-1}^{1} R(x,y)dy = \int_{0}^{x} \left[(y-x)\frac{\varphi'(y)}{\varphi(y)}\right]_y' dy = x\frac{\varphi'(0)}{\varphi(0)} = 0.$$

By analogy for $x < 0$ $\int_{-1}^{1} R(x,y)dy = 0$.

Part c) follows from (5) by direct computation.

The operator A is invertible and we give an explicit formula for its inverse operator A^{-1}. Let every $f \in L_1[-1,1]$ and $-1 < x < 1$ we set

$$(A^{-1}f)(x) = f(x) + \int_{0}^{x} \left(\varphi''(y)\int_{y}^{x} \frac{dt}{\varphi(t)} - \frac{\varphi'(y)}{\varphi(y)}\right) f(y)dy.$$

In the next statement we collect some properties of A^{-1}.

Theorem 4

a) A^{-1} *is a linear operator,* $\left\|A^{-1}f\right\|_p \le c\,\|f\|_p$ *for every* $1 \le p < \infty$.

b) $A^{-1}f = f$ *for every* $f \in \Pi_0$.

c) $A^{-1}Af = AA^{-1}f = f$ *for every* $f \in L_1[-1,1]$.

d) If $f,\ f' \in AC_{loc}(-1,1)$, *then* $(A^{-1}f)(0) = f(0)$, $(A^{-1}f)'(0) = f'(0)$ *and*

$$\begin{aligned}\frac{1-x^2}{\varphi(x)}(\varphi(x)(A^{-1}f)'(x))' &= \left(1-x^2\right)f''(x), \qquad -1<x<1,\\ \textit{i.e. } D_1A^{-1}f &= D_2 f \qquad \textit{for} \quad -1<x<1.\end{aligned}$$

Proof. a) Let $x > 0$. First we estimate $\varphi''(y)\int_y^x \frac{dt}{\varphi(t)}$. As $0 < y < x$ we have $\left|\int_y^x \frac{dt}{\varphi(t)}\right| \le \left|\int_y^1 \frac{dt}{\varphi(t)}\right| \le c\int_y^1 \frac{dt}{(1-t)^\lambda} \le c(1-y)^{1-\lambda}$ for $\lambda \in (0,1)$.Let $\lambda \in (0,1)$. We have

$$\begin{aligned}&\int_0^1 \left|\int_0^x \left(\varphi''(y)\int_y^x \frac{dt}{\varphi(t)}\right) f(y)dy\right|^p dx\\
\le\ & c\int_0^1 \left(\int_0^x (1-y)^{\lambda-2}(1-y)^{1-\lambda}\,|f(y)|\,dy\right)^p dx = c\int_0^1 \left(\int_0^x \frac{1}{1-y}\,|f(y)|\,dy\right)^p dx\\
=\ & c\int_0^1 \left(\int_{1-x}^1 \frac{1}{y}\,|f(1-y)|\,dy\right)^p dx \qquad (y \to 1-y)\ \text{(Hardy inequality)}\\
\le\ & c\int_0^1 \left(y\frac{|f(1-y)|}{y}\right)^p dy = c\int_0^1 |f(1-y)|^p\,dy = c\,\|f\|^p_{p[0,1]}\,.\end{aligned}$$

Similarly, using Hardy inequality we get

$$\int_0^1 \left|\int_0^x \frac{\varphi'(y)}{\varphi(y)} f(y)dy\right|^p dx \le c\int_0^1 \left(\int_0^x \frac{|f(y)|}{1-y}dy\right)^p dx \le c\,\|f\|^p_{p[0,1]}\,.$$

Similar estimates hold for $x < 0$

$$\int_{-1}^0 \left|\int_0^x \left(\varphi''(y)\int_y^x \frac{dt}{\varphi(t)}\right) f(y)dy\right|^p dx \le c\,\|f\|^p_{p[-1,0]} \qquad \text{and}$$

$$\int_{-1}^0 \left|\int_0^x \frac{\varphi'(y)}{\varphi(y)} f(y)dy\right|^p dx \le c\,\|f\|^p_{p[-1,0]}\,.$$

From these inequalities we get

$$\left\{\int_{-1}^{1}\left|\int_{0}^{x}\left[\varphi''(y)\int_{y}^{x}\frac{dt}{\varphi(t)}-\frac{\varphi'(y)}{\varphi(y)}\right]f(y)dy\right|^{p}dx\right\}^{\frac{1}{p}}\le c\,\|f\|_{p[-1,1]}\quad\text{for }\lambda\in(0,1).$$

This proves a) in the case $\lambda\in(0,1)$.

Let now $\lambda=1$. We have $\left|\int_y^x\frac{dt}{\varphi(t)}\right|\le c\left|\int_y^x\frac{dt}{1-t}\right|=c\left|\ln\frac{1-y}{1-x}\right|, |\varphi''(y)|\le c$ for $y\in[-1,1]$. We estimate

$$\int_{0}^{1}\left|\int_{0}^{x}\left(\varphi''(y)\int_{y}^{x}\frac{dt}{\varphi(t)}\right)f(y)dy\right|^{p}dx\le c\int_{0}^{1}\left|\int_{0}^{x}\left|\ln\frac{1-y}{1-x}\right||f(y)|\,dy\right|^{p}dx$$

$$\le c\int_{0}^{1}\left|\int_{0}^{x}\left(|\ln(1-y)|+|\ln(1-x)|\right)|f(y)|\,dy\right|^{p}dx$$

$$\le c\left(\int_{0}^{1}\left|\int_{0}^{x}|\ln(1-y)|\,|f(y)|\,dy\right|^{p}dx+\int_{0}^{1}\left|\int_{0}^{x}|\ln(1-x)|\,|f(y)|\,dy\right|^{p}dx\right)$$

$=I_1+I_2$. Using Hardy inequality as above we get

$$I_1=\int_{0}^{1}\left|\int_{0}^{x}|\ln(1-y)|\,|f(y)|\,dy\right|^{p}dx\le c\int_{0}^{1}\left|\int_{0}^{x}\frac{1}{1-y}\,|f(y)|\,dy\right|^{p}dx\le c\,\|f\|_{p[0,1]}^{p}.$$

$$I_2=\int_{0}^{1}\left|\int_{0}^{x}|\ln(1-x)|\,|f(y)|\,dy\right|^{p}dx=\int_{0}^{1}|\ln(1-x)|^{p}\left|\int_{0}^{x}|f(y)|\,dy\right|^{p}dx$$

$$\le\int_{0}^{1}|\ln(1-x)|^{p}\,dx\left|\int_{0}^{1}|f(y)|\,dy\right|^{p}\le c\,\|f\|_{1,[0,1]}^{p}\le c\,\|f\|_{p[0,1]}^{p},\ \text{as}\ \|f\|_{1}\le c\,\|f\|_{p},\ p\ge 1.$$

The case $x\in(-1,0), y\in(x,0)$ is similar. This proves a).

Part b) follows from

$$\int_{0}^{x}\left(\varphi''(y)\int_{y}^{x}\frac{dt}{\varphi(t)}-\frac{\varphi'(y)}{\varphi(y)}\right)dy=0.$$

Finally, c) and d) can be obtained by direct computation.

The action of the operators A and A^{-1} for $\lambda=\frac{1}{2}$ on the function $f(x)=x$ is given bellow:

$(A(.))(x)=\frac{1}{2}(1+x)\ln(1+x)-\frac{1}{2}(1-x)\ln(1-x)$,
$(A^{-1}(.))(x)=\arcsin x$.

Definition 2 *Set*

$$Z_1 = \left\{ f \in C^2[-1,1] : f'(-1) = 0, f'(1) = 0 \right\},$$

$$Z_2 = \left\{ f \in C^2[-1,1] : \int_{-1}^{0} \varphi'(x) f'(x)dx = 0, \int_{0}^{1} \varphi'(x) f'(x)dx = 0 \right\}.$$

Theorem 5 *Let $\lambda \in (0,1]$. Then*

a) $(Af)''(x)$ is continuous at $x = -1$ and at $x = 1$ for every $f \in Z_1$.

b) $\int_{-1}^{0} \varphi'(x)(Af)'(x)dx = \int_{0}^{1} \varphi'(x)(Af)'(x)dx = 0$ for every $f \in Z_1$.

c) $(A^{-1}f)''(x)$ is continuous at $x = -1$ and at $x = 1$ for every $f \in Z_2$.

d) $(A^{-1}f)'(-1) = (A^{-1}f)'(1) = 0$ for every function $f \in Z_2$.

e) $A(Z_1) = Z_2$ and $A^{-1}(Z_2) = Z_1$.

Proof. For every function $f \in Z_1$ we have that $f'(x) = (x-1)f''(1) + o(1-x)$ and $f'(x) = (x+1)f''(-1) + o(1+x)$. From Theorem 3 c) we have

$$(Af)''(x) = f''(x) + \frac{\varphi'(x)}{\varphi(x)} f'(x).$$

As in the proof of Theorem 3 we use presentations $\frac{\varphi'(x)}{\varphi(x)} = \frac{-\lambda}{1-x} + g_1(x)$ and $\frac{\varphi'(x)}{\varphi(x)} = \frac{\lambda}{1+x} + g_2(x)$. Then

$$\lim_{x \to 1-0} \frac{\varphi'(x)}{\varphi(x)} f'(x) = \lim_{x \to 1-0} \left(\frac{-\lambda}{1-x} + g_1(x) \right) \left[(x-1)f''(1) + o(1-x) \right] = \lambda f''(1),$$

and

$$\lim_{x \to -1+0} \frac{\varphi'(x)}{\varphi(x)} f'(x) = \lim_{x \to -1+0} \left(\frac{\lambda}{1+x} + g_2(x) \right) \left[(x+1)f''(-1) + o(1+x) \right] = \lambda f''(-1)$$

which together with the above representations gives for x close to 1

$$(Af)''(x) = (1+\lambda)f''(1) + o(1)$$

and

$$(Af)''(x) = (1+\lambda)f''(-1) + o(1)$$

for x close to -1. This proves a).

We write the derivative of Af as

$$(Af)'(x) = f'(x) + \int_{0}^{x} \frac{\varphi'(y)}{\varphi(y)} f'(y)dy.$$

Then we have

$$\begin{aligned}
&\int_0^1 \varphi'(x)(Af)'(x)dx \\
&= \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \varphi'(x)(Af)'(x)dx \\
&= \lim_{\varepsilon\to 0+} \left(\int_0^{1-\varepsilon} \varphi'(x)f'(x)dx + \int_0^{1-\varepsilon} \varphi'(x)\int_0^x \frac{\varphi'(y)}{\varphi(y)} f'(y)dydx \right).
\end{aligned}$$

We consider the last term.

$$\begin{aligned}
\int_0^{1-\varepsilon} \varphi'(x)\int_0^x \frac{\varphi'(y)}{\varphi(y)} f'(y)dydx &= \int_0^{1-\varepsilon} \frac{\varphi'(y)}{\varphi(y)} f'(y) \left(\int_y^{1-\varepsilon} \varphi'(x)dx \right) dy \\
&= \int_0^{1-\varepsilon} \frac{\varphi'(y)}{\varphi(y)} \left(\varphi(1-\varepsilon) - \varphi(y)\right) f'(y)dy \\
&= \varphi(1-\varepsilon)\int_0^{1-\varepsilon} \frac{\varphi'(y)}{\varphi(y)} f'(y)dy - \int_0^{1-\varepsilon} \varphi'(y)f'(y)dy.
\end{aligned}$$

Then

$$\int_0^{1-\varepsilon} \varphi'(x)(Af)'(x)dx = \varphi(1-\varepsilon)\int_0^{1-\varepsilon} \frac{\varphi'(y)}{\varphi(y)} f'(y)dy.$$

For every $f \in Z_1$, $\left| \int_0^1 \frac{\varphi'(y)}{\varphi(y)} f'(y)dy \right| \le c \int_0^1 \frac{\left|f'(y)\right|}{1-y} dy \le c$ as $f'(1) = 0$ and $f'(y) = (y-1)f''(1) + o(1-y)$. We use that $\lim_{\varepsilon\to 0} \varphi(1-\varepsilon) = \varphi(1) = 0$ and hence eventually

$$\lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \varphi'(x)(Af)'(x)dx = \int_0^1 \varphi'(x)(Af)'(x)dx = 0.$$

Similar arguments prove the other claim of b).

We have

$$(A^{-1}f)'(x) = f'(x) - \frac{1}{\varphi(x)}\int_0^x \varphi'(y)f'(y)dy. \tag{6}$$

$$\begin{aligned}(A^{-1}f)''(x) &= f''(x) + \frac{\varphi'(x)}{\varphi^2(x)}\int_0^x \varphi'(y)f'(y)dy - \frac{\varphi'(x)}{\varphi(x)}f'(x)\\ &= f''(x) - \frac{\varphi'(x)}{\varphi(x)}(A^{-1}f)'(x). \qquad (7)\end{aligned}$$

For every $f \in Z_2$, $x \in [0,1]$ $\int_0^x \varphi'(y)f'(y)dy + \int_x^1 \varphi'(y)f'(y)dy = 0$ and we rewrite (6) as

$$(A^{-1}f)'(x) = f'(x) + \frac{1}{\varphi(x)}\int_x^1 \varphi'(y)f'(y)dy.$$

Using Taylor's expansion of f' around 1 we get from the above

$$(A^{-1}f)'(x) = f'(x) + \frac{1}{\varphi(x)}\int_x^1 \varphi'(y)(f'(1) + (y-1)f''(1) + o(1-y))dy.$$

Now we compute the last integral, which is equal to

$$f'(1)\int_x^1 \varphi'(y)dy + f''(1)\int_x^1 (y-1)\varphi'(y)dy + \int_x^1 \varphi'(y)o(1-y))dy.$$

As $\int_x^1 \varphi'(y)dy = \varphi(1) - \varphi(x) = -\varphi(x)$ we have

$$(A^{-1}f)'(x) = f'(x) - f'(1) + \frac{f''(1)}{\varphi(x)}\int_x^1 (y-1)\varphi'(y)dy + \frac{1}{\varphi(x)}\int_x^1 \varphi'(y)o(1-y))dy.$$

Applying the L'Hôpital's rule we obtain

$$\begin{aligned}&\lim_{x\to 1-0}\frac{\int_x^1 (y-1)\varphi'(y)dy}{(1-x)\varphi(x)}\\ =& \lim_{x\to 1-0}\frac{(1-x)\varphi'(x)}{(1-x)\varphi'(x) - \varphi(x)}\\ =& \lim_{x\to 1-0}\frac{(1-x)\lambda\left(1-x^2\right)^{\lambda-1}(-2x)}{(1-x)\lambda\left(1-x^2\right)^{\lambda-1}(-2x) - (1-x^2)^{\lambda}} = \frac{\lambda}{1+\lambda}\end{aligned}$$

and we can write $\int_x^1 (y-1)\varphi'(y)dy = \frac{\lambda}{1+\lambda}(1-x)\varphi(x) + o\left((1-x)\varphi(x)\right)$ for x close to 1. Similarly $\int_x^1 \varphi'(y)o(1-y)dy = o\left(1-x^2\right)^{\lambda+1}$ for x close to 1. Above computations and Taylor's expantion of f' around 1 imply

$$(A^{-1}f)'(x) = -\frac{1}{1+\lambda}(1-x)f''(1) + o(1-x). \tag{8}$$

We use $\frac{\varphi'(x)}{\varphi(x)} = \frac{-\lambda}{1-x} + g_1(x)$. Equations (7), (8) and $\frac{\varphi'(x)}{\varphi(x)}(1-x) = -\lambda + o(1)$ give $(A^{-1}f)''(x) = \frac{1}{\lambda+1}f''(1) + o(1)$ for x close to 1. In a similar way we get

$$(A^{-1}f)'(x) = \frac{1}{1+\lambda}(1+x)f''(-1) + o(1+x). \tag{9}$$

Hence $(A^{-1}f)''(x) = \frac{1}{1+\lambda}f''(-1) + o(1)$ for x close to -1, which proves c).

Part d) follows from (8) and (9).

For every $f \in Z_1$ from a) we get $(Af)'$, $Af \in AC[-1,1]$ and hence $Af \in C^2$. Now using b) we get $Af \in Z_2$, i.e., $A(Z_1) \subset Z_2$. Similarly, from c) and d) we get $A^{-1}(Z_2) \subset Z_1$. Using Theorem 4 c) we get $Z_1 = A^{-1}(A(Z_1)) \subset A^{-1}(Z_2)$ and $Z_2 = A(A^{-1}(Z_2)) \subset A(Z_1)$. Hence $A^{-1}(Z_2) = Z_1$ and $A(Z_1) = Z_2$.

3 Proofs of the Theorems

Lemma 1 *a) For every $t > 0$ and $f \in L_p[-1,1], 1 \le p < \infty$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) = K(f,t;L_p[-1,1],Z_1,D_1).$$

b) For every $t > 0$ and $f \in C[-1,1]$ we have

$$K(f,t;C[-1,1],C^2,D_1) \sim K(f,t;C[-1,1],Z_1,D_1).$$

Proof. Let $\mu \in C^\infty(\mathbb{R})$ be such that $\mu(x) = 1$ for $x \le 0$, $\mu(x) = 0$ for $x \ge 1$ and $0 < \mu(x) < 1$ for $0 < x < 1$. For given $\delta \in (0, \frac{1}{2})$ we set $\mu_{-1}(x) = \mu(\frac{1+x}{\delta})$ and $\mu_1(x) = \mu(\frac{1-x}{\delta})$ for every $x \in [-1,1]$. Thus, supp $\mu_{-1}(x) = [-1,-1+\delta]$, supp $\mu_1(x) = [1-\delta,1]$ and $\left\|\mu_j^{(k)}\right\|_\infty = O(\delta^{-k})$ for $j = -1,1$ and $k = 1,2$.

Let $g \in C^2[-1,1]$. For $x \in [-1,1]$ set

$$G(x) = [1-\mu_{-1}(x)-\mu_1(x)]g(x) + \mu_{-1}(x)g(-1) + \mu_1(x)g(1). \tag{10}$$

Then $G \in Z_1$. From $G(x) - g(x) = \mu_{-1}(x)[g(-1)-g(x)] + \mu_1(x)[g(1)-g(x)]$ we get $\|G-g\|_p \le 2^{1/p}\|G-g\|_\infty \le 2^{1/p}\omega_1(g,\delta)_\infty = O(\delta)$.From (10) and the form of the operator $(D_1 g)(x) = (1-x^2)g''(x) - 2\lambda x g'(x)$ we obtain

$$\begin{aligned}(D_1G)(x) = {} & [1-\mu_{-1}(x)-\mu_1(x)](D_1g)(x) - 2(1-x^2)[\mu'_{-1}(x)+\mu'_1(x)]g'(x) \\ & +(D_1\mu_1)(x)[g(1)-g(x)] + (D_1\mu_{-1})(x)[g(-1)-g(x)].\end{aligned} \tag{11}$$

From (11) for $1 \le p < \infty$ we get $\|D_1G\|_p \le \|D_1g\|_p + O(\delta^{1/p})$, which proves part a) in view of Lemma 2 in [3, p.116].

For $p = \infty$ (11) implies

$$\|D_1G\|_\infty \le \|D_1g\|_\infty + c[|g'(-1)| + |g'(1)|] + O(\delta) \le c\,\|D_1g\|_\infty + O(\delta),$$

because of $|(D_1g)(-1)| = c\;|g'(-1)|$ and $|(D_1g)(1)| = c\,|g'(1)|$. Applying Lemma 2 again in [3, p.116] we prove part b).

Lemma 2 *For every $t > 0$ and $f \in L_p[-1,1]$, $1 \le p < \infty$, we have*

$$K(f,t;L_p[-1,1],Z_1,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2).$$

Proof. For a given $g \in Z_1$ we set $G = Ag \in Z_2$ (see Theorem 5 e)). Then Theorem 4 c), a) and d) implies $\|f-g\|_p = \left\|A^{-1}(Af-Ag)\right\|_p \le c\,\|Af-G\|_p$ and $\|D_1g\|_p = \left\|D_1A^{-1}G\right\|_p = \|D_2G\|_p$. Hence,

$$\|f-g\|_p + t\,\|D_1g\|_p \le c(\|Af-G\|_p + t\,\|D_2G\|_p),$$

which gives $K(f,t;L_p,Z_1,D_1) \le cK(Af,t;L_p,Z_2,D_2)$.

For a given $G \in Z_2$ we set $g = A^{-1}G \in Z_1$ (see Theorem 5 e)). Using Theorem 3 a), c) and Theorem 4 c), we get

$$\|Af-G\|_p = \left\|A(f-A^{-1}G)\right\|_p \le c\,\|f-g\|_p\,, \;\; \|D_2G\|_p = \|D_2Ag\|_p = \|D_1g\|_p\,.$$

Hence,

$$\|Af-G\|_p + t\,\|D_2G\|_p \le c(\|f-g\|_p + t\,\|D_1g\|_p),$$

which gives $K(Af,t;L_p[-1,1],Z_2,D_2) \le cK(f,t;L_p[-1,1],Z_1,D_1)$.

From Lemmas 1, 2 we obtain

Corollary 2 *For every $t > 0$ and $f \in L_p[-1,1]$, $1 \le p < \infty$, we have*

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2).$$

Lemma 3 *a) For every $t \in (0,1]$ and $F \in L_p[-1,1]$, $\frac{1}{\lambda} < p < \infty, \lambda \in (0,1)$ and for $\lambda = 1$, $1 \le p < \infty$ we have*

$$K(F,t;L_p[-1,1],Z_2,D_2) \le c\left[K(F,t^{\frac{1}{p}+(1-\lambda)};L_p[-1,1],C^2,D_2) + t^{\frac{1}{p}+(1-\lambda)}E_0(F)_p\right].$$

b) For every $t \in (0,1]$ and $F \in L_p[-1,1]$, $1 \le p \le \frac{1}{\lambda}$, $\lambda \in (0,1)$ we have

$$K(F,t;L_p[-1,1],Z_2,D_2) = K(F,t;L_p[-1,1],C^2,D_2).$$

Proof. Let $\lambda \in (0,1)$. For $\delta \in (0,1/2)$ we set $\mu(x) = (1-x\delta^{-1})^3_+$, where $(y)_+ = y$ if $y \ge 0$ and $(y)_+ = 0$ if $y \le 0$. For $g \in C^2[-1,1]$ we set

(12) $$G(x) = g(x) + \alpha\mu(x+1) + \beta\mu(1-x)$$

where

$$\alpha = \frac{\delta}{3}\frac{\int_{-1}^{0}\varphi'(y)g'(y)dy}{\int_{-1}^{0}\varphi'(y)(1-\frac{y+1}{\delta})_+^2 dy}, \quad \beta = -\frac{\delta}{3}\frac{\int_{0}^{1}\varphi'(y)g'(y)dy}{\int_{0}^{1}\varphi'(y)(1-\frac{1-y}{\delta})_+^2 dy}.$$

Parameters α, β and δ are chosen in such way that $G \in Z_2$. From (12) we get $\|G-g\|_p \le c\delta^{1/p}(|\alpha|+|\beta|)$ and

$$G''(x) = g''(x) + 6\delta^{-2}\left[\alpha\left(1-\frac{x+1}{\delta}\right)_+ + \beta\left(1-\frac{1-x}{\delta}\right)_+\right].$$

Hence $\left\|\left(1-x^2\right)G''\right\|_p \le \left\|\left(1-x^2\right)g''\right\|_p + c\delta^{-1+1/p}(|\alpha|+|\beta|)$, and

$$\begin{aligned} K(F,t;L_p,Z_2,D_2) &\le \|F-G\|_p + t\|D_2G\|_p \\ &\le \|F-g\|_p + t\|D_2g\|_p + c\delta^{1/p}(1+t\delta^{-1})(|\alpha|+|\beta|). \end{aligned}$$

In order to estimate $|\alpha|+|\beta|$ we calculate

$$\begin{aligned} \left|\int_0^1 \varphi'(y)(1-\frac{1-y}{\delta})_+^2 dy\right| &= \frac{1}{\delta^2}\int_{1-\delta}^{1}\varphi'(y)(\delta-1+y)^2 dy \\ &\ge \frac{c}{\delta^2}\int_{1-\delta}^{1}\frac{1}{(1-y)^{1-\lambda}}(\delta-(1-y))^2 dy \\ &= \frac{c}{\delta^2}\int_0^{\delta}\frac{(\delta-t)^2}{t^{1-\lambda}}dt = c\delta^{\lambda}, \end{aligned}$$

because $\int_0^{\delta}\frac{(\delta-t)^2}{t^{1-\lambda}}dt = \delta^{2+\lambda}\int_0^1\frac{(1-t)^2}{t^{1-\lambda}}dt,\ (t\to\delta t)$.
In a similar way we get

$$\left|\int_{-1}^{0}\varphi'(y)(1-\frac{y+1}{\delta})_+^2 dy\right| \ge c\delta^{\lambda}.$$

Then

$$|\alpha| \le c\delta^{1-\lambda}\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right|, \qquad |\beta| \le c\delta^{1-\lambda}\left|\int_0^1\varphi'(y)g'(y)dy\right|.$$

Using Hölder inequality we estimate

$$\begin{aligned}|\alpha| &\leq c\delta^{1-\lambda}\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right| \\ &\leq c\delta^{1-\lambda}\left\{\int_{-1}^{0}\left|\varphi'(y)\right|^{q}dy\right\}^{1/q}\left\{\int_{-1}^{0}\left|g'(y)\right|^{p}dy\right\}^{1/p} \\ &\leq c\delta^{1-\lambda}\left\{\int_{-1}^{0}\left|g'(y)\right|^{p}dy\right\}^{1/p} \\ &= c\delta^{1-\lambda}\left\|g'\right\|_{p[-1,0]} \text{ for } p>\frac{1}{\lambda}, \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

We used that $\int_{-1}^{0}|\varphi'(y)|^q\,dy \leq c\int_{-1}^{0}\left|\frac{1}{(1+y)^{1-\lambda}}\right|^q dy \leq c$, as $q=\frac{p}{p-1}=\frac{1}{1-\frac{1}{p}}<\frac{1}{1-\lambda}$ for $p>\frac{1}{\lambda}$, i.e. $q(1-\lambda)<1$. Similarly

$$|\beta| \leq c\delta^{1-\lambda}\left\|g'\right\|_{p[0,1]} \text{ for } p>\frac{1}{\lambda}.$$

Hence

$$|\alpha|+|\beta| \leq c\delta^{1-\lambda}\left\|g'\right\|_{p} \text{ for } p>\frac{1}{\lambda}.$$

In order to estimate the norm of g' we apply the inequality

$$\left\|g'\right\|_p \leq c\left(\|D_2 g\|_p + E_0(g)_p\right),$$

which, for instance, follows from [2, p.135, assertion (a)]. Then we get

$$|\alpha|+|\beta| \leq c\delta^{1-\lambda}\left(\|D_2 g\|_p + E_0(g)_p\right) \leq c\delta^{1-\lambda}\left(\|D_2 g\|_p + \|F-g\|_p + E_0(F)_p\right).$$

Now we take $\delta = t/2$. Thus

$$\begin{aligned}&K(F,t;L_p,Z_2,D_2) \\ \leq\ & \|F-g\|_p + t\,\|D_2 g\|_p + ct^{1/p+1-\lambda}\left(\|D_2 g\|_p + \|F-g\|_p + E_0(F)_p\right) \\ \leq\ & c\left(\|F-g\|_p + t^{1/p+1-\lambda}\,\|D_2 g\|_p + t^{1/p+1-\lambda}E_0(F)_p\right),\end{aligned}$$

for every $g\in C^2[-1,1]$, as $0<\frac{1}{p}+1-\lambda=1+\left(\frac{1}{p}-\lambda\right)<1$ and then for $t\in(0,1]$ we have $t^{1/p+1-\lambda}\geq t$ which proves part a) in the case $\lambda\in(0,1), \frac{1}{\lambda}<p<\infty$.

Let $\lambda=1$. We have as above

$$\left|\int_{0}^{1}\varphi'(y)\Big(1-\frac{1-y}{\delta}\Big)_+^2 dy\right| \geq c\delta, \quad \left|\int_{-1}^{0}\varphi'(y)\Big(1-\frac{1+y}{\delta}\Big)_+^2 dy\right| \geq c\delta.$$

Then as $|\varphi'(y)| \le c$ for y$\in [-1,1], \lambda = 1$ we have

$$\begin{aligned} |\alpha| + |\beta| &\le c\left(\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right| + \left|\int_{0}^{1}\varphi'(y)g'(y)dy\right|\right) \\ &\le c\left(\|g'\|_{1,[-1,0]} + \|g'\|_{1,[0,1]}\right) \le c\|g'\|_1 \le c\|g'\|_p \text{ for } p \ge 1. \end{aligned}$$

The proof completes as in the case $\lambda \in (0,1)$.

In order to prove part b) it is sufficient to show (see Lemma 2 in [3, p.116]) that for every $g \in C^2[-1,1]$ and every $\varepsilon > 0$ there exists $G \in Z_2$ such that $\|G-g\|_p < \varepsilon$ and $\left\|(1-x^2)G''\right\|_p < \|(1-x^2)g''\|_p + \varepsilon$. For $1 \le p < \frac{1}{\lambda}$ we can define G by (12).We have

$$\begin{aligned} \|G-g\|_p &\le c\delta^{1/p}(|\alpha|+|\beta|) \\ &\le c\delta^{1/p+1-\lambda}\left(\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right| + \left|\int_{0}^{1}\varphi'(y)g'(y)dy\right|\right) \underset{\delta\to 0}{\to} 0. \end{aligned}$$

We used that $\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right| \le c\int_{-1}^{0}\frac{1}{(1+y)^{1-\lambda}}|g'(y)|\,dy \le c$ and $\left|\int_{0}^{1}\varphi'(y)g'(y)dy\right| \le c\int_{0}^{1}\frac{1}{(1-y)^{1-\lambda}}|g'(y)|\,dy \le c$ for $\lambda > 0$.

$$\begin{aligned} \left\|(1-x^2)G''\right\|_p &\le \|(1-x^2)g''\|_p + c\delta^{-1+1/p}(|\alpha|+|\beta|) \\ &\le \|(1-x^2)g''\|_p + c\delta^{1/p-\lambda}\left(\left|\int_{-1}^{0}\varphi'(y)g'(y)dy\right| + \left|\int_{0}^{1}\varphi'(y)g'(y)dy\right|\right). \end{aligned}$$

When $\frac{1}{p} - \lambda > 0$ the last term tends to zero as $\delta \to 0$, which proves part b) in case $1 \le p < \frac{1}{\lambda}$.

The case $p = \frac{1}{\lambda}$ needs special consideration and different definition of G.

Let $\delta \in (0, \frac{1}{2})$. We set

$$\psi_\delta''(x) = \begin{cases} 0 & \text{for } x \in [-1,0]; \\ \frac{x}{(1-x^2)^{1+\lambda}} & \text{for } x \in (0, 1-\delta]; \\ \frac{1-\delta}{(2\delta-\delta^2)^{1+\lambda}} & \text{for } x \in (1-\delta, 1]. \end{cases}$$

By integration we have

$$\psi_\delta'(x) = \begin{cases} 0 & \text{for } x \in [-1,0]; \\ \frac{1}{2\lambda(1-x^2)^\lambda} - \frac{1}{2\lambda} & \text{for } x \in (0, 1-\delta]; \\ \frac{1}{2\lambda(2\delta-\delta^2)^\lambda} - \frac{1}{2\lambda} + \frac{1-\delta}{(2\delta-\delta^2)^{1+\lambda}}[x-(1-\delta)] & \text{for } x \in (1-\delta, 1], \end{cases}$$

$$\psi_\delta(x) = \begin{cases} 0 & \text{for } x \in [-1,0]; \\ \frac{1}{2\lambda}\int\limits_0^x \frac{dt}{(1-t^2)^\lambda} - \frac{1}{2\lambda}x & \text{for } x \in (0,1-\delta]; \\ \frac{1}{2\lambda}\int\limits_0^{1-\delta} \frac{dt}{(1-t^2)^\lambda} - \frac{1}{2\lambda}(1-\delta) & \\ +\left(\frac{1}{2\lambda(2\delta-\delta^2)^\lambda} - \frac{1}{2\lambda}\right)[x-(1-\delta)] & \text{for } x \in (1-\delta,1]; \\ +\frac{1-\delta}{2(2\delta-\delta^2)^{1+\lambda}}[x-(1-\delta)]^2 & \end{cases}$$

$\psi''_\delta(x)$, $\psi'_\delta(x)$ and $\psi_\delta(x)$ are continuous and increasing functions.

We set now $\mu(x) = \psi_\delta(x)$. For $g \in C^2[-1,1]$ we set

$$G(x) = g(x) + \alpha\mu(x) + \beta\mu(-x). \tag{13}$$

Parameters α and β are chosen in such way that $G \in Z_2$:

$$\begin{aligned} 0 &= \int\limits_0^1 \varphi'(x)G'(x)dx = \int\limits_0^1 \varphi'(x)g'(x)dx + \alpha\int\limits_0^1 \varphi'(x)\psi'_\delta(x)dx. \text{ Hence} \\ \alpha &= -\frac{\int\limits_0^1 \varphi'(x)g'(x)dx}{\int\limits_0^1 \varphi'(x)\psi'_\delta(x)dx}. \text{ Similarly } \beta = \frac{\int\limits_{-1}^0 \varphi'(x)g'(x)dx}{\int\limits_{-1}^0 \varphi'(x)\psi'_\delta(-x)dx} = \frac{\int\limits_{-1}^0 \varphi'(x)g'(x)dx}{\int\limits_0^1 \varphi'(-x)\psi'_\delta(x)dx}. \end{aligned}$$

From (13) we get

$$\begin{aligned} \|G-g\|_{\frac{1}{\lambda}} &\le (|\alpha|+|\beta|)\,\|\psi_\delta\|_{\frac{1}{\lambda}} \text{ and} \\ \left\|(1-x^2)G''\right\|_{\frac{1}{\lambda}} &\le \left\|(1-x^2)g''\right\|_{\frac{1}{\lambda}} + (|\alpha|+|\beta|)\left\|(1-x^2)\psi''_\delta\right\|_{\frac{1}{\lambda}}. \end{aligned}$$

In order to estimate the last expressions we use some properties of ψ_δ given in the following

Assertion 1

Let $\delta \in (0, \frac{1}{2})$. Then we have

a) $\left|\int\limits_0^1 \varphi'(x)\psi'_\delta(x)dx\right| \sim \ln\frac{1}{\delta}$.

b) $\left\|(1-x^2)\psi''_\delta\right\|_{\frac{1}{\lambda}} \sim \left(\ln\frac{1}{\delta}\right)^\lambda$.

c) $\|\psi_\delta\|_{\frac{1}{\lambda}} \sim 1$.

Using Assertion 1 we obtain

$$\begin{aligned}
\|G-g\|_{\frac{1}{\lambda}} &\le (|\alpha|+|\beta|)\,\|\psi_\delta\|_{\frac{1}{\lambda}} \le c(|\alpha|+|\beta|) \\
&\le c\left(\frac{\left|\int\limits_0^1 \varphi'(x)g'(x)dx\right|}{\left|\int\limits_0^1 \varphi'(x)\psi_\delta'(x)dx\right|}+\frac{\left|\int\limits_{-1}^0 \varphi'(x)g'(x)dx\right|}{\left|\int\limits_0^1 \varphi'(-x)\psi_\delta'(x)dx\right|}\right) \\
&\le \frac{c}{\ln\frac{1}{\delta}}\left(\left|\int\limits_{-1}^{0}\varphi'(x)g'(x)dx\right|+\left|\int\limits_{0}^{1}\varphi'(x)g'(x)dx\right|\right).
\end{aligned}$$

$$\begin{aligned}
&\left\|(1-x^2)G''\right\|_{\frac{1}{\lambda}} \\
\le\ & \left\|(1-x^2)g''\right\|_{\frac{1}{\lambda}}+(|\alpha|+|\beta|)\left\|(1-x^2)\psi_\delta''\right\|_{\frac{1}{\lambda}} \\
\le\ & \left\|(1-x^2)g''\right\|_{\frac{1}{\lambda}}+\left(\frac{\left|\int\limits_0^1 \varphi'(x)g'(x)dx\right|}{\left|\int\limits_0^1 \varphi'(x)\psi_\delta'(x)dx\right|}+\frac{\left|\int\limits_{-1}^0 \varphi'(x)g'(x)dx\right|}{\left|\int\limits_0^1 \varphi'(-x)\psi_\delta'(x)dx\right|}\right)\left(\ln\frac{1}{\delta}\right)^{\lambda} \\
\le\ & \left\|(1-x^2)g''\right\|_{\frac{1}{\lambda}}+\frac{\left(\ln\frac{1}{\delta}\right)^{\lambda}}{\ln\frac{1}{\delta}}\left(\left|\int\limits_{0}^{1}\varphi'(x)g'(x)dx\right|+\left|\int\limits_{-1}^{0}\varphi'(x)g'(x)dx\right|\right).
\end{aligned}$$

Let $g\in C^2[-1,1]$, $\varepsilon>0$ is a small number. For a given function g and $\varepsilon>0$ we may choose $\delta>0$ such that

$$\frac{c}{\left(\ln\frac{1}{\delta}\right)^{1-\lambda}}\left(\left|\int\limits_{-1}^{0}\varphi'(x)g'(x)dx\right|+\left|\int\limits_{0}^{1}\varphi'(x)g'(x)dx\right|\right)\le\varepsilon,$$

which proves b) in case $p=\frac{1}{\lambda}$ in view of [3, Lemma 2, p.116]).

Lemma 4 *For every* $t\in(0,1]$ *and* $f\in L_p[-1,1]$, $\frac{1}{\lambda}<p<\infty$ *for* $\lambda\in(0,1)$ *and* $1\le p<\infty$ *for* $\lambda=1$ *we have*

$$tE_0(f)_p\le cK(f,t;L_p[-1,1],C^2,D_1).$$

Proof. Let first $\lambda\in(0,1)$. For every $g\in\mathbf{C}^2[-1,1]$ we have

$$|g(x)-g(0)|\le\left|\int\limits_0^x\frac{dt}{\varphi(t)}\right|\left\|\varphi g'\right\|_\infty,\ x\in(-1,1).$$

We obtain the last inequality using the Generalized Mean Value Theorem

$$\frac{g(x)-g(0)}{\int\limits_0^x \frac{dt}{\varphi(t)}} = \frac{g'(\xi)}{\frac{1}{\varphi(\xi)}} = \varphi(\xi)g'(\xi) \text{ for some } \xi \text{ in } (0,x).$$

$\left|\int\limits_0^x \frac{dt}{\varphi(t)}\right| \le \max\left\{\left|\int\limits_0^{-1} \frac{dt}{\varphi(t)}\right|, \left|\int\limits_0^{1} \frac{dt}{\varphi(t)}\right|\right\} \le c$, for $\lambda \in (0,1)$. Hence $\|g-g(0)\|_p \le c\,\|\varphi g'\|_\infty$. Since $\varphi(1)g'(1) = \varphi(-1)g'(-1) = 0$ using Hölder inequality we get for every $x \in [-1,1]$

$$\begin{aligned}
\left|\varphi(x)g'(x)\right| &= \left|\int\limits_{-1}^x \left(\varphi(t)g'(t)\right)' dt\right| \\
&= \left|\int\limits_{-1}^x \frac{\varphi(t)}{1-t^2}\frac{1-t^2}{\varphi(t)}\left(\varphi(t)g'(t)\right)' dt\right| = \left|\int\limits_{-1}^x \frac{\varphi(t)}{1-t^2}\left(D_1 g\right)(t)dt\right| \\
&\le \left\{\int\limits_{-1}^x \left(\frac{1}{\left(1-t^2\right)^{1-\lambda}}\right)^q dt\right\}^{1/q} \left\{\int\limits_{-1}^x \left|\left(D_1 g\right)(t)\right|^p dt\right\}^{1/p} \\
&\le c\,\|D_1 g\|_p \text{ for } p > \frac{1}{\lambda} \text{ and } \frac{1}{p}+\frac{1}{q}=1, \text{ as } q(1-\lambda)<1. \text{ Thus,}
\end{aligned}$$

$$tE_0(f)_p \le t\,\|f-g(0)\|_p \le t\,\|f-g\|_p + t\,\|g-g(0)\|_p \le c\left[\|f-g\|_p + t\,\|D_1 g\|_p\right],$$

which proves Lemma 4 in the case $p > \frac{1}{\lambda}, \lambda \in (0,1)$.

Now we consider the case $\lambda = 1$. For every $g \in \mathbf{C}^2[-1,1]$ we have

$$\begin{aligned}
|g(x)-g(0)| &\le \left|\int\limits_0^x \frac{dt}{\varphi(t)}\right| \|\varphi g'\|_\infty = \left|\int\limits_0^x \frac{dt}{1-t^2}\right| \|\varphi g'\|_\infty \\
&= \frac{1}{2}\left|\ln\left|\frac{1+x}{1-x}\right|\right| \|\varphi g'\|_\infty, \; x \in (-1,1).
\end{aligned}$$

Then $\|g-g(0\|_p \le \frac{1}{2}\|\varphi g'\|_\infty \left\|\ln\left|\frac{1+x}{1-x}\right|\right\|_p \le c\,\|\varphi g'\|_\infty$ for $1 \le p < \infty$. Using that

$\varphi(1)g'(1) = \varphi(-1)g'(-1) = 0$ and Hölder inequality we get for every $x \in [-1,1]$

$$\begin{aligned}
\left|\varphi(x)g'(x)\right| &= \left|\int_{-1}^{x} \left(\varphi(t)g'(t)\right)' dt\right| \\
&= \left|\int_{-1}^{x} \frac{\varphi(t)}{1-t^2}\frac{1-t^2}{\varphi(t)} \left(\varphi(t)g'(t)\right)' dt\right| = \left|\int_{-1}^{x} \frac{\varphi(t)}{1-t^2} \left(D_1 g\right)(t) dt\right| \\
&= \left|\int_{-1}^{x} 1.\left(D_1 g\right)(t) dt\right| \le \left\{\int_{-1}^{x} (1)^q\, dt\right\}^{1/q} \left\{\int_{-1}^{x} \left|\left(D_1 g\right)(t)\right|^p dt\right\}^{1/p} \\
&\le c\left\|D_1 g\right\|_p \text{ for } p \ge 1 \text{ and } \frac{1}{p}+\frac{1}{q}=1. \text{ Thus,}
\end{aligned}$$

$$tE_0(f)_p \le t\left\|f-g(0)\right\|_p \le t\left\|f-g\right\|_p + t\left\|g-g(0)\right\|_p \le c\left[\left\|f-g\right\|_p + t\left\|D_1 g\right\|_p\right],$$

which proves the lemma.

Proof of Theorems 1 and 2. From parts a) and b) of Theorems 3 and 4 we get $E_0(f)_p \sim E_0(Af)_p$ for $\lambda \in (0,1]$, $1 \le p < \infty$. Using Corollary 2 and Lemma 3 part a) with $F = Af$ we get

$$\begin{aligned}
& K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],Z_2,D_2) \\
\le\ & c\left[K(Af,t^{\frac{1}{p}+1-\lambda};L_p[-1,1],C^2,D_2) + t^{\frac{1}{p}+1-\lambda}E_0(f)_p\right],
\end{aligned}$$

$$\text{for } \frac{1}{\lambda} < p < \infty\ ,\ \lambda \in (0,1)\ \textit{and}\quad \textit{for } 1 \le p < \infty\ ,\ \lambda = 1,$$

which proves the first inequality of Theorem 2.
From Corollary 2 and Lemma 3 part b) we obtain

$$K(f,t;L_p[-1,1],C^2,D_1) \sim K(Af,t;L_p[-1,1],C^2,D_2),\ \text{for } 1 \le p \le \frac{1}{\lambda},\ \lambda \in (0,1),$$

which proves Theorem 1 a).
From Corollary 2 and Lemma 4 we obtain for $\frac{1}{\lambda} < p < \infty$, $\lambda \in (0,1)$ *and for* $1 \le p < \infty$, $\lambda = 1$

$$\begin{aligned}
K(Af,t;L_p[-1,1],C^2,D_2) + tE_0(f)_p &\le K(Af,t;L_p[-1,1],Z_2,D_2) + tE_0(f)_p \\
&\le cK(f,t;L_p[-1,1],C^2,D_1),
\end{aligned}$$

which proves the second inequality of Theorem 2 and in the case $\lambda = 1, p = 1$ Theorem 1 b).

Remark 1 Theorem 1 a) is not true for $\lambda = 1$ (see Theorem B). To make it true on the right hand side of the relation we have to add the term $tE_0(f)_1$, what is exactly the result in [3] for $p = 1$. That is not strange, because for $\lambda = 1$ after we integrate by parts the integral conditions (describing the space Z_2) we obtain the conditions considered by Ivanov in [3, p.120] and for $\lambda = 1$ the differential operator $(D_1 g)(x) = (1 - x^2)g^{''} - 2xg^{'}(x)$ what is exactly the analogue of D_3 in the inteval $[-1, 1]$.

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Teodora Dimova Zapryanova
University of Economics
Department of Math. Sciences
Blvd. Kniaz Boris I ,77, 9002 Varna, Bulgaria
e-mail:teodorazap@abv.bg