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Contents

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Academician Professor D.D. Stancu - a life time dedicated to numerical analysis and theory of approximation [1]

Voichiţa Adriana Radu

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

This article wants to be a tribute dedicated to the academician professor Dimitrie D. Stancu. His life (1927 – 2014) was a beautiful but a hard life, full of work, honor and success. This article reflects the author personal point of view and is far from complete, but it is a prospective through the eyes of one of his last PhD students.

2010 Mathematics Subject Classification: 01A65, 01A70, 41A36.
Key words and phrases: Approximation by positive linear operators, Stancu operators.

1 Academician Professor D.D. Stancu

This spring, on April 17, the mathematical community suffered a big loss: the decease of Academician Professor D.D. Stancu, a Romanian distinguish mathematician.

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He was an Emeritus member of American Mathematical Society and an Honorary member of the Romanian Academy. He was also a member of the German society Gesellschaft fur Angewandte Mathematik und Mechanik.

Professor D.D. Stancu was born on February 11, 1927, in a farmer family, from the township Călacea, situated not far from Timişoara, the capital of Banat, a southwest province of Romania. In his shoolage he had many difficulties being orphan and very poor, but with the help of his mathematics teachers he succeeded to make progress in studies at the prestigious Liceum Moise Nicoara from the large city Arad.

In the period 1947-1951 he studied at the Faculty of Mathematics of the University Victor Babeş, from Cluj, Romania. When he was a student he was under the influence of professor Tiberiu Popoviciu (1906-1975), a great master of Numerical Analysis and Approximation Theory. He stimulated him to do research work (see [3]).

The main contributions of research work of Professor D.D. Stancu fall into the following list of topics: Approximation of functions by means of linear and positive operators, Representation of remainders in linear approximation procedures, Probabilistic methods for construction and investigation of linear positive operators, Interpolation theory, Spline approximation, Numerical differentiation, Orthogonal polynomials, Numerical quadratures and cubatures, Taylor-type expansions, Use of Interpolation and Calculus of finite differences in Probability theory and Mathematical statistics.

He has obtained the Ph.D. in Mathematics in 1956 and his scientific advisor for the doctoral dissertation was professor Tiberiu Popoviciu.

From 1951 he had a continuous academic career at the Babes-Bolyai University of Cluj. At the university, professor D.D. Stancu has taught several courses: Mathematical Analysis, Numerical Analysis, Approximation Theory, Informatics, Probability Theory and Constructive Theory of Functions.

From 1968 (in 46 years) he had 46 PhD students from Romania, Germany and Vietnam. In the following, we will give the list of the mathe- maticians who's PhD thesis was conducted by Academician Professor D.D. Stancu, in cronological order.

From 1970 to 1979 :

1 Gligor Moldovan – 1971
2 Ştefan Măruşter – 1975
3 Ioan Gânscă – 1975
4 Alexandru Lupaş – 1976
5 Ion Mihoc – 1976
6 Trung Du Hoang – 1976
7 Ştefan Cobzaş – 1978
8 Maria Micula – 1978
9 Octavian Dogaru – 1979

From 1980 to 1989 :

1 Dumitru Acu – 1980
2 Dumitru Adam – 1980
3 Aurel Gaidici – 1980
4 Horst Kramer – 1980
5 Maria Mihoc – 1981
6 Ioan Gavrea – 1982
7 Petru Blaga – 1983
8 Adrian Diaconu – 1983
9 Crăciun Iancu – 1983
10 Ioan Valeriu Şerb – 1983
11 Floare Elvira Kramer – 1984
12 Constantin Manole – 1984
13 Traian Augustin Mureşan – 1984
14 Zoltan Kasa – 1985
15 Leon Ţâmbulea – 1985
16 Cristina Sanda Cismaşiu – 1986
17 Tiberiu Vladislav – 1986
18 Maria Dumitrescu – 1989
19 Teodor Toadere – 1989

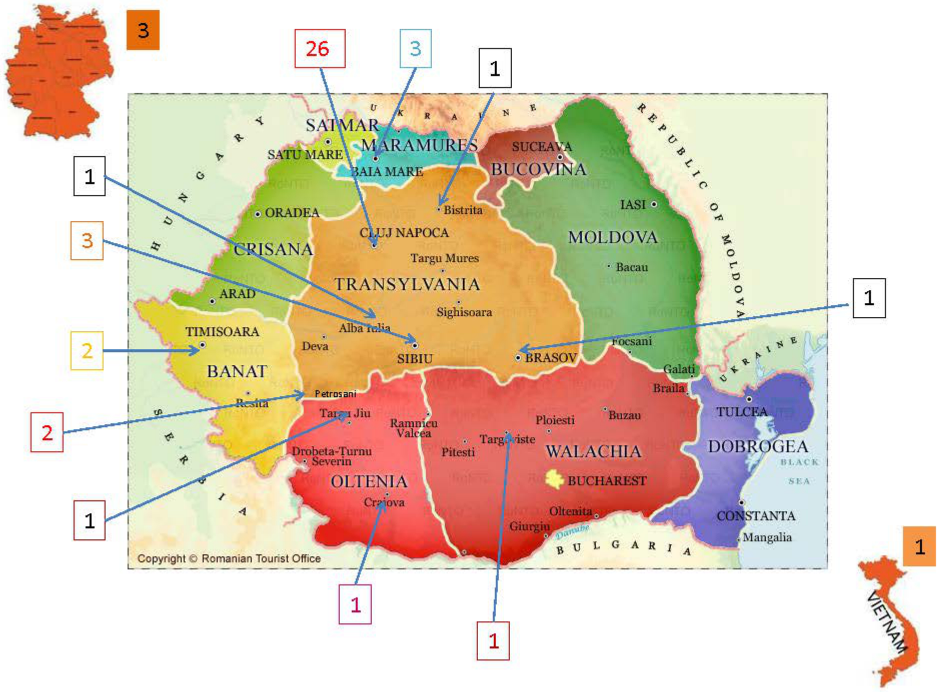
From 1990 to 1999 :

1 Dumitru Dumitrescu – 1990
2 Ioana Chiorean – 1994
3 Octavian Agratini – 1995
4 Alexandra Ana Ciupa – 1995
5 Reiner Dunnbeil – 1996
6 Alexandru Dan Bărbosu – 1997
7 Gabriela Vlaic – 1998
8 Emil Cătinaş – 1999
9 Daria Elena Dumitraş – 1999
10 Silvia Toader – 1999

From 2000 to 2010 :

1 Andrei Vernescu – 2000
2 Lucia Căbulea – 2002
3 Ioana Marcela Gorduza (Taşcu) – 2004
4 Daniel Marius Vladislav – 2004
5 Maria Crăciun – 2005
6 Voichiţa Adriana Cleciu (Radu) – 2006
7 Alina Ofelia Beian (Puţura) – 2007
8 Elena Iulia Stoica (Laze) – 2010

The mathematical legacy of professor D.D. Stancu is spread all over Romania and even more, as we can see in the following distribution:



2 An attempt to portray D.D. Stancu

In 2002, professor Octavian Agratini (see[1]), using the following sets

- P_1={caring father, youth protector}
- P_2={gifted teacher, minute & skilled tutor, wise & witty speaker}
- P_3={good organizator, honest judge, ambitious & challenging partner}

define the space

$$\mathbf{DD} := P_1 \cup P_2 \cup P_3.$$

Then, he give us the following result:

Theorem 1 *We have*

$$\boldsymbol{DD}{=}sp\ \{internationally\ apreciated\ mathematician,\ great\ heart\}.$$

Now, we try to give a natural consequence of O. Agratini's Theorem.

Let us consider the following :

- *PhD*={dedicated & serious hard workers PhD students}
- f= function of knowledge

- $||f|| = sup|f(x)|, x \in PhD$
- $B(PhD) = \{f : PhD \to \mathbb{R}|$ f bounded by time$\}$
- $C(PhD) = \{f : PhD \to \mathbb{R}|$ f continuous for a lifetime$\}$

Then, we have

$$C_B(PhD) := C(PhD) \cap B(PhD).$$

Theorem 2 $(C_B(PhD), \|\cdot\|)$ *is a subspace of* ***DD*** *space.*

And if we have the set

$$F\text{=}\{\text{trusted, challenging \& dearest friends }\},\ PhD \subset F$$

then

Theorem 3 $(C_B(F), \|\cdot\|)$ *is a* ***DD*** *space.*

3 Stancu operators

In 1968, professor D.D. Stancu introduced and studied a new sequence of linear and positive operators, constructed using Polya-Markov scheme

$S_n^\alpha : C[0,1] \to C[0,1]$,

$$(1)\qquad \left(S_n^\alpha f\right)(x) = \sum_{k=0}^{n} \omega_{(n,k)}^\alpha(x) f\left(\frac{k}{n}\right)$$

where

$$(2)\qquad \omega_{(n,k)}^\alpha(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},$$

$n \in \mathbb{N}$ and α a real parameter depending only on n (see [13], [16], [7], [2] and [17]).

First, we recall the construction of the operator.

Let U be an urn with a white balls & b black balls, $N := a + b$. A trial consists in taking a ball from the urn, recording its colour and replacing the ball into the urn, together with another c balls with the same color.

We want to determine the probability that in n repeated trials to get k white balls.

We denote by X_n^k the desired event. We have to compute $P(X_n^k)$.

Also, W_j will be the event of getting a white ball at the j^{th} trial, $j = \overline{1,n}$.

The desired event can be written as

$$X_n^k = \bigcup \{W_{i_1} \cap W_{i_2} \cap \ldots \cap W_{i_k} \cap \overline{W_{i_{k+1}}} \cap \ldots \cap \overline{W_{i_n}}\}$$

where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$.

The previous union contains $\binom{n}{k}$ terms since we can obtain k white balls at any k trials from n trials.

The desired probability is :

$$P(X_n^k) = P\left(\bigcup\{W_{i_1} \cap W_{i_2} \cap \ldots \cap W_{i_k} \cap \overline{W_{i_{k+1}}} \cap \ldots \cap \overline{W_{i_n}}\}\right)$$
$$= \sum P(W_{i_1}) \cdot P(W_{i_2}) \ldots P(W_{i_k}) \cdot P(\overline{W_{i_{k+1}}}) \ldots P(\overline{W_{i_n}})$$
$$= \sum p_{i_1} \cdot p_{i_2} \ldots p_{i_k} \cdot q_{i_{k+1}} \ldots q_{i_n}$$

then we have

$$P(X_n^k) = \binom{n}{k} \frac{a(a+c)\ldots(a+\overline{k-1}c)b(b+c)\ldots(b+\overline{n-k-1}c)}{N(N+c)(N+2c)\ldots(N+\overline{n-1}c)}.$$

If we use the following notation $\frac{a}{N} = x, x \in (0,1)$, $\frac{b}{N} = 1-x$, $\frac{c}{N} = \alpha$ then we have the *Stancu fundamental polynomials*

$$\omega_{(n,k)}^{\alpha}(x) = \binom{n}{k} \frac{x(x+\alpha)\ldots(x+\overline{k-1}\alpha)(1-x)(1-x+\alpha)\ldots(1-x+\overline{n-k-1}\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{n-1}\alpha)} \ ,$$

and using the notation of the factorial powers, with natural exponent and real step, we have

$$\omega_{(n,k)}^{\alpha}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}.$$

This initial Stancu operators have been intensivly studied by numerous foreign and Romanian mathematicians. In 1968,1970 and 1971, in [13], [14] and [15] professor D.D. Stancu proved the fallowing properties:

- this operators are linear and positive,
- are interpolating at the ends of the interval $[0,1]$,
- for $\alpha = 0$, we find the classical Bernstein operators,
- for $\alpha = -\frac{1}{n}$, (2) becomes the Lagrange interpolation polynomials,
- the following identities hold true:

$$\begin{cases} \left(S_n^{\alpha} e_0\right)(x) = 1 \\ \left(S_n^{\alpha} e_1\right)(x) = x \\ \left(S_n^{\alpha} e_2\right)(x) = \frac{1}{n}x + \frac{n-1}{n} \cdot \frac{x(x+\alpha)}{1+\alpha}, x \in [0,1] \end{cases}$$

- the convergence theorem are proved,
- estimations of the rate of convergence in terms of modulus of continuity are given,

- the operators (1) can be represented by means of the Beta function, the Bernstein operators and by finite and divided differences,
- the norm of the operator is $||S_n^\alpha|| = 1$,
- the operators generate the approximation formula

$$f(x) = (S_n^\alpha f)(x) + (R_n^\alpha f)(x),$$

- different representations of the remainder are given.

In addition, other mathematicians have completed the list of properties of Stancu operators.

In 1978 and 1980, in [11] and [12] G. Mastroianni and M. Occorsio disscused about the variation diminishing property and about the derivatives of Stancu's polynomials.

In 1984, in [9] H.H. Gonska and J. Meier found better constants for estimations of the rate of convergence, in terms of modulus of continuity.

In 1988 and 1989, in [5] and [6] B. Della Vechia gave some recurrence formula and an elementary proof of the preservation of Lipschitz constants by the Stancu operators.

In 2002, in [10] A. Lupaş and L. Lupaş proved a representation of the remainder term and also some mean value theorems. In addition they discussed a quadrature formula for Stancu operators.

Also, in 2002, in [8] Z. Finta present direct and converse results for the operators (1). Moreover, he proved the equivalence of $\|S_n^\alpha f - f\|$ and $\|B_n f - f\|$.

In 2003, in [4] the author revealed the preservation of global smoothness of the Stancu operators, using the modulus of continuity and K- functionals.

4 Conclusions

This paper tried to gather (in chronological order) the most significant properties of the operators discovered and proved either by professor D.D. Stancu or by those who have investigated them.

After the pioneer work of professor D.D. Stancu, these operators have been successfully used by other mathematicians to study properties of linear positive methods of approximation. There are numerous combinations and generalizations of Stancu operators: Bernstein-Stancu operators, Schurer-Stancu operators, Durrmeyer-Stancu operators, Baskakov-Durrmeyer-Stancu operators, Stancu-Hurwitz operators, Kantorovich-Stancu operators and many others.

Professor D.D. Stancu publication lists about 160 items (papers and books). The intensive research work and the important results obtained by professor D.D. Stancu has brought to him international recognition and appreciation. Serch for "Stancu operators" on the Web of Science, returned more then 150 results as journal articles (there are more than 70 papers containing his name in their titles).

We conclude with the words of William Arthur Ward:

"The good teacher explains.
The superior teacher demonstrates.
The great teacher inspires."

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Academician Professor Dimitrie D. Stancu, a respectful remember and a deep homage [1]

Andrei Vernescu

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

The presentation consists in a remembering of the life and the work of our beloved master Acad. Prof. D. D. Stancu.

2010 Mathematics Subject Classification: 01A60, 41A10, 41A25, 41A36, 41A80, 65D25, 65D30.

Key words and phrases: Numerical Analysis, Approximation Theory, Interpolation theory, Numerical differentiation, Orthogonal Polynomials, Numerical quadratures and cubatures, Taylor-type expansions, Linear positive operators of approximation, finite operatorial calculus.

This spring, on April 17, we lost our beloved master and adviser, Professor **Dimitrie D. Stancu**. The chief of the Romanian School of Numerical Analysis and Approximation Theory was a very important professor of "Babeş-Bolyai" University, an honorary member of the Romanian Academy and a Doctor Honoris Causa of the University "Lucian Blaga" of Sibiu and of the North University of Baia Mare (now the North Universitary Center of the Technical University of Cluj-Napoca).

The death of Academician Professor Dimitrie D. Stancu is a painful shock not only for his family, but also for all his friends, contributors, disciples, former students and former Ph. D. students and for the entire mathematical community. We all are very sad and we remember now not only the great mathematician, the head of the important Romanian Mathematical School of Numerical Analysis and Approximation Theory, but also we think to the man, to his pleasant, warm, optimistic and generous personality.

Let's remember some events and facts related to the life and work of Professor Dimitrie D. Stancu. It constitutes a model of the action against the difficulties of

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the unfavourable "initial conditions" of the life, to accomplish a beautiful life, career and mathematical work. Indeed, he was born on February 11, 1927 in a very poor family in the village Călacea, situated near Timişoara, the capital of the province of Banat, he remained orphan and was forced to work when he was too young. Fortunately, his oldest brother took him in Arad at "Regina Maria" ("The Queen Mary") orphanage, where there was included in an elementary school. After this, he studied during the years 1943 and 1947 at the prestigious highschool "Moise Nicoară" of Arad. This educational institution has already given to the cultural and scientific community several personalities, including two academicians in the domain of mathematics: Tiberiu Popoviciu (1906-1975) and Caius Iacob (1912-1992). Dimitrie D. Stancu was a brilliant highschool student and his native talent in mathematics was noticed by his teachers and especially by the one in mathematics, Ascaniu Crişan, the school principal. In 1947 Dimitrie D. Stancu began his four-year study in mathematics at the "Babeş-Bolyai" University from Cluj (today Cluj-Napoca), a town with a very important cultural life, a famous universitary center and the capital of the historical Romanian province of Transylvania. Immediately the very talented but also very sympathetic student was discovered by the great Professor Tiberiu Popoviciu. This professor, one of the most important and influential professors of the University, a great master of the Theory of Approximation, influenced and stimulated the work of research in mathematics of the young Stancu, which was named "preparator" since his third year of studies. In 1951, after his graduation, he was engaged assistant at the Department of Mathematical Analysis, of the University of Cluj. The pleasant town of Cluj will become the new residence of D. D. Stancu. He obtained the Ph. D. in mathematics in 1956, for which the thesis (ready since 1955) was intitled "A study of the polynomial interpolation of functions of several variables, with applications to the numerical differentiation and integration; methods for evaluating the remainders" (in Romanian) and had 192 pages. His adviser was Tiberiu Popoviciu and in the examination comission were Miron Nicolescu, Dumitru V. Ionescu and Adolf Haimovici.

In a normal succession, D. D. Stancu advanced up to the rank of full professor in 1969. He holds a continuous academic career at the University "Babeş-Bolyai", excepting the year 1961-1962, when he had a fellowship at the Numerical Analysis Department of the University of Wisconsin, Madison, conducted by late professor Preston C. Hammer. Here he spent at the University of Wisconsin during the academic year 1961-1962 and had direct scientific contacts with important mathematicians of the time: J. Favard, A. Ostrovski, I. J. Schoenberg, G. G. Lorentz, A. Cheney, P. L. Butzer, A. Sharma and other. Dimitrie D. Stancu also participated to several mathematical events organized by the American Mathematical Society at Milwaukee, Chicago and New York.

When he returned in Romania, he was named deputy dean of the faculty of Mathematics of his University and head of the new created Chair of Numerical and Statistical Calculus; he was the head of this chair between 1962 and 1995, when he retired.

He had a very nice family: his wife dr. Felicia Stancu also worked in mathematics in the same University. His daughters Angela (1957) and Mirela (1958) are teaching mathematics in highschools of Cluj-Napoca. He has three grandsons: Alexandru-Mircea Scridon (1983) and George Scridon (1992), the sons of Mirela and Ştefana Munteanu (1991), the daughter of Angela. For his family and friends, Dimitrie D. Stancu was also named sometimes Didi (from his initiales D. D.)

Professor D. D. Stancu taught Mathematical Analysis, Numerical Analysis, Approximation Theory, Constructive Theory of Functions, Computer Science, Probability Theory and Mathematical Statistics. He published at Babeş-Bolyai University in 1977 his course of Numerical Analysis and Approximation Theory and later he published in cooperation the fundamental treatise [9], [10], [11], a veritable "bible" in the domain.

In the 60's, having a large vision of the future evolutions, he was one of the promotors of the use of the computers, as Grigore C. Moisil at Bucharest.

Professor D. D. Stancu had a very rich research work in Interpolation Theory, Numerical differentiation, Orthogonal Polynomials, Numerical quadratures and cubatures, Taylor-type expansions, Approximation of functions by linear positive operators, Representation of remainders in linear approximation formulas, probabilistic methods for construction and investigation of linear positive operators of approximation, use of interpolation and of calculs of finite differences in probability theory and mathematical statistics, use of the finite operatorial calculus (umbral calculus) in the construction of operators of approximation. He introduced a famous approximation operator named today the operator of Stancu. Consequently, many operators may be considered by a point of vue of an operator of Stancu and the operators of Bernstein-Stancu, Kantorovich-Stancu, Schurer-Stancu, Stancu-Mülbach and other are intensively studied.

An list of selected publications of Professor Dimitrie D. Stancu is given in [1].

Also, note that the name of D. D. Stancu appears in many titles of research papers of foreign authors; a list of these until 2002 is given in [7], pp. 10-15. After this year, also other many papers contain in the title the name of D. D. Stancu.

Professor Dimitrie D. Stancu guided many Ph. D. students (Romanians and foreign). A list of the first 40 doctoral students of our professor and of the corresponding titles of the thesis is given in [7], pp. 6-9.

He was a member of the American Mathematical Society (since 1961), of the German Society "Gesellschaft für Angewandte Mathematik und Mechanik" (GAMM). He was the Editor in Chief of the prestigious journal published by the Romanian Academy "Revue d'Analyse Numérique et de Théorie de l'Approximation" and many years ago he becomes a member of the Editorial Board of the famous Italian journal "Calcolo", published now by "Springer-Verlag". He was a member of the editorial comitees of "Studia Univ. Babeş-Bolyai" and "Mathematica" He has done extensive reviewing, especially in "Mathematical Reviews" and "Zentralblatt für Mathematik".

Professor D. D. Stancu took part to numerous scientific events: Gattlinburg (TN) in USA, Lancaster and Durham in England, Stuttgart, Hannover, Hamburg, Göttingen, Dortmund, München, Siegen, Würzburg, Berlin and Oberwolfach in Germany, Roma, Napoli, Potenza, L'Aquila in Italy, Budapest, Paris, Sofia and other.

He has been invited to present lectures at several Universities from USA, Germany, Holland and other.

Under the leadership of professor D. D. Stancu were organized several scientific events in Cluj-Napoca: "International Conference on Approximation and Optimization" (1996), "International Symposium on Numerical Analysis and Approximation Theory" (2002) and "International Conference on Numerical Analysis and Approximation Theory" (2006). Professor Stancu also took part at several editions of the RoGer Seminars.

We remember always with respect and gratitude the wonderful personality of our brilliant mathematician, of our beloved Professor and adviser, DIMITRIE D. STANCU.

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General Mathematics Vol. 22, No. 1 (2014), 19–26

On Bullen's and related inequalities [1]

Ana Maria Acu, Heiner Gonska

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

The estimate in Bullen's inequality will be extended for continuous functions using the second order modulus of smoothness. A different form of this inequality will be given in terms of the least concave majorant. Also, the composite case of Bullen's inequality is considered.

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1 Motivation

Over the years it happened during several editions of RoGer - the Romanian-German Seminar on Approximation Theory - that the second author learned about inequalities the validity of which was known for regular (i.e., differentiable) functions only. Using tools from Approximation Theory, we showed in [1] and [2] that such restriction can sometimes be dropped and that the estimates can be extended to (at least) continuous functions on the given compact intervals, for example.

Our more general estimates in [1] and [2] were given in terms of the least concave majorant of the usual first order modulus of continuity. Already this is rather a complicated quantity. There we focussed on Ostrowski- and Grüss-type directions. The best way seems to be that via a certain K-functional. This road was recently and thoroughly described in a paper by Păltănea [10]. But the knowledge about this method is much older. See papers by Peetre [11], Mitjagin and Semenov [9] as well as the diploma thesis of Sperling [12], for example.

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In Sections 2 and 3 we will consider classical and composite Bullen functionals. It will become clear there that it is quite natural to use the second order modulus of smoothness of a continuous function and a related K-functional.

At the end of this note we will return to the concave majorant and its significance in the field of Inequalities. For the composite case and continuously differentiable functions Section 4 contains a very precise estimate in terms of the majorant.

2 Bullen's inequality revisited

This paper is mostly meant to be a contribution to the ever-lasting discussion on an inequality given by Bullen in [3] (see also an earlier paper by Hammer [7]) in the following form:

Theorem A. *If f is convex and integrable, then*

$$\left(\int_{-1}^{1} f\right) - 2f(0) \le f(-1) + f(1) - \int_{-1}^{1} f.$$

If transformed to an arbitrary compact interval $[a,b] \subset \mathbb{R}$, $a < b$, the equivalent form of the inequality reads

$$\frac{2}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right).$$

This is the form we learned about at "RoGer 2014 - Sibiu". In the talk of Petrică Dicu we also learned that in 2000 Dragomir and Pearce [4] had given the following inequality for functions f in $C^2[a,b]$ with known bounds for f'':

Theorem 1 *Let $f : [a,b] \to R$ be a twice differentiable function for which there exist real constants m and M such that*

$$m \le f''(x) \le M, \quad \textit{for all} \quad x \in [a,b].$$

Then

$$\begin{aligned} m\frac{(b-a)^2}{24} &\le \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)dx \\ &\le M\frac{(b-a)^2}{24}. \end{aligned} \tag{1}$$

If we define the Bullen functional B by

$$B(f) := \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)dx,$$

we note that B is defined for all functions in $C[a,b]$ and that - so far - we have the following

Proposition 1 *The Bullen functional $B : C[a, b] \to \mathbb{R}$ satisfies*

(i) $|B(f)| \leq 4||f||_\infty$ *for all* $f \in C[a, b]$, $||\cdot||_\infty$ *indicating the sup norm on* $[a, b]$.

(ii) $|B(g)| \leq \dfrac{(b-a)^2}{24} \cdot ||g''||_\infty$ *for all* $g \in C^2[a, b]$.

We will next explain how to turn this into a more general statement using the following result from [6]:

Theorem 2 *Let* $(F, ||\cdot||)_F)$ *be a Banach space, and let* $H : C[a, b] \to (F, ||\cdot||_F)$ *be an operator, where*

a) $||H(f+g)||_F \leq \gamma(||Hf||_F + ||Hg||_F)$ *for all* $f, g \in C[a, b]$;

b) $||Hf||_F \leq \alpha \cdot ||f||_\infty$ *for all* $f \in C[a, b]$;

c) $||Hg||_F \leq \beta_0 \cdot ||g||_\infty + \beta_1 \cdot ||g'||_\infty + \beta_2 \cdot ||g''||_\infty$ *for all* $g \in C^2[a, b]$.

Then for all $f \in C[a, b]$, $0 < h \leq (b-a)/2$, *the following inequality holds:*

$$||Hf||_F \leq \gamma \left\{ \beta_0 \cdot ||f||_\infty + \frac{2\beta_1}{h}\omega_1(f; h) + \frac{3}{4}\left(\alpha + \beta_0 + \frac{2\beta_1}{h} + \frac{2\beta_2}{h^2}\right)\omega_2(f; h) \right\}.$$

Taking $H = B$ and $F = \mathbb{R}$ in Theorem 2 we have the following list of constants:

$$\gamma = 1, \;\; \alpha = 4, \;\; \beta_0 = 0, \;\; \beta_1 = 0, \;\; \beta_2 = \frac{(b-a)^2}{24}.$$

This takes us to the following

Proposition 2 *For the Bullen functional B and all $f \in C[a, b]$ we have*

$$|B(f)| \leq \left(3 + \left(\frac{b-a}{4h}\right)^2\right) \cdot \omega_2(f, h), \;\; 0 < h \leq \frac{b-a}{2}.$$

The special choice $h = \dfrac{b-a}{k}$, $k \geq 2$ *yields*

$$|B(f)| \leq \left(3 + \frac{k^2}{16}\right) \cdot \omega_2\left(f, \frac{b-a}{k}\right).$$

Remark 1 *For $f \in C^2[a, b]$ the latter inequality implies*

$$|B(f)| \leq \left(\frac{1}{16} + \frac{3}{k^2}\right)(b-a)^2||f''||_\infty, k \geq 2.$$

As far as the constant $\left(\dfrac{1}{16} + \dfrac{3}{k^2}\right)$ *is concerned, this is much worse than what was invested for C^2 functions.*

An alternative estimate is given in the next proposition. Note that it also follows from Theorem 6 in Gavrea's paper [5].

Proposition 3 *If the second K-functional on $C[a,b]$ is defined by*

$$K(f;t^2;C[a,b],C^2[a,b]) := \inf\{||f-g||_\infty + t^2||g''||_\infty : g\in C^2[a,b]\},\ \ t\geq 0,$$

then

$$|B(f)| \leq 4K\left(f;\frac{(b-a)^2}{96};C[a,b],C^2[a,b]\right).$$

Proof. In [1] the following result was obtained:

$$\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt\right|$$
$$\leq 2(b-a)K\left(f;\frac{(x-a)^3+(b-x)^3}{24(b-a)};C[a,b],C^2[a,b]\right).$$

The proposition is proved if we substitute $x=\dfrac{a+b}{2}$ in the above inequality.

Remark 2 *(i) For $h\in C^2[a,b]$, we have*

$$\begin{aligned}
|B(h)| &\leq 4\cdot K\left(h;\frac{(b-a)^2}{96};C[a,b],C^2[a,b]\right)\\
&= 4\cdot\inf\left\{||h-g||_\infty+\frac{(b-a)^2}{96}||g''||_\infty, g\in C^2[a,b]\right\}\\
&\leq 4\cdot\frac{(b-a)^2}{96}||h''||_\infty \ \textit{(taking g=h)}\\
&= \frac{(b-a)^2}{24}||h''||_\infty,\ \ \textit{i.e.,}
\end{aligned}$$

the original inequality for C^2 functions.

(ii) It is known that, for $f\in C[a,b]$,

$$K(f;t^2;C[a,b],C^2[a,b]) \leq c\cdot\omega_2(f,t),\ \ 0\leq t\leq\frac{b-a}{2}$$

with a constant $c\neq c(f,t)$. According to our knowledge the best possible value of c is unknown.

Zhuk showed in [13] that, for $t\leq\dfrac{b-a}{2}$, one has

$$K(f;t^2;C[a,b],C^2[a,b]) \leq \frac{9}{4}\cdot\omega_2(f;t).$$

Using the latter we arrive, for $h\in C^2[a,b]$, at

$$|B(h)| \leq \frac{3(b-a)^2}{32}\|h''\|_\infty.$$

3 The composite case

Here we consider the composite case of the Bullen functional, i.e., the functional which arises when comparing the composite trapezoidal and midpoint rules. To this end the interval $[a,b]$ is divided in $n \geq 1$ subintervals as follows

$$a = x_0 < \cdots < x_i < x_{i+1} < \cdots < x_n = b.$$

Let the composite Bullen functional $B_c : C[a,b] \to \mathbb{R}$ be given by

$$B_c(f) = \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[\frac{f(x_i)+f(x_{i+1})}{2} + f\left(\frac{x_i+x_{i+1}}{2}\right)\right.$$
$$\left. - \frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right].$$

Proposition 4 *In the composite case there holds*

(i) $|B_c(f)| \leq 4||f||_\infty$ *for all* $f \in C[a,b]$,

(ii) $|B_c(g)| \leq \dfrac{1}{24(b-a)}\displaystyle\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3||g''||_\infty$ *for all* $g \in C^2[a,b]$.

Using Theorem 2 and Proposition 4 we obtain the following inequality for the composite Bullen functional involving the second modulus of continuity:

Proposition 5 *For the composite Bullen functional one has*

$$|B_c(f)| \leq \left(3 + \frac{1}{16(b-a)h^2}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\right)\omega_2(f,h),\ 0 < h \leq \frac{b-a}{2}.$$

For $h = \dfrac{1}{k}\sqrt{\dfrac{\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3}{b-a}}$, $k \geq 2$, *this yields*

$$|B_c(f)| \leq \left(3 + \frac{k^2}{16}\right)\omega_2\left(f, \frac{1}{k}\sqrt{\frac{\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3}{b-a}}\right),\ k \geq 2.$$

Remark 3 *For* $f \in C^2[a,b]$ *the latter inequality implies*

$$|B_c(f)| \leq \left(\frac{1}{16}+\frac{3}{k^2}\right)\frac{\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3}{b-a}\|f''\|_\infty \leq \left(\frac{1}{16}+\frac{3}{k^2}\right)(b-a)^2\|f''\|_\infty.$$

The requirement $F(x_0,x_1,\ldots,x_n) = \displaystyle\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \to$ *minimum entails* $x_{i+1}-x_i = \dfrac{b-a}{n}$, $i = \overline{0,n-1}$.

The inequality involving a K-functional is given next.

Proposition 6 *For the functional B_c given as above, $f \in C[a,b]$, we have*

$$|B_c(f)| \leq 4K\left(f; \frac{1}{96(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3; C[a,b], C^2[a,b]\right). \tag{2}$$

Proof. Let $g \in C^2[a,b]$ arbitrary. Then, for $f \in C[a,b]$,

$$\begin{aligned}
|B_c(f)| &\leq |B_c(f-g)| + |B_c(g)| \\
&\leq 4\|f-g\|_\infty + \frac{1}{24(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty \\
&= 4\left\{\|f-g\|_\infty + \frac{1}{96(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty\right\}.
\end{aligned}$$

Passing to the infimum over g yields inequality (2).

4 Composite Bullen functional for $f \in C^1[a,b]$

Using the least concave majorant of the modulus of continuity in this section we consider Bullen's inequality for $f \in C^1[a,b]$.

Proposition 7 *If $f \in C^1[a,b]$, then*

$$|B_c(f)| \leq \tilde{\omega}\left(f', \frac{1}{24(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\right).$$

Proof. We have

$$\begin{aligned}
|B_c(f)| \leq{}& \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2} + f\left(\frac{x_i+x_{i+1}}{2}\right)\right. \\
&\left. - \frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \\
={}& \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right. \\
&\left. + f\left(\frac{x_i+x_{i+1}}{2}\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \\
={}& \frac{1}{b-a}\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}\left(\frac{f(x_i)-f(x)}{2} + \frac{f(x_{i+1})-f(x)}{2}\right.\right. \\
&\left.\left. + f\left(\frac{x_i+x_{i+1}}{2}\right) - f(x)\right)dx\right| \leq 2\|f'\|_\infty.
\end{aligned}$$

Let $g \in C^2[a,b]$. Using Proposition 4 and the latter inequality implies

$$\begin{aligned}|B_c(f)| &= |B_c(f-g+g)| \le |B_c(f-g)| + |B_c(g)| \\ &\le 2\|(f-g)'\|_\infty + \frac{1}{24(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty \\ &= 2\left\{\|(f-g)'\|_\infty + \frac{1}{48(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty\right\}.\end{aligned}$$

Passing to the infimum over $g \in C^2[a,b]$ we have

$$|B_c(f)| \le 2K\left(f'; \frac{1}{48(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3; C^1[a,b], C^2[a,b]\right),$$

so the result follows as a consequence of the relation (see [10])

$$K\left(f'; t; C^1[a,b], C^2[a,b]\right) = \frac{1}{2}\tilde{\omega}(f', 2t), 0 \le t \le \frac{b-a}{2}.$$

A particular consequence of Proposition 7 is the following version of Bullen's inequality.

Proposition 8 *If $f \in C^1[a,b]$, then*

$$|B(f)| \le \tilde{\omega}\left(f', \frac{(b-a)^2}{24}\right).$$

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Modified Bernstein-Durrmeyer operators [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this work, we introduce a new type of Bernstein-Durrmeyer operators based on a function $\tau(x)$ which is continuously differentiable $\infty-$ times on $[0,1]$, such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$ for $x\in[0,1]$. We present an asymptotic formula and a quantitative type asymptotic formula for the operators. Later we give comparison with classical Bernstein-Durrmeyer operators. We obtain some direct results for the new operators with the aid of Ditzian-Totik modulus of smoothness. Finally a graphical example is given. All results in this work show that our new operators are flexible and sensitive to the rate of convergence to f, depending on our selection of $\tau(x)$.

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1 Introduction

Many well-known operators preserve the linear as well as constant functions. For example, Bernstein, Baskakov, Szasz-Mirakyan operators posses these properties, i.e. $L_n(e_i;x)=e_i(x)$, where $e_i(x)=x^i$ $(i=0,1)$. To make the convergence faster King [18] proposed an approach to modify the classical Bernstein polynomial for $f\in C[0,1]$ by

$$((B_nf)\circ r_n)(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(r_n(x))^k(1-r_n(x))^{n-k},$$

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where r_n is a sequence of continuous functions defined on $[0,1]$ with $0 \leq r_n(x) \leq 1$ for each $x \in [0,1]$ and $n \in \mathbb{N} = \{1,2,...\}$. King considered a particular case of $r_n(x)$ so that the corresponding operators preserve the test functions e_0 and e_2 of Bohman-Korovkin theorem. Moreover, Gonska et al. [15] constructed sequences of King-type operators which are based on a function τ such that $\tau \in C[0,1]$ is strictly increasing, $\tau(0) = 0$, $\tau(1) = 1$. These operators are defined by $V_n : C[0,1] \to C[0,1]$

$$V_n^\tau f = (B_n f) \circ \tau_n = (B_n f) \circ (B_n \tau)^{-1} \circ \tau.$$

Inspired by the above ideas, the authors of [6] defined the sequence of linear Bernstein type operators for $f \in C[0,1]$ by

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}, \tag{1}$$

for any functions τ being continuously differentiable $\infty-$ times on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. They investigated its shape preserving and convergence properties as well as its asymptotic behavior and saturation. This type of approximation operators generalizes the Korovkin set from $\{1, e_1, e_2\}$ to $\{1,\ \tau,\ \tau^2\}$ and also presents a better degree of approximation depending on τ. These advantages of the above construction lead to an interesting area of research, so that generalized Szasz type operators depending on τ and their approximation properties were recently studied in [2]. In [7], the authors considered the special case $\tau = (e_2 + \alpha e_1)/(1+\alpha)$ and studied some shape preserving approximation and convergence properties. It was shown that these operators represent a good shape preserving approximation process making a comparison with Bernstein polynomials. Recently, the authors of [8], using the referred operators, studied some modified Bernstein-Durrmeyer type operators that reproduce certain test functions.

In the present paper, we deal with Durrmeyer type generalization of (1). Durrmeyer [12] was first who generalized the Bernstein polynomials for integrable functions on $[0,1]$. Later on, Durrmeyer type operators and further generalizations presenting better approximation results have been intensively studied. Among the others, we refer the readers to [9], [8], [10], [17], [16], [21] and the references therein. The operators B_n^τ are meaningful for continuous functions whereas for functions belonging to Lebesgue space, the Durrmeyer modifications of them are more useful. In this direction, we consider the following modified Durrmeyer operators:

$$D_n^\tau(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^\tau(x) \int_0^1 \left(f \circ \tau^{-1}\right)(t)\, p_{n,k}(t)\, dt, \tag{2}$$

where

$$p_{n,k}^\tau(x) := \binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}$$

and

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

If we choose $\tau(x) = x$, we have classical Durrmeyer operators [12] given by

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(t)\, p_{n,k}(t)\, dt. \tag{3}$$

The rest of the paper is organized as follows. In the next section, we give some auxiliary results that will be used throughout the paper. Later, we focus on a Voronovskaya type asymptotic formula, as well as its quantitative version, using a general construction from which our new operators can be obtained as a special case. We also investigate local approximation properties of D_n^τ in quantitative form using an appropriate K-functional and Ditzian-Totik moduli of smoothness. Finally we give a comparison of the new operators with classical Bernstein-Durrmeyer operators in terms of a graphical example.

2 Some Lemmas

For our main results, we shall need some auxiliary results. Since they are similar to the corresponding results for the Bernstein-Durrmeyer operators D_n, we give the following lemmas without proofs. Also they can be checked just by taking $\tau = e_1$ (see [14]).

Lemma 1 *We have*

$$D_n^\tau e_0 = e_0, \quad D_n^\tau \tau = \frac{1+\tau n}{n+2}, \quad D_n^\tau \tau^2 = \frac{\tau^2 n(n-1) + 4n\tau + 2}{(n+2)(n+3)}.$$

Lemma 2 *If we define the central moment operator by*

$$\begin{aligned} \mu_{n,m}^\tau(x) &= D_n^\tau\left((\tau(t) - \tau(x))^m ; x\right) \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}^\tau(x) \int_0^1 (t-\tau(x))^m\, p_{n,k}(t)\, dt, \ m \in \mathbb{N}, \end{aligned}$$

then we have

$$\mu_{n,0}^\tau(x) = 1, \ \mu_{n,1}^\tau(x) = \frac{1-2\tau(x)}{n+2} \tag{4}$$

$$\mu_{n,2}^\tau(x) = \frac{\tau(x)(1-\tau(x))(2n-6)+2}{(n+2)(n+3)}. \tag{5}$$

for all $n, m \in \mathbb{N}$.

Lemma 3 *For all $n \in \mathbb{N}$ we have*

$$D_n^\tau \left(e_{\tau,2}^x; x\right) \leq \frac{2}{n+2} \delta_{n,\tau}^2 (x), \tag{6}$$

where $e_{\tau,i}^x (t) = (\tau (t) - \tau (x))^i$ *and* $\delta_{n,\tau}^2 (x) := \varphi_\tau^2 (x) + \frac{1}{n+3}$, $\varphi_\tau^2 (x) := \tau (x) (1 - \tau (x))$, $x \in [0,1]$.

Lemma 4 *For $f \in C[0,1]$, we have $\|D_n^\tau f\| \leq \|f\|$, where $\|.\|$ is the uniform norm on $C[0,1]$.*

Proof. By the definition of the operator D_n^τ and using Lemma 1 we have

$$\begin{aligned} \left|D_n^\tau (f; x)\right| &\leq (n+1) \sum_{k=0}^{n} p_{n,k}^\tau (x) \int_0^1 \left|\left(f \circ \tau^{-1}\right)(t)\right| p_{n,k} (t)\, dt \\ &\leq \left\|f \circ \tau^{-1}\right\| D_n^\tau (1; x) = \|f\|. \end{aligned}$$

We observe that these operators are positive and linear. Furthermore, in the case of $\tau (x) = x$, the operators (2) reduce to the classical Bernstein-Durrmeyer operators. We can also observe that the operators (2) preserve the linear space of $\tau-$polynomials $\mathbb{P}_m^\tau = \left\{\tau^i : 1 = 1, 2, ...\right\}$ of degree at most m, that is $D_n^\tau (\mathbb{P}_m^\tau) \subset \mathbb{P}_m^\tau$.

The uniform convergence of the operators D_n^τ can be obtained by the well known Korovkin theorem using Lemma 1 and the fact that $\left\{e_0,\ \tau,\ \tau^2\right\}$ is an extended complete Tchebychev system on $[0,1]$ (see [6]). From the result of Shisha and Mond [22] and using Lemma 1 we have

$$\left|D_n^\tau (f; x) - f (x)\right| \leq \left(1 + \frac{\mu_{n,2}^\tau (x)}{\delta^2}\right) \omega (f, \delta)$$

for $\delta > 0$, where $\omega (f, \cdot)$ is the classical modulus of smoothness. For $\tau (x) = x$ the similar result was obtained for the Bernstein-Durrmeyer operator D_n in [14].

We can use this result to compare the rates of convergence of $D_n^\tau f$ and D_n depending on the selection of the function $\tau (x)$. For example if we choose $\tau (x) = \sin^2 \frac{\pi x}{2}$, we can easily see that

$$\tau (x) (1 - \tau (x)) - x (1 - x) \leq 0$$

for $x \in [0,1]$. This inequality tells us that depending on the selection of $\tau (x)$, the rate of convergence of $D_n^\tau f$ to f is at least so fast as the rate of convergence for the classical Bernstein-Durrmeyer operator D_n (see Remark 1).

2.1 A General Construction

We first consider a general construction from which we can obtain the operators D_n^τ as a special case.

Suppose that $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, are positive linear operators and $L_n e_0 = e_0$. Let $\tau \in C^2[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$, $\tau'(x) > 0$, $x \in [0,1]$. We define the operator $K_n : C[0,1] \to C[0,1]$ by

$$K_n g := \left(L_n\left(g \circ \tau^{-1}\right)\right) \circ \tau, \quad n \geq 1,$$

for $g \in C[0,1]$. It is obvious that K_n are linear positive operators and $K_n e_0 = e_0$.

It is easy to verify that

$$K_n^m g := \left(L_n^m\left(g \circ \tau^{-1}\right)\right) \circ \tau, \quad m, n \geq 1, \tag{7}$$

$g \in C[0,1]$. Suppose that for $f \in C[0,1]$ there exist $P_n : C[0,1] \to C[0,1]$ such that

$$\lim_{m\to\infty} L_n^m f = P_n f, \ m \geq 1, \tag{8}$$

uniformly on $[0,1]$. Then (7) and (8) yield

$$\lim_{m\to\infty} K_n^m g = \left(P_n\left(g \circ \tau^{-1}\right)\right) \circ \tau \quad n \geq 1, \tag{9}$$

uniformly on $[0,1]$.

2.2 Transferring the Asymptotic Formula

For the Bernstein-Durrmeyer operators we have

$$\lim_{n\to\infty} n\left(D_n(f,x) - f(x)\right) = (1-2x) f'(x) + x(1-x) f''(x). \tag{10}$$

We use this result to obtain a Voronovskaya type formula for the modified operators D_n^τ and then the obtained result will be used to compare the operators D_n and D_n^τ in the next section.

Theorem 1 *Let $f \in C[0,1]$ with $f''(x)$ finite for $x \in [0,1]$. If there exist $\alpha, \beta \in C[0,1]$ such that*

$$\lim_{n\to\infty} n\left(L_n(f,x) - f(x)\right) = \alpha(x) f''(x) + \beta(x) f'(x), \tag{11}$$

then we have

$$\lim_{n\to\infty} n\left(K_n(g,t) - g(t)\right) = \frac{\alpha(\tau(t))}{\tau'(t)^2} g''(t) + \left(\frac{\beta(\tau(t))}{\tau'(t)} - \frac{\alpha(\tau(t))\,\tau''(t)}{\tau'(t)^3}\right) g'(t) \tag{12}$$

for $g \in C[0,1]$ with $g''(x)$ finite for $x \in [0,1]$.

Proof. Since

$$n\left(K_n(g,t) - g(t)\right) = n\left(\left(L_n\left(g \circ \tau^{-1}\right)\right)(\tau(t)) - \left(g \circ \tau^{-1}\right)(\tau(t))\right)$$

we can write from (11)

$$\lim_{n\to\infty}\left(K_n\left(g,t\right)-g\left(t\right)\right)=\alpha\left(\tau\left(t\right)\right)\left(g\circ\tau^{-1}\right)^{''}\left(\tau\left(t\right)\right)+\beta\left(\tau\left(t\right)\right)\left(g\circ\tau^{-1}\right)^{'}\left(\tau\left(t\right)\right). \tag{13}$$

Since

$$\left(g\circ\tau^{-1}\right)^{'}=\left(g^{'}\circ\tau^{-1}\right)\left(\tau^{-1}\right)^{'}$$

and

$$\left(\tau^{-1}\right)^{'}\left(\tau\left(t\right)\right)=\frac{1}{\tau'\left(t\right)}$$

we have

$$\left(g\circ\tau^{-1}\right)^{'}\left(\tau\left(t\right)\right)=\frac{g^{'}\left(t\right)}{\tau'\left(t\right)}. \tag{14}$$

Also since

$$\left(g\circ\tau^{-1}\right)^{''}=\left(g^{''}\circ\tau^{-1}\right)\left(\left(\tau^{-1}\right)^{'}\right)^{2}+\left(g^{'}\circ\tau^{-1}\right)\left(\tau^{-1}\right)^{''}$$

and

$$\frac{d}{dt}\left(\left(\tau^{-1}\right)^{'}\left(\tau\left(t\right)\right)\right)=\left(\tau^{-1}\right)^{''}\left(\tau\left(t\right)\right)\tau^{'}\left(t\right)=-\frac{\tau^{''}\left(t\right)}{\left(\tau'\left(t\right)\right)^{2}}$$

we have

$$\left(g\circ\tau^{-1}\right)^{''}\left(\tau\left(t\right)\right)=\frac{g^{''}\left(t\right)}{\left(\tau'\left(t\right)\right)^{2}}-g^{'}\left(t\right)\frac{\tau^{''}\left(t\right)}{\left(\tau'\left(t\right)\right)^{3}} \tag{15}$$

Using (13), (14) and (15) we obtain the desired result.

2.3 Quantitative type Asymptotic Formula

The estimation of the Peano remainder was given in [19] using the modulus of continuity of n-th derivative $f^{(n)}$ of the function f and the least concave majorant of the modulus. As an application the following quantitative variant of the classical Voronovskaya theorem was given.

Theorem 2 *([19])Let $L_n : C\left[0,1\right]\to C\left[0,1\right]$, $n\geq 1$, be positive linear operators such that $L_ne_0=e_0$. If $f\in C^2\left[0,1\right]$ and $x\in\left[0,1\right]$, then*

$$\left|L_n\left(f;x\right)-f\left(x\right)-f^{'}\left(x\right)L_n\left(\left(e_1-x\right);x\right)-\frac{1}{2}f''\left(x\right)L_n\left(\left(e_1-x\right)^2;x\right)\right|$$

$$\leq\ \frac{1}{2}L_n\left(\left(e_1-x\right)^2;x\right)\widetilde{\omega}\left(f'',\frac{1}{3}\sqrt{\frac{L_n\left(\left(e_1-x\right)^4;x\right)}{L_n\left(\left(e_1-x\right)^2;x\right)}}\right),$$

where $\widetilde{\omega}\left(f'',\cdot\right)$ denotes the least concave majorant of $\omega\left(f;\cdot\right)$ given by

$$\widetilde{\omega}\left(f'',\varepsilon\right)=\left\{\begin{array}{ll}\sup_{0\leq x\leq\varepsilon\leq y\leq 1}\frac{(\varepsilon-x)\omega(f;y)+(y-\varepsilon)\omega(f;x)}{y-x}, & if\quad 0\leq\varepsilon\leq 1\\ \widetilde{\omega}\left(f,1\right)=\omega\left(f;1\right) & ,\ if\quad \varepsilon>1\end{array}\right.. \tag{16}$$

2.4 Applications

a) Let $L_n = D_n$. It is known that (11) is satisfied choosing $\alpha(x) = x(1-x)$, $\beta(x) = 1 - 2x$; see (10). Since $K_n = D_n^\tau$, we get the following theorem.

Theorem 3 *If* $f \in C^2[0,1]$, *then*

$$\lim_{n\to\infty} n\left[D_n^\tau(f;x) - f(x)\right] = \frac{\alpha(\tau(x))}{(\tau'(x))^2} f''(x) + \left(\frac{\beta(\tau(x))}{\tau'(x)} - \frac{\alpha(\tau(x))\tau''(x)}{(\tau'(x))^3}\right) f'(x)$$

uniformly on $[0,1]$, *with* α *and* β *defined above.*

b) The iterates of Durrmeyer operators are investigated in [1]. We know from [1, Theorem 3.3, (2)] that

$$\lim_{m\to\infty} D_n^m f = \left(\int_0^1 f(x)\,dx\right) e_0,$$

$f \in C[0,1]$. From (9) it follows for $g \in C[0,1]$ that

$$\lim_{m\to\infty} (D_n^\tau)^m g = \left(\int_0^1 g\left(\tau^{-1}(t)\right) dt\right) e_0,$$

uniformly on $[0,1]$.

Theorem 4 *Let* $f \in C^2[0,1]$. *Then the following inequality holds*

$$\begin{aligned} &\left|\left[D_n^\tau(f;x) - f(x)\right] - \frac{f'(x)}{\tau'(x)}\mu_{n,1}^\tau(x)\right. \\ &\left. -\frac{1}{\tau'(x)}\left(\frac{f''(x)\tau'(x) - f'(x)\tau''(x)}{(\tau'(x))^2}\right)\mu_{n,2}^\tau(x)\right| \\ \le\ & \frac{1}{2}\mu_{n,2}^\tau(x)\,\widetilde{\omega}\left(\left(f\circ\tau^{-1}\right)'', \frac{1}{3}n^{-1/2}\right), \end{aligned}$$

where $\widetilde{\omega}(f,\delta)$ *is given in (16).*

Proof. If we apply the Theorem 2 with $L_n f = \left(D_n\left(f\circ\tau^{-1}\right)\right)\circ\tau = D_n^\tau f$, then we have

$$\begin{aligned} &\left|D_n^\tau(f;x) - f(x) - \left(f\circ\tau^{-1}\right)'(\tau(x))\,\mu_{n,1}^\tau(x) - \frac{1}{2}\left(f\circ\tau^{-1}\right)''(\tau(x))\,\mu_{n,2}^\tau(x)\right| \\ \le\ & \frac{1}{2}\mu_{n,2}^\tau(x)\,\widetilde{\omega}\left(\left(f\circ\tau^{-1}\right)'', \frac{1}{3}n^{-1/2}\right). \end{aligned}$$

Corollary 1 *If we chose* $\tau(x) = x$ *in Theorem 3, we have the equality (10).*

3 Local Approximation

To prove the local approximation result let us recall some definitions. For $\eta > 0$ and $\mathcal{W}^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$, the Peetre's K-functional [20] is defined by

$$K(f,\eta) = \inf_{g \in \mathcal{W}^2} \{\|f - g\| + \eta \|g\|_{\mathcal{W}^2}\}, \tag{17}$$

where

$$\|f\|_{\mathcal{W}^2} = \|f\| + \|f'\| + \|f''\|.$$

By [4, Proposition 3.4.1], there exists a positive constant $C_1 > 0$ independent of f and η such that

$$K(f,\eta) \leq C_1 \left(\omega_2 \left(f, \sqrt{\eta}\right) + \min\{1,\eta\} \|f\|\right), \tag{18}$$

for all $x \in [0,1]$ and $f \in C[0,1]$, where the second order modulus of continuity of f is defined as

$$\omega_2(f,\eta) = \sup_{|h| < \eta} \sup_{x, x+2h \in [0,1]} |f(x+2h) - f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is defined as

$$\omega(f,\eta) = \sup_{|h| < \eta} \sup_{x, x+h \in [0,1]} |f(x+h) - f(x)|.$$

Throughout the rest of the paper we assume that $\inf_{x \in [0,1]} \tau'(x) \geq a$, $a \in \mathbb{R}^+$.

Theorem 5 *For the operator $D_n^\tau f$, there exist absolute constants $C, C_f > 0$ (C is independent of f and n, C_f is depend only on f) such that*

$$|D_n^\tau(f;x) - f(x)| \leq CK\left(f, \frac{\delta_{n,\tau}^2(x)}{(n+2)}\right) + \omega\left(f; \frac{|1 - 2\tau(x)|}{(n+2)a}\right).$$

Proof. We first define an auxiliary operator for $f \in C[0,1]$ by

$$\tilde{D}_n^\tau(f;x) = D_n^\tau(f;x) + f(x) - \left(f \circ \tau^{-1}\right)\left(\frac{1 + n\tau(x)}{n+2}\right). \tag{19}$$

It is clear by Lemma 1 that

$$D_n^\tau(1;x) = \tilde{D}_n^\tau(1;x) = 1, \tag{20}$$

and

$$\tilde{D}_n^\tau(\tau;x) = D_n^\tau(\tau;x) + \tau(x) - \frac{1 + n\tau(x)}{n+2} = \tau(x). \tag{21}$$

The classical Taylor expansion of the function $g \in \mathcal{W}^2$ yields for $t \in [0,1]$ that

$$\begin{aligned} g(t) &= \left(g\circ\tau^{-1}\right)(\tau(t)) = \left(g\circ\tau^{-1}\right)(\tau(x)) + D\left(g\circ\tau^{-1}\right)(\tau(x))(\tau(t)-\tau(x)) \\ &\quad + \int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\, D^2\left(g\circ\tau^{-1}\right)(u)\,du. \end{aligned} \tag{22}$$

We can write the last term in the above equality, with change of variable $u = \tau(y)$, as

$$\begin{aligned} &\int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\, D^2\left(g\circ\tau^{-1}\right)(u)\,du \\ &= \int\limits_{x}^{t} (\tau(t)-\tau(y))\, D^2\left(g\circ\tau^{-1}\right)\tau(y)\,\tau'(y)\,dy. \end{aligned} \tag{23}$$

From (15)

$$D^2\left(g\circ\tau^{-1}\right)(\tau(y)) = \frac{1}{\tau'(y)}\left[\frac{g''(y)\,\tau'(y) - g'(y)\,\tau''(y)}{(\tau'(y))^2}\right],$$

we have

$$\begin{aligned} &\int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\, D^2\left(g\circ\tau^{-1}\right)(u)\,du \qquad (24)\\ &= \int\limits_{x}^{t} (\tau(t)-\tau(y))\left[\frac{g''(y)\,\tau'(y) - g'(y)\,\tau''(y)}{(\tau'(y))^2}\right] dy \\ &= \int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\,\frac{g''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^2}du - \int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\,\frac{g'\left(\tau^{-1}(u)\right)\tau''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^3}du. \end{aligned}$$

Applying the operator $\tilde{D}_n^\tau$ to both sides of equality (22) and using (21) and (24), we deduce

$$\begin{aligned} \tilde{D}_n^\tau(g;x) &= g(x) + \tilde{D}_n^\tau\left(\int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\,\frac{g''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^2}du; x\right) \\ &\quad - \tilde{D}_n^\tau\left(\int\limits_{\tau(x)}^{\tau(t)} (\tau(t)-u)\,\frac{g'\left(\tau^{-1}(u)\right)\tau''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^3}du; x\right) \end{aligned}$$

$$\begin{aligned}
&= g(x)+D_n^{\tau}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^2}du;x\right)\\
&\quad-\int_{\tau(x)}^{\frac{1+n\tau(x)}{n+2}}\left(\frac{1+n\tau(x)}{n+2}-u\right)\frac{g''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^2}du\\
&\quad+D_n^{\tau}\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'\left(\tau^{-1}(u)\right)\tau''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^3}du;x\right)\\
&\quad-\int_{\tau(x)}^{\frac{1+n\tau(x)}{n+2}}\left(\frac{1+n\tau(x)}{n+2}-u\right)\frac{g'\left(\tau^{-1}(u)\right)\tau''\left(\tau^{-1}(u)\right)}{\left(\tau'\left(\tau^{-1}(u)\right)\right)^3}du.
\end{aligned}$$

Since τ is strictly increasing on the interval $(0,1)$ and $\inf_{x\in[0,1]}\tau'(x)\geq a$, $a\in\mathbb{R}^+$, we get

$$\begin{aligned}
\left|\tilde{D}_n^{\tau}(g;x)-g(x)\right| &\leq D_n^{\tau}\left(e_{\tau,2}^{x};x\right)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right]\\
&\quad+\left(\frac{1+n\tau(x)}{n+2}-\tau(x)\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right].
\end{aligned}$$

We also can write with (6) that

$$\begin{aligned}
\left|\tilde{D}_n^{\tau}(g;x)-g(x)\right| &\leq \left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right]\frac{2}{(n+2)}\delta_{n,\tau}^2(x)\\
&\quad+\left(\frac{1+n\tau(x)}{n+2}-\tau(x)\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right]\\
&= \left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right]\frac{2}{(n+2)}\delta_{n,\tau}^2(x)\\
&\quad+\left(\frac{1-2\tau(x)}{n+2}\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\,\|\tau''\|}{a^3}\right]\\
&\leq \frac{3}{(n+2)a^2}\delta_{n,\tau}^2(x)\|g''\|+\frac{3}{(n+2)a^3}\delta_{n,\tau}^2(x)\|g'\|\,\|\tau''\|. \qquad (25)
\end{aligned}$$

Furthermore, by Lemma 4, we have

$$\left|\tilde{D}_n^{\tau}(g;x)\right|\leq\left|D_n^{\tau}(f;x)\right|+|f(x)|+\left|\left(f\circ\tau^{-1}\right)\left(\frac{1+n\tau(x)}{n+2}\right)\right|\leq 3\|f\|. \qquad (26)$$

Hence, for $f \in C[0,1]$ and $g \in \mathcal{W}^2$, we obtain

$$\begin{aligned}
& \left|D_n^\tau(f;x) - f(x)\right| \\
= & \left|\tilde{D}_n^\tau(f;x) - f(x) + \left(f \circ \tau^{-1}\right)\left(\frac{1+n\tau(x)}{n+2}\right) - f(x)\right| \\
\leq & \left|\tilde{D}_n^\tau(f-g;x)\right| + \left|\tilde{D}_n^\tau(g;x) - g(x)\right| + |g(x) - f(x)| \\
& + \left|\left(f \circ \tau^{-1}\right)\left(\frac{1+n\tau(x)}{n+2}\right) - \left(f \circ \tau^{-1}\right)(\tau(x))\right| \\
\leq & \; 4\|f-g\| + \frac{3}{(n+2)a^2}\delta_{n,\tau}^2(x)\|g''\| + \frac{3}{(n+2)a^3}\delta_{n,\tau}^2(x)\|g'\|\|\tau''\| \\
& + \omega\left(f \circ \tau^{-1}; \left|\frac{1-2\tau(x)}{n+2}\right|\right),
\end{aligned}$$

and if we choose $C := \max\left\{4, \frac{3}{a^2}, \frac{3\|\tau''\|}{a^3}\right\}$ then we have

$$\begin{aligned}
\left|D_n^\tau(f;x) - f(x)\right| \leq \; & C\left\{\|f-g\| + \frac{\delta_{n,\tau}^2(x)\|g''\|}{n+2} + \frac{\delta_{n,\tau}^2(x)\|g'\|}{n+2} + \frac{\delta_{n,\tau}^2(x)\|g\|}{n+2}\right\} \\
& + \omega\left(f \circ \tau^{-1}; \left|\frac{1-2\tau(x)}{n+2}\right|\right). \qquad (27)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\omega\left(f \circ \tau^{-1}; t\right) &= \sup\left\{\left|f\left(\tau^{-1}(y)\right) - f\left(\tau^{-1}(x)\right)\right| : 0 \leq y - x \leq t\right\} \\
&= \sup\left\{|f(\bar{y}) - f(\bar{x})| : 0 \leq \tau(\bar{y}) - \tau(\bar{x}) \leq t\right\}.
\end{aligned}$$

If $0 \leq \tau(\bar{y}) - \tau(\bar{x}) \leq t$, then $0 \leq (\bar{y} - \bar{x})\tau'(u) \leq t$, for some $u \in (\bar{x}, \bar{y})$, i.e., $0 \leq \bar{y} - \bar{x} \leq \frac{t}{\tau'(u)} \leq \frac{t}{a}$. Thus we have

$$\omega\left(f \circ \tau^{-1}; t\right) \leq \sup\left\{|f(\bar{y}) - f(\bar{x})| : 0 \leq \bar{y} - \bar{x} \leq \frac{t}{a}\right\} = \omega\left(f; \frac{t}{a}\right). \qquad (28)$$

Using (28) in (27) we have

$$\begin{aligned}
\left|D_n^\tau(f;x) - f(x)\right| \leq \; & C\left\{\|f-g\| + \frac{\delta_{n,\tau}^2(x)\|g''\|}{n+2} + \frac{\delta_{n,\tau}^2(x)\|g'\|}{n+2} + \frac{\delta_{n,\tau}^2(x)\|g\|}{n+2}\right\} \\
& + \omega\left(f; \frac{1}{a}\left|\frac{1-2\tau(x)}{n+2}\right|\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in \mathcal{W}^2$ we obtain

$$\left|D_n^\tau(f;x) - f(x)\right| \leq CK\left(f, \frac{\delta_{n,\tau}^2(x)}{n+2}\right) + \omega\left(f; \frac{|1-2\tau(x)|}{(n+2)a}\right). \qquad (29)$$

Corollary 2 *On the other hand, if we use the relation (18) in (29) we get*

$$
\begin{aligned}
\left|D_n^{\tau}(f;x)-f(x)\right| &\leq CC_1\left[\omega_2\left(f,\frac{\delta_{n,\tau}(x)}{(n+2)^{\frac{1}{2}}}\right)+\min\left\{1,\frac{\delta_{n,\tau}^2(x)}{n+2}\right\}\|f\|\right] \\
&\quad +\omega\left(f;\frac{|1-2\tau(x)|}{(n+2)\,a}\right),
\end{aligned}
$$

Since

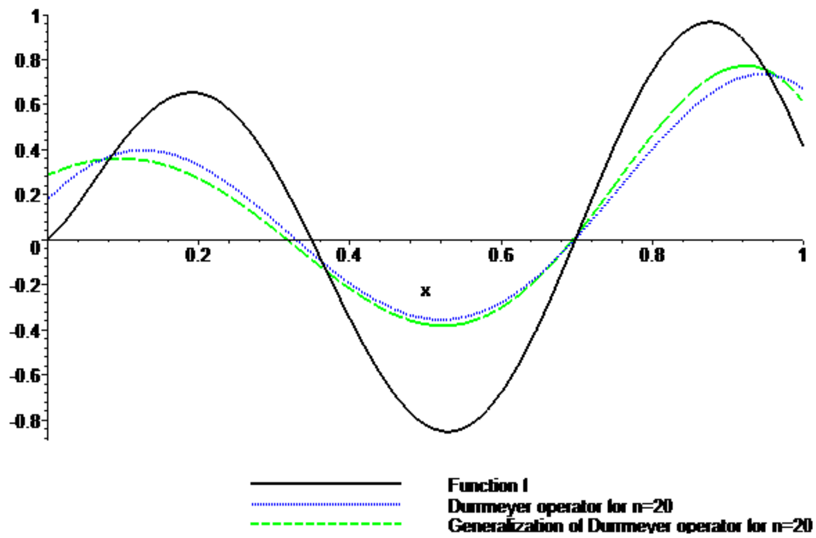
$$
\frac{\delta_{n,\tau}^2(x)}{n+2}=\frac{\tau(x)(1-\tau(x))+\frac{1}{n+2}}{n+2}\leq\frac{1+\frac{1}{n+2}}{n+2}=\frac{n+3}{(n+2)^2}<1
$$

we can write

(30)

$$
\left|D_n^{\tau}(f;x)-f(x)\right|\leq CC_1\left[\omega_2\left(f,\frac{\delta_{n,\tau}(x)}{(n+2)^{\frac{1}{2}}}\right)+\frac{\delta_{n,\tau}^2(x)}{n+2}\|f\|\right]+\omega\left(f;\frac{|1-2\tau(x)|}{(n+2)\,a}\right).
$$

Remark 1 *Let us give some examples for these approximation processes. Let $f(x)=x^{1/4}\sin(9x)$, $\tau(x)=\frac{x^2+x}{2}$. When we choose $n=20$, the approximation to the function f by D_n and D_n^{τ} is shown in the following figure.*



Let us take $f(x)=e^{x^2}$, $\tau(x)=x^{98}$. We compute the error of approximation by using the modulus of continuity for D_n and D_n^{τ} at the point $x_0=0.2$ in the following table.

n	Estimation for the operator D_n	Estimation for the operators D_n^{τ}
10	1.561785518	1.222692211
10^2	0.624679640	0.475437534
10^3	0.216656743	0.164213119
10^4	0.070701794	0.053564561
10^5	0.022586191	0.017110834
10^6	0.007165514	0.005428422
10^7	0.002268260	0.001718379
10^8	0.000717519	0.000543575
10^9	0.000226921	0.000171910
10^{10}	0.000071761	0.000054365

If we compare the results, we can say that D_n^{τ} convergence faster than D_n to the function f at the point $x_0 = 0,2$.

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Some Hermite-Hadamard inequalities for convex functions on the co-ordinates [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

The paper deals with the weighted Hermite-Hadamard-type inequalities. We study the case of co-ordinated convex functions defined on multi-dimensional intervals. We present a natural extension for n-dimensional intervals of an inequality due to Fink. In the same framework, we also treat the special case of convex-concave symmetric functions.

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Key words and phrases: convex function, co-ordinated convex function, Hermite-Hadamard's inequality, Fejér's inequality.

1 Introduction

The first weighted version of the famous Hermite-Hadamard's inequality, due to Fejér [6], is the following:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)p(x)\,dx}{\int_a^b p(x)\,dx} \leq \frac{f\left(a\right)+f\left(b\right)}{2}, \tag{1}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function and $p:[a,b]\to[0,\infty)$ is an integrable and symmetric about $\frac{a+b}{2}$ function, with $\int_a^b p(x)\,dx>0$. The Hermite-Hadamard's inequality is obtained by taking $p=1$.

On the basis of the Choquet's theory, a general Hermite-Hadamard-type inequality can be formulated in the framework of positive Radon measures (see [10]). Fink [7] establishes a relevant weighted version of the Hermite-Hadamard's inequality

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for real Borel measures, subject to some positivity conditions (see also [8]). In particular, the Fink's result provides the following inequalities

$$f(m) \leq \int_a^b f(x)p(x)\,dx \leq \frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b), \tag{2}$$

for a convex function $f : [a,b] \to \mathbb{R}$ and for an integrable function $p : [a,b] \to [0,\infty)$, such that $\int_a^b p(x)\,dx = 1$ (namely, p is a density function of a random variable X taking values in $[a,b]$), where $m = \int_a^b xp(x)\,dx$ is the barycenter of p (or $m = \mathbb{E}[X]$). Note that the above inequalities are sharp. Also remark that (2) strongly extends (1).

Such kinds of inequalities are treated by a very rich literature. In this paper we study the multi-dimensional case. More specifically, we refer to the convex functions on the co-ordinates.

Definition 1 *Let $\Delta := \times_{i=1}^n \left[a_i^{(0)}, a_i^{(1)}\right]$ be a n-dimensional interval of $\mathbb{R}^n$. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates (componentwise convex) on Δ if f is convex in each argument when the other arguments are held fixed, that is the inequality*

$$f(x_1, \cdots, x_{i-1}, (1-\lambda)u + \lambda v, x_{i+1}, \cdots, x_n)$$
$$\leq (1-\lambda)f(x_1, \cdots, x_{i-1}, u, x_{i+1}, \cdots, x_n) + \lambda f(x_1, \cdots, x_{i-1}, v, x_{i+1}, \cdots, x_n)$$

holds for all $i \in \{1, \cdots, n\}$, $x_k \in \left[a_k^{(0)}, a_k^{(1)}\right]$ $(k \neq i)$, $u, v \in \left[a_i^{(0)}, a_i^{(1)}\right]$, and $\lambda \in [0,1]$.

Clearly, any convex function is also convex on the co-ordinates, but the reverse statement is not true. A Hermite-Hadamard-type inequality for convex functions on the co-ordinates in a bi-dimensional interval has been proved by Dragomir [4]. In the same framework, some Fejér-type inequalities for double integrals were recently obtained by Alomari and Darus [1] and Latif [9]. Many results on multi-dimensional Hermite-Hadamard inequalities can also be found in [5]. In the spirit of the Fink's inequalities (2), Cal and Cárcamo [2] provide a very nice and instructive probabilistic approach of weighted Hermite-Hadamard inequalities for convex functions defined on compact convex sets of $\mathbb{R}^n$.

Our purpose is to present some new Hermite-Hadamard inequalities for convex functions on the co-ordinates. So, we obtain a natural extension of (2) for n-dimensional intervals. We also discuss a special case of co-ordinated convex-concave functions.

2 Main results

The Fink's result (2) admits the following extension for convex on the co-ordinates functions in n-dimensional intervals.

Theorem 1 *Assume a n-dimensional interval* $\Delta = \prod_{i=1}^{n} \left[a_i^{(0)}, a_i^{(1)}\right]$ *in* $\mathbb{R}^n$ *and* $f : \Delta \to \mathbb{R}$ *a convex on the co-ordinates function. Let* $p_i : \left[a_i^{(0)}, a_i^{(1)}\right] \to [0,\infty)$ *be the density function of a random variable* X_i*, with the mean* $m_i = \mathbb{E}[X_i] = \int_{a_i^{(0)}}^{a_i^{(1)}} x p_i(x)\, dx$*, for all* $i \in \{1, \cdots, n\}$*. Denote* $\lambda_i^{(k)} = \left(m_i - a_i^{(1-k)}\right) / \left(a_i^{(k)} - a_i^{(1-k)}\right)$*, for* $i = 1, \cdots, n$*,* $k = 0, 1$*. Then*

$$f(m_1, \cdots, m_n) \le \int_{\Delta} f(x_1, \cdots, x_n) \prod_{i=1}^{n} p_i(x_i)\; dx_1 \cdots dx_n$$

$$\le \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} \left(\prod_{i=1}^{n} \lambda_i^{(k_i)}\right) f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}\right).$$

Proof. We will prove the result by induction. For $n = 1$, the conclusion directly follows from the relation (2). Suppose that the property is true for a positive integer n. Let $\Delta_1 = \Delta \times \left[a_{n+1}^{(0)}, a_{n+1}^{(1)}\right]$ a $n+1$-dimensional interval in $\mathbb{R}^{n+1}$, where $\Delta = \prod_{i=1}^{n} \left[a_i^{(0)}, a_i^{(1)}\right] \subset \mathbb{R}^n$. Let us consider the density functions $p_i : \left[a_i^{(0)}, a_i^{(1)}\right] \to [0,\infty)$ with associated means m_i, for all $i \in \{1, \cdots, n, n+1\}$, and define the coefficients $\lambda_i^{(k)}$ as in the enunciation of the theorem. Assume a convex on the co-ordinates function $f : \Delta_1 \to \mathbb{R}$. From above assumptions, we have

$$\begin{aligned}
(3) \qquad & p_{n+1}(x_{n+1}) f(m_1, \cdots, m_n, x_{n+1}) \\
& \le p_{n+1}(x_{n+1}) \int_{\Delta} f(x_1, \cdots, x_n, x_{n+1}) \prod_{i=1}^{n} p_i(x_i)\; dx_1 \cdots dx_n \\
& \le p_{n+1}(x_{n+1}) \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} \left(\prod_{i=1}^{n} \lambda_i^{(k_i)}\right) f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, x_{n+1}\right),
\end{aligned}$$

for each fixed number $x_{n+1} \in \left[a_{n+1}^{(0)}, a_{n+1}^{(1)}\right]$.

Since $f(m_1, \cdots, m_n, \cdot) : \left[a_{n+1}^{(0)}, a_{n+1}^{(1)}\right] \to \mathbb{R}$ is convex, we obtain

$$f(m_1, \cdots, m_n, m_{n+1}) \le \int_{a_{n+1}^{(0)}}^{a_{n+1}^{(1)}} p_{n+1}(x_{n+1}) f(m_1, \cdots, m_n, x_{n+1})\; dx_{n+1},$$

as a consequence of (2). For similar reasons, we find

$$\int_{a_{n+1}^{(0)}}^{a_{n+1}^{(1)}} p_{n+1}(x_{n+1}) \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, x_{n+1}\right) \prod_{i=1}^{n} \lambda_i^{(k_i)}\, dx_{n+1}$$

$$= \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} \prod_{i=1}^{n} \lambda_i^{(k_i)} \int_{a_{n+1}^{(0)}}^{a_{n+1}^{(1)}} f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, x_{n+1}\right) p_{n+1}(x_{n+1})\; dx_{n+1}$$

$$\leq \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} \prod_{i=1}^{n} \lambda_i^{(k_i)} \left[\frac{a_{n+1}^{(1)} - m_{n+1}}{a_{n+1}^{(1)} - a_{n+1}^{(0)}} f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, a_{n+1}^{(0)}\right) \right.$$

$$\left. + \frac{m_{n+1} - a_{n+1}^{(0)}}{a_{n+1}^{(1)} - a_{n+1}^{(0)}} f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, a_{n+1}^{(1)}\right) \right]$$

$$= \sum_{(k_1,\cdots,k_n,k_{n+1})\in\{0,1\}^n} \left(\prod_{i=1}^{n+1} \lambda_i^{(k_i)} \right) f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}, a_{n+1}^{(k_{n+1})}\right).$$

Now, we integrate (3) on $\left[a_{n+1}^{(0)}, a_{n+1}^{(1)}\right]$ and apply the Fubini's theorem for the middle term. Then, from above last relations, we get the conclusion for the $n+1$-dimension. This ends the proof by induction of the theorem.

Remark that the result of our theorem can be deduced from Corollary 2 of [2] (where the Hermite-Hadamard majorant is not presented in an explicit form), but only under the more restrictive assumption of the usual convexity.

P. Czinder and Z. Páles [3] proved a Hermite-Hadamard inequality for a particular class of symmetric convex-concave functions. Florea and Niculescu [8]) extended this result in the Fink's meaning. A version of this extension is presented below.

For an interval $[a, b]$ and a fixed point $c \in [(a + b)/2, b]$, let us consider a density function $p : [a, b] \to [0, \infty)$, with the mean m and the symmetry property $p(x) = p(2c - x)$, $\forall\ x \in [2c - b, b]$. Assume $f : [a, b] \to \mathbb{R}$, such that $f(x) + f(2c - x) = 2f(c), \forall\ x \in [2c - b, b]$. In what follows, we will say that the function f is $c-$*symmetric* in $x \in [2c-b, b]$. If f is convex over the interval $[a, c]$ and concave over the interval $[c, b]$, then the inequalities (2) hold.

Using similar arguments as in the proof of Theorem 1, we obtain the following n-dimensional extension of Fink's inequality for convex-concave symmetric functions (the proof is omitted).

Theorem 2 *We keep the notations of Theorem 1. Let c_i be a fixed point in $\left[\left(a_i^{(0)} + a_i^{(1)}\right)/2, a_i^{(1)}\right]$, for $i = 1, \cdots, n$. Suppose that the density function p_i on the interval $\left[a_i^{(0)}, a_i^{(1)}\right]$, with the mean m_i, has the symmetry property $p_i(x_i) = p_i(2c_i - x_i),\ \forall\ x_i \in \left[2c_i - a_i^{(1)}, a_i^{(1)},\right]$, for all i. Also, assume that, for all i, the function $f(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)$ is c_i-symmetric in $t \in \left[2c_i - a_i^{(1)}, a_i^{(1)}\right]$, for fixed $x_j,\ j \in \{1, \cdots, n\} \setminus \{i\}$. If $f(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n)$ is convex in t over the interval $\left[a_i^{(0)}, c_i\right]$ and concave in t over the interval $\left[c_i, a_i^{(1)}\right]$ (for fixed $x_j,\ j \neq i$), then*

$$f(m_1, \cdots, m_n) \leq \int_\Delta f(x_1, \cdots, x_n) \prod_{i=1}^{n} p_i(x_i)\ dx_1 \cdots dx_n$$

$$\leq \sum_{(k_1,\cdots,k_n)\in\{0,1\}^n} \left(\prod_{i=1}^{n} \lambda_i^{(k_i)} \right) f\left(a_1^{(k_1)}, \cdots, a_n^{(k_n)}\right).$$

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A Bernstein-Durrmeyer operator which preserves e_0 and e_2 [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this paper we construct a Bernstein-Durrmeyer operator which preserves e_0 and e_2. We give a recurrence relation for the moments of this operator and we study the convergence. A Voronovskaya formula is presented. Also, we give comparisons with the classical Durrmeyer operator related to the approximation order and the approximation error.

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Key words and phrases: polynomial operator, moments, Voronovskaya formula, approximation order.

1 Introduction

Let the polynomial operators of Bernstein type $P_n : C[0,1] \to \Pi_n$, $n \geq 1$ given by

$$P_n f(x) = \sum_{k=0}^{n} p_{n,k}(x)\lambda_{n,k}(f),\ x \in [0,1], \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, ..., n.$$

Examples of such operators which preserve some test functions $e_i(x) = x^i$, $i \in I \subset \mathbb{N}$ (i.e. $P_n(e_i) = e_i$) are:

• if

$$\lambda_{n,k}(f) = f\left(\frac{k}{n}\right),\ k = 0, ..., n,$$

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then we get the classical Bernstein operator B_n; it preserves the functions e_0 and e_1

• if

$$\lambda_{n,k}(f) = f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}\right),\ k = 0, ..., n,$$

then we get an operator which preserves the functions e_0 and e_2 (see I. Gavrea [6])

• if

$$\lambda_{n,k}(f) = f\left(\left(\frac{k(k-1)\cdot ... \cdot (k-j+1)}{n(n-1)\cdot ... \cdot (n-j+1)}\right)^{1/j}\right),\ k = 0, ..., n,$$

then we get the Bernstein type operator $B_{n,0,j}$ considered in [1]; it preserves the functions e_0 and e_j

• if

$$\lambda_{n,k}(f) = (n+1)\int_0^1 p_{n,k}(t)f(t)dt,\ k = 0, ..., n,$$

then we get the Durrmeyer operator M_n introduced by J. L. Durrmeyer in [4] and studied by M. Deriennic (see [3]); it preserves the function e_0

• if

$$\begin{aligned} \lambda_{n,0}(f) &= f(0),\ \lambda_{n,n}(f) = f(1), \\ \lambda_{n,k}(f) &= (n-1)\int_0^1 p_{n-2,k-1}(t)f(t)dt,\ k = 1, ..., n-1, \end{aligned}$$

then we get the genuiune Bernstein-Durrmeyer operator U_n introduced by W. Chen in [2] and T.N.T. Goodman, A. Sharma in [7]; it preserves the functions e_0 and e_1.

In this article we present a Bernstein-Durrmeyer operator which preserves the test functions e_0 and e_2. In Section 2 we construct the operator and we give recurrence formulas for the images of the monomials and for the moments of this operator. In Section 3 we study the convergence of the operator. We present a Voronovskaya formula. Comparisons with the classical Durrmeyer operator from the point of view of the approximation order and the approximation error are given in Section 4.

2 The construction of the operator and the moments

We look after a Bernstein type approximation operator P_n of the form (1) with

$$\lambda_{n,k}(f) = c\int_0^1 x^a(1-x)^b f(x)dx,$$

where $a, b, c \in \mathbb{R}$, which preserves the test functions e_0 and e_2. We need that the following conditions to be satisfied (see Z. Finta [5]):

$$\begin{aligned} \lambda_{n,k}(e_0) &= 1,\ \lambda_{n,k}(e_1) \in \left[\frac{k(k-1)}{n(n-1)}, \sqrt{\frac{k(k-1)}{n(n-1)}}\right], \\ \lambda_{n,k}(e_2) &= \frac{k(k-1)}{n(n-1)},\ k = 0, ..., n. \end{aligned}$$

We obtain

$$a = k-2,\ b = n-k-1,\ c = (n-2)\binom{n-3}{k-2},\ k = 2,...,n-1,\ n > 3.$$

If we take

$$\lambda_{n,0}(f) = f(0),\ \lambda_{n,1}(f) = f(0),\ \lambda_{n,n}(f) = f(1),$$

then we get the Bernstein-Durrmeyer operator

$$D_n f(x) = p_{n,0}(x)f(0) + p_{n,1}(x)f(0)$$
$$+(n-2)\sum_{k=2}^{n-1} p_{n,k}(x)\int_0^1 p_{n-3,k-2}(t)f(t)dt + p_{n,n}(x)f(1),\ x \in [0,1].$$

We have

$$D_n e_0(x) = 1, \tag{2}$$
$$D_n e_1(x) = \frac{nx - 1 + (1-x)^n}{n-1}, \tag{3}$$
$$D_n e_2(x) = x^2. \tag{4}$$

Theorem 1 *The following recurrence relation for the images of the monomials under the operator D_n holds*

$$D_n e_j(x) = \frac{nx+j-2}{n+j-2} D_n e_{j-1}(x) + \frac{x(1-x)}{n+j-2}(D_n e_{j-1})'(x),$$

for $j \geq 2$.

Proof. We have

$$D_n e_j(x) = \sum_{k=2}^{n} p_{n,k}(x)\frac{(k-1)...(k+j-2)}{(n-1)...(n+j-2)}.$$

Using the equalities

$$x(1-x)p'_{n,k}(x) = (k-nx)p_{n,k}(x),\ k = 0,...,n, \tag{5}$$

we obtain

$$\frac{k+j-2}{n+j-2}p_{n,k}(x) = \frac{nx+j-2}{n+j-2}p_{n,k}(x) + \frac{x(1-x)}{n+j-2}p'_{n,k}(x).$$

It follows that

$$\begin{aligned} D_n e_j(x) &= \frac{nx+j-2}{n+j-2}\sum_{k=2}^{n} p_{n,k}(x)\frac{(k-1)...(k+j-3)}{(n-1)...(n+j-3)} \\ &+ \frac{x(1-x)}{n+j-2}\sum_{k=2}^{n} p'_{n,k}(x)\frac{(k-1)...(k+j-3)}{(n-1)...(n+j-3)}, \end{aligned}$$

and the recurrence relation is provided.

The moments of the operator D_n are defined by

$$T_{n,m}(x) = D_n((e_1 - xe_0)^m)(x).$$

Theorem 2 *We have the following recurrence relation for the moments of the operator D_n:*

$$(6) \qquad (n+m-1)T_{n,m+1}(x) = x(1-x)\left(T_{n,m}(x)\right)'$$

$$+2mx(1-x)T_{n,m-1}(x) + (m(1-2x)-1+x)T_{n,m}(x) + (1-x)^n(-x)^m, \ m \geq 1.$$

Proof. Let

$$T^1_{n,m}(x) = p_{n,0}(x)(-x)^m + p_{n,1}(x)(-x)^m + p_{n,n}(x)(1-x)^m$$

and

$$T^2_{n,m}(x) = (n-2)\sum_{k=2}^{n-1} p_{n,k}(x)\int_0^1 p_{n-3,k-2}(t)(t-x)^m dt.$$

It follows that

$$T_{n,m}(x) = T^1_{n,m}(x) + T^2_{n,m}(x).$$

We observe that $T^1_{n,m}(x)$ satisfies the recurrence relation (6).

Next, we will show that

$$(7) \qquad (n+m-1)T^2_{n,m+1}(x)$$

$$= x(1-x)\left(T^2_{n,m}(x)\right)' + 2mx(1-x)T^2_{n,m-1}(x) + (m(1-2x)-1+x)T^2_{n,m}(x).$$

We have

$$(8) \qquad (T^2_{m,n}(x))' = (n-2)\sum_{k=2}^{n-1} p'_{n,k}(x)\int_0^1 p_{n-3,k-2}(t)(t-x)^m dt - mT^2_{n,m-1}(x).$$

Using the formula (5) we get

$$(9) \qquad x(1-x)\sum_{k=2}^{n-1} p'_{n,k}(x)\int_0^1 p_{n-3,k-2}(t)(t-x)^m dt$$

$$= \sum_{k=2}^{n-1}(k-nx)p_{n,k}(x)\int_0^1 p_{n-3,k-2}(t)(t-x)^m dt.$$

From the equalities

$$\begin{aligned} k - nx &= k - nt + n(t-x) \\ &= (k-2) - (n-3)t - 3(t-x) - 3x + 2 + n(t-x) \end{aligned}$$

the last sum from (9) becomes

$$\sum_{k=2}^{n-1} p_{n,k}(x) \int_0^1 t(1-t)p'_{n-3,k-2}(t)(t-x)^m dt \tag{10}$$

$$+((2-3x)T^2_{n,m}(x) + (n-3)T^2_{n,m+1}(x))/(n-2).$$

From (8), (9) and (10) we get

$$x(1-x)(T^2_{n,m}(x))' + mx(1-x)T^2_{n,m-1}(x) - (2-3x)T^2_{n,m}(x) - (n-3)T^2_{n,m+1}(x)$$
$$= (n-2)\sum_{k=2}^{n-1} p_{n,k}(x) \int_0^1 t(1-t)p'_{n-3,k-2}(t)(t-x)^m dt.$$

Using the equality

$$t(1-t) = -(x-t)^2 - (1-2x)(x-t) + x(1-x)$$

we obtain

$$(n-2)\sum_{k=2}^{n-1} p_{n,k}(x) \int_0^1 p'_{n-3,k-2}(t) \times (-(t-x)^{m+2} + (1-2x)(t-x)^{m+1} + x(1-x)(t-x)^m)dt$$
$$= -(n-2)\sum_{k=2}^{n-1} p_{n,k}(x) \int_0^1 p_{n-3,k-2}(t)$$
$$\times (-(m+2)(t-x)^{m+1} + (m+1)(1-2x)(t-x)^m + mx(1-x)(t-x)^{m-1})dt$$
$$= (m+2)T^2_{n,m+1}(x) - (m+1)(1-2x)T^2_{n,m}(x) - mx(1-x)T^2_{n,m-1}(x).$$

It follows that (7) holds and the proof is completed.

We list the moments up to order four

$$\begin{aligned}
T_{n,0}(x) &= 1,\\
T_{n,1}(x) &= \frac{x-1+(1-x)^n}{n-1},\\
T_{n,2}(x) &= -\frac{2x(x-1+(1-x)^n)}{n-1},\\
T_{n,3}(x) &= \frac{3x^2(2(x-1)+(n+1)(1-x)^n)}{(n-1)(n+1)},\\
T_{n,4}(x) &= -\frac{4x^2(x(1-x)^n(n+1)(n+2) - 3n(1-x)^2 + 9x^2 - 12x + 3)}{(n-1)(n+1)(n+2)}.
\end{aligned}$$

Using the recurrence formula we can prove

$$T_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right),\ x \in (0,1]. \tag{11}$$

3 The convergence properties

Theorem 3 *For all $f \in C[0,1]$ we have*

$$\lim_{n\to\infty} D_n f = f, \ \textit{uniformly on } [0,1].$$

Proof. The conclusion follows from (2)-(4) by using the well-known Korovkin theorem.

Theorem 4 *If $f \in C[0,1]$, then*

$$|D_n f(x) - f(x)| \le \left(1 + \frac{2x(1-x-(1-x)^n)}{(n-1)\delta^2}\right)\omega(f,\delta), \tag{12}$$

where $x \in [0,1]$, $\delta > 0$ and $\omega(f,\cdot)$ is the modulus of continuity

$$\omega(f,\delta) = \sup\{|f(x+h) - f(h)| : \ x, x+h \in [0,1], \ 0 \le h \le \delta\}.$$

Proof. From the result of Shisha and Mond [9] we have

$$|D_n f(x) - f(x)| \le |f(x)||D_n e_0(x) - e_0(x)| +$$
$$\left(D_n e_0(x) + \frac{1}{\delta^2} D_n(e_2 - 2xe_1 + x^2 e_0)(x)\right)\omega(f,\delta), \ x \in [0,1].$$

Using (2)-(4) we get the conclusion.

The following theorem give a Voronovskaya formula for the operator D_n.

Theorem 5 *If $f \in C[0,1]$ has a second derivative at a point $x \in (0,1)$, then we have*

$$\lim_{n\to\infty} n\left(D_n(f)(x) - f(x)\right) = (x-1)f'(x) + x(1-x)f''(x). \tag{13}$$

Proof. From Taylor formula we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \mu(t-x)$$

where μ is a integrable function, bounded on $[-x, 1-x]$ and with the property $\lim_{u\to 0}\mu(u) = 0$. From the linearity of the operator D_n we obtain

$$D_n f(x) = f(x) + D_n((\cdot - x))(x) f'(x) + \frac{D_n((\cdot - x)^2)}{2} f''(x) + D_n((\cdot - x)^2 \mu(\cdot - x))$$

and

$$n(D_n f(x) - f(x))$$
$$= nD_n((\cdot - x))(x) f'(x) + \frac{nD_n((\cdot - x)^2)}{2} f''(x) + nD_n((\cdot - x)^2 \mu(\cdot - x)).$$

Using the formulas for the moments of order one and two of the operator D_n from the end of the Section 2, we get

$$\lim_{n\to\infty} nD_n((\cdot - x))(x) = x - 1, \tag{14}$$

$$\lim_{n\to\infty} nD_n((\cdot - x)^2)(x) = 2x(1-x). \tag{15}$$

It remains to prove that

$$\lim_{n\to\infty} R_n(x) = 0, \tag{16}$$

where

$$R_n(x) = nD_n((\cdot - x)^2\mu(\cdot - x)).$$

Let $M = \sup_{u\in[x,1-x]} |\mu(u)|$ and $\epsilon, \delta > 0$ such that $|\mu(u)| < \epsilon$ for every u with the property $|u| < \delta$. We have

$$|\mu(t-x)| < \epsilon + \frac{M}{\delta^2}(t-x)^2$$

for every $t \in [0,1]$. It follows

$$\begin{aligned} |R_n(x)| &\leq \epsilon n D_n((\cdot - x)^2)(x) + \frac{M}{\delta^2} nD_n((\cdot - x)^4)(x) \\ &\leq \frac{\epsilon}{2(n-1)} + \frac{MC_4}{n\delta^2}, \end{aligned}$$

where C_4 is a constant which follows from (11). For n sufficiently large we get

$$|R_n(x)| \leq \epsilon.$$

It follows that (16) is true and therefore the limit in (13) exists.

4 Comparisons with the classical Durrmeyer operator

In this section we present comparisons of the operator D_n with the classical Durrmeyer operator related to the approximation order and approximation error.

First, we compare the approximation orders. For the classical Durrmeyer operator we have (see [3])

$$|M_n f(x) - f(x)| \leq \left(1 + \frac{2 + 2(n-3)x(1-x)}{(n+2)(n+3)\delta^2}\right)\omega(f,\delta), \tag{17}$$

where $x \in [0,1]$ and $\delta > 0$, while for the operator D_n we have estimation given by Theorem 4.

In order to get a better approximation order for the operator D_n we need

$$\frac{2x(1-x)}{n-1} \le \frac{2+2(n-3)x(1-x)}{(n+2)(n+3)}. \tag{18}$$

It follows

$$x \in [0, x_1(n)] \cup [x_2(n), 1]$$

with

$$x_1(n) = \frac{1}{2} - \frac{\sqrt{3(7+26n+15n^2)}}{6(1+3n)}, x_2(n) = \frac{1}{2} + \frac{\sqrt{3(7+26n+15n^2)}}{6(1+3n)}.$$

We have

$$\lim_{n\to\infty} x_1(n) = \frac{1}{2} - \frac{\sqrt{5}}{6} = 0.1273...$$

$$\lim_{n\to\infty} x_2(n) = \frac{1}{2} + \frac{\sqrt{5}}{6} = 0.8726...$$

Next, we compare the approximation error.

Theorem 6 *If f is decreasing and convex on $[0,1]$, then there exists $n_1 \in \mathbb{N}$ such that for $n \ge n_1$ we have*

$$f(x) \le D_n f(x) \le M_n f(x), \ x \in [2/3, 1]. \tag{19}$$

Proof. As the operator D_n preserves the test functions e_0 and e_2, from [10, Th. 2] we have

$$f(x) \le D_n f(x), \ x \in [0,1] \tag{20}$$

for every f which is convex with respect to the function e_2. For more about generalized convexity see [8].

We define the function

$$g : [0,1] \to \mathbb{R}, \ g(x) = f(\sqrt{x}). \tag{21}$$

The function f is convex with respect to the function e_2 if and only if the function g is convex in the classical sense. We have

$$g''(x) = \frac{1}{4x\sqrt{x}} \left(\sqrt{x} f''(\sqrt{x}) - f'(\sqrt{x}) \right), \ x \in (0,1].$$

It follows that if f is decreasing and convex then g is convex and the inequality (20) holds (we have equality in 0 due the interpolation property of the operator D_n at the endpoints).

For the second inequality we compare the Voronovskaya formula for the classical Durrmeyer operator (see [3])

$$\lim_{n\to\infty} n \left(M_n f(x) - f(x) \right) = (1-2x) f'(x) + x(1-x) f''(x), \ x \in [0,1],$$

with the formula (13).

Theorem 7 *Let $f \in C^3[0,1]$ be a function satisfying one of the conditions*
a)f is increasing on $[0,1]$ and $f'''(x) \geq 0$, $x \in [0,1]$ and $f'(0) = 0$,
b)f is increasing on $[0,1]$ and $f'''(x) \leq 0$, $x \in [0,1]$ and $f''(1) - f'(1) \geq 0$,
then there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$ we have

$$f(x) \leq D_n f(x) \leq M_n f(x), \ x \in [0, 2/3]. \tag{22}$$

Proof. We follow the same steps as in the proof of the previous theorem.

For the first inequality we study the convexity of the function g given by (21). We define the function $h : [0,1] \to \mathbb{R}, h(x) = xf''(x) - f'(x)$. We have $h'(x) = xf'''(x)$, $x \in [0,1]$.

It follows that if one of the conditions a) and b) are satisfied then g is convex. Thus, the first inequality holds.

For the second inequality we compare the Voronovskaya formulas for these two operators.

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A de Casteljau type algorithm in matrix form [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

We describe a de Casteljau type algorithm in matrix form for some linear operators that appear in Approximation Theory. Some monotonicity preserving properties of the operators are proved by using this algorithm.

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Key words and phrases: linear operators, de Casteljau algorithm, monotonicity preserving properties.

1 Introduction

For $x \in [0,1]$ and $k \in \mathbb{N}$ consider the matrix

$$A_k(x) := \begin{pmatrix} 1-x & x & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-x & x & \dots & 0 & 0 & 0 \\ 0 & 0 & 1-x & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1-x & x & 0 \\ 0 & 0 & 0 & \dots & 0 & 1-x & x \end{pmatrix} \in \mathcal{M}_{k,k+1}.$$

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Let $f \in C[0,1]$, $n \in \mathbb{N}$ and $F_n := \Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f\Big(\frac{n-1}{n}\Big),\ f(1)\Big)^t$. Denote $B_n f(x) := A_1(x)A_2(x)\dots A_n(x)F_n$. Then

$$\begin{aligned}
&A_1(x) = \big(1-x,\ x\big)\\
&A_1(x)A_2(x) = \big((1-x)^2,\ 2x(1-x),\ x^2\big)\\
&\dots\dots\dots\\
&A_1(x)A_2(x)\dots A_n(x) = \Bigg((1-x)^n,\ \binom{n}{1}x(1-x)^{n-1},\dots\\
&\qquad\qquad\qquad\qquad\dots,\ \binom{n}{k}x^k(1-x)^{n-k},\dots,x^n\Bigg).
\end{aligned}$$

It follows that $B_n f(x)$, defined above, is the classical Bernstein polynomial associated with the function f.

On the other hand,

$$A_n(x)F_n,\ A_{n-1}(x)A_n(x)F_n,\ \dots,\ A_1(x)A_2(x)\dots A_n(x)F_n$$

represent the successive steps in the classical de Casteljau algorithm for computing the value $B_n f(x)$.

These remarks can be found in [2], where they are applied also to the Bernstein operators of second kind.

In this paper we generalize the above approach and use the de Casteljau algorithm in matrix form to get monotonicity preserving properties for other operators appearing in Approximation Theory.

2 Monotonicity preserving properties

For each $n \in \mathbb{N}$, we denote by V_n the set of vectors in $\mathbb{R}^n$ having the components in increasing order:

$$V_n = \{v = (v_1, v_2, \dots, v_n)^t \mid v_j \in \mathbb{R},\ v_1 \le v_2 \le \dots \le v_n\}$$

and by V_n^+ the set of vectors in V_n having all the components non-negative

$$V_n^+ = \{v = (v_1, v_2, \dots, v_n)^t \mid v_j \in \mathbb{R},\ 0 \le v_1 \le v_2 \le \dots \le v_n\}.$$

If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ are two vectors in $\mathbb{R}^n$ by $u \le v$ we mean that $u_j \le v_j$ for all $j \in \{1, \dots n\}$.

For $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $x \in [0,1]$ let $A_k(x)$ be a matrix with k rows and $k+1$ columns, with real elements $a_{ij}^k(x)$. If $a_{ij}^k : [0,1] \to \mathbb{R}$ are continuous functions, we can define an operator $L_n : C[0,1] \to C[0,1]$ by

$$L_n(f)(x) = A_1(x)A_2(x)\dots A_n(x)\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f\Big(\frac{n-1}{n}\Big),\ f(1)\Big)^t,$$

for every $f \in C[0,1]$, $x \in [0,1]$.

Theorem 1 *Suppose that for every $k \in \{1, \dots, n\}$ the following conditions are satisfied:*

(H1) If $v \in V_{k+1}$, then $A_k(x)v \in V_k$, for every $x \in [0,1]$.

(H2) If $x, y \in [0,1]$ such that $x < y$ and $u, v \in V_{k+1}$ with $u \leq v$, then $A_k(x)u \leq A_k(y)v$.

Then, if $f \in C[0,1]$ is an increasing function, $L_n(f)$ is increasing too.

Proof. Let $f : [0,1] \to \mathbb{R}$ be an increasing function. Let $x, y \in [0,1]$ such that $x < y$. We want to prove that $L_n(f)(x) \leq L_n(f)(y)$, i.e.,

$$
\begin{aligned}
(1) \qquad & A_1(x)A_2(x)\dots A_n(x)\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f(1)\Big)^t \\
& \leq A_1(y)A_2(y)\dots A_n(y)\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f(1)\Big)^t.
\end{aligned}
$$

Since f is increasing we have that $\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f(1)\Big)^t \in V_{n+1}$ and by (H1) $u_n(x) := A_n(x)\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f(1)\Big)^t \in V_n$ and $u_n(y) := A_n(y)\Big(f(0),\ f\Big(\frac{1}{n}\Big),\ \dots,\ f(1)\Big)^t \in V_n$.

Also by (H2) we get $u_n(x) \leq u_n(y)$.

Continuing in the same way it follows that $u_{n-1}(x) := A_{n-1}(x)u_n(x) \in V_{n-1}$, $u_{n-1}(y) := A_{n-1}(y)u_n(x) \in V_{n-1}$ and $u_{n-1}(x) \leq u_{n-1}(y)$.

Step by step we get to prove (1).

A similar result can be proved for increasing functions that take only nonnegative values:

Theorem 2 *Suppose that for every $k \in \{1, \dots, n\}$ and every $x \in [0,1]$ the matrix $A_k(x)$ has only non-negative elements and the following conditions are satisfied:*

(H1') If $v \in V_{k+1}^+$, then $A_k(x)v \in V_k^+$, for every $x \in [0,1]$

(H2') If $x, y \in [0,1]$ such that $x < y$ and $u, v \in V_{k+1}^+$ with $u \leq v$, then $A_k(x)u \leq A_k(y)v$.

Let $f \in C[0,1]$ be an increasing function such that $f(x) \geq 0$ for every $x \in [0,1]$. Then $L_n(f)$ is increasing and $L_n(f)(x) \geq 0$ for every $x \in [0,1]$.

Next we give some characterizations for matrices that satisfy hypotheses (H1) and (H1').

Theorem 3 *Let $A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq k+1} \in \mathcal{M}_{k,k+1}$ be a matrix such that $a_{ij} \geq 0$ for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, k+1\}$. The following assertions are equivalent:*

(i) $Av \in V_k^+$ for every $v \in V_{k+1}^+$.

(ii) $\Big(\sum_{j=l}^{k+1} a_{1j}, \sum_{j=l}^{k+1} a_{2j}, \dots, \sum_{j=l}^{k+1} a_{kj}\Big)^t \in V_k^+$, for every $l \in \{1, \dots, k+1\}$.

Proof. (i) $\Rightarrow$ (ii) Let $l \in \{1, \dots, k+1\}$. The vector $(0, \dots, 0, 1, \dots, 1)^t$ (first $l-1$ positions equal to zero) belongs to V_{k+1}^+, so according to (i),

$$A(0, \dots, 0, 1, \dots, 1)^t = \Big(\sum_{j=l}^{k+1} a_{1j}, \sum_{j=l}^{k+1} a_{2j}, \dots, \sum_{j=l}^{k+1} a_{kj} \Big)^t \in V_k^+.$$

(ii) $\Rightarrow$ (i) Let $v = (x_1, \dots, x_{k+1})^t$ be an arbitrary vector from V_{k+1}^+, i.e., $0 \le x_1 \le x_2 \le \dots \le x_{k+1}$. Then we can write v in the form

$$v = x_1(1, 1, \dots, 1)^t + (x_2 - x_1)(0, 1, \dots, 1)^t + \dots + (x_{k+1} - x_k)(0, 0, \dots, 1)^t$$

and then

$$\begin{aligned} Av = x_1 \left(\sum_{j=1}^{k+1} a_{1j}, \dots, \sum_{j=1}^{k+1} a_{kj} \right)^t + (x_2 - x_1) \left(\sum_{j=2}^{k+1} a_{1j}, \dots, \sum_{j=2}^{k+1} a_{kj} \right)^t \\ + \dots + (x_{k+1} - x_k) \left(\sum_{j=k+1}^{k+1} a_{1j}, \dots, \sum_{j=k+1}^{k+1} a_{kj} \right)^t \end{aligned}$$

From (ii) and the fact that $x_{l+1} - x_l \ge 0$ for all $l \in \{1, \dots, k\}$ it follows that $Av \in V_k^+$.

Theorem 4 *Let $A = (a_{ij})_{1 \le i \le k, 1 \le j \le k+1} \in \mathcal{M}_{k,k+1}$ be a matrix such that $a_{ij} \ge 0$ for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, k+1\}$. The following assertions are equivalent:*

(i) $Av \in V_k$ for every $v \in V_{k+1}$.

(ii) $\Big(\sum_{j=l}^{k+1} a_{1j}, \sum_{j=l}^{k+1} a_{2j}, \dots, \sum_{j=l}^{k+1} a_{kj} \Big)^t \in V_k^+$, for every $l \in \{1, \dots, k+1\}$ and $\sum_{j=1}^{k+1} a_{1j} = \sum_{j=1}^{k+1} a_{2j} = \dots = \sum_{j=1}^{k+1} a_{kj} = a$.

Proof. (i) $\Rightarrow$ (ii) The first part follows as in the previous result. To prove the second part, we consider the vector $v = (-1, -1, \dots, -1)^t \in V_{k+1}$. From (i) we have $Av \in V_k$, that is $-\sum_{j=1}^{k+1} a_{1j} \le -\sum_{j=1}^{k+1} a_{2j} \le \dots \le -\sum_{j=1}^{k+1} a_{kj}$. Since we have also $\sum_{j=1}^{k+1} a_{1j} \le \sum_{j=1}^{k+1} a_{2j} \le \dots \le \sum_{j=1}^{k+1} a_{kj}$, the equality of the sums follows.

(ii) $\Rightarrow$ (i) Let $v \in V_{k+1}$. Then there exists $m \ge 0$ such that $v + m(1, 1, \dots, 1)^t = (x'_1, \dots, x'_{k+1})^t \in V_{k+1}^+$. This vector can be written as

$$\begin{aligned} v + m(1, 1, \dots, 1)^t = x'_1(1, 1, \dots, 1)^t + (x'_2 - x'_1)(0, 1, \dots, 1)^t \\ + \dots + (x'_{k+1} - x'_k)(0, 0, \dots, 1)^t, \end{aligned}$$

so $v = m(-1,-1,\dots,-1)^t + x_1'(1,1,\dots,1)^t + (x_2' - x_1')(0,1,\dots,1)^t + \dots + (x_{k+1}' - x_k')(0,0,\dots,1)^t$. Then the product between the matrix A and the vector v is

$$\begin{aligned} Av &= m(-a,-a,\dots,-a)^t + x_1'(a,a,\dots,a)^t \\ &+ (x_2' - x_1')\Big(\sum_{j=2}^{k+1} a_{1j}, \sum_{j=2}^{k+1} a_{2j}, \dots, \sum_{j=2}^{k+1} a_{kj}\Big)^t + \dots \\ &+ (x_{k+1}' - x_k')(a_{1,k+1}, a_{2,k+1}, \dots, a_{k,k+1})^t \end{aligned}$$

and it belongs to V_k.

3 Applications

I. Let $(\lambda_n)_{n\ge 1}$, $(\rho_n)_{n\ge 1}$ be two sequences of positive numbers. For $x \in [0,1]$, let $A_1(x) = (\lambda_1(1-x) \quad \rho_1 x)$ and for $k \in \{2,\dots,n\}$

$$A_k(x) = \begin{pmatrix} \frac{\lambda_k}{\lambda_{k-1}}(1-x) & x & 0 & \dots & 0 & 0 \\ 0 & 1-x & x & \dots & 0 & 0 \\ 0 & 0 & 1-x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & \dots & 1-x & \frac{\rho_k}{\rho_{k-1}} x \end{pmatrix} \in \mathcal{M}_{k,k+1}$$

It is easy to see that, for any function $f \in C[0,1]$,

$$\begin{aligned} &A_1(x)A_2(x)\dots A_n(x)\Big(f(0), f\big(\frac{1}{n}\big), \dots, f(1)\Big)^t \\ &= \sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big) =: M_n(f)(x), \end{aligned}$$

where $\alpha_{n,0} = \lambda_n$, $\alpha_{n,n} = \rho_n$ and $\alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k+1}$ for $k \in \{1,\dots,n\}$.

We obtained in this way the operator $M_n(f)$ introduced by Campiti and Metafune in [1] as a generalization of the classical Bernstein operators. Using the results of Section 2 we can prove:

Theorem 5 *Let $(\lambda_n)_{n\ge 1}$ be a decreasing sequence of positive numbers and $(\rho_n)_{n\ge 1}$ an increasing sequence of positive numbers, such that $\lambda_1 \le \rho_1$. If $f : [0,1] \to \mathbb{R}$ is increasing and $f(x) \ge 0$ for each $x \in [0,1]$, then $M_n(f)$ is an increasing function and $M_n(f)(x) \ge 0$ for all $x \in [0,1]$.*

Proof. First we check assertion (ii) from Theorem 3 for every A_k, $k \in \{1,\dots,n\}$.

We have

$$\Big(\sum_{j=1}^{k+1} a_{1j},\ldots,\sum_{j=1}^{k+1} a_{kj}\Big) = \Big(\frac{\lambda_k}{\lambda_{k-1}}(1-x)+x,1,\ldots,1,1-x+\frac{\rho_k}{\rho_{k-1}}x\Big),$$
$$\Big(\sum_{j=2}^{k+1} a_{1j},\ldots,\sum_{j=2}^{k+1} a_{kj}\Big) = \Big(x,1,\ldots,1,1-x+\frac{\rho_k}{\rho_{k-1}}x\Big),\ \ldots\ldots,$$
$$\Big(\sum_{j=k}^{k+1} a_{1j},\ldots,\sum_{j=k}^{k+1} a_{kj}\Big) = \Big(0,0,\ldots,x,1-x+\frac{\rho_k}{\rho_{k-1}}x\Big),$$
$$\Big(\sum_{j=k+1}^{k+1} a_{1j},\ldots,\sum_{j=k+1}^{k+1} a_{kj}\Big) = \Big(0,0,\ldots,0,\frac{\rho_k}{\rho_{k-1}}x\Big)$$

These vectors belong to V_k^+ if and only if $\lambda_k \le \lambda_{k-1}$ and $\rho_{k-1} \le \rho_k$. Hypothesis (H1') follows now directly.

We check next hypothesis (H2') of Theorem 2 for each $k \in \{1,\ldots,n\}$. Let $x<y$ be arbitrary fixed, $u,v \in V_{k+1}^+$ with $u \le v$.

For $k=1$, let $u=(u_1,u_2)^t$. $A_1(x)u \le A_1(y)u$ is equivalent to $\lambda_1(1-x)u_1 + \rho_1 x u_2 \le \lambda_1(1-y)u_1+\rho_1 y u_2$, that is $(y-x)(\lambda_1 u_1 - \rho_1 u_2) \le 0$. This is true for any $0 \le u_1 \le u_2$ if and only if $\lambda_1 \le \rho_1$. Obviously, if $u \le v$ then $A_1(y)u \le A_1(y)v$, so also $A_1(x)u \le A_1(y)v$.

For $k \in \{2,\ldots,n\}$, $A_k(x)u \le A_k(y)u$ means that three types of inequalities have to be satisfied:

1) $\frac{\lambda_k}{\lambda_{k-1}}(1-x)u_1 + xu_2 \le \frac{\lambda_k}{\lambda_{k-1}}(1-y)u_1 + yu_2$, which is equivalent to $\Big(\frac{\lambda_k}{\lambda_{k-1}}u_1 - u_2\Big)(y-x) \le 0$, insured by the conditions $\lambda_k \le \lambda_{k-1}$ and $0 \le u_1 \le u_2$;

2) $(1-x)u_j + xu_{j+1} \le (1-y)u_j + yu_{j+1}$, equivalent to $(u_j - u_{j+1})(y-x) \le 0$ which is obviously true;

3) $(1-x)u_k + \frac{\rho_k}{\rho_{k-1}}xu_{k+1} \le (1-y)u_k + \frac{\rho_k}{\rho_{k-1}}yu_{k+1}$, which is equivalent to $\Big(u_k - \frac{\rho_k}{\rho_{k-1}}u_{k+1}\Big)(y-x) \le 0$, insured by the conditions $\rho_k \ge \rho_{k-1}$ and $0 \le u_k \le u_{k+1}$.

Finally, if $u \le v$ it follows also that $A_k(x)u \le A_k(y)v$.

Remark 1 *The conditions of Theorem 5 are not sufficient to insure the transformation of any increasing function into an increasing function by the Campiti-Metafune operator. For instance, if we choose $n=2$, $\lambda_k = 1$ and $\rho_k = 4$ for all $k \in \{1,2\}$ the conditions of Theorem 5 are satisfied and we obtain the operator of order two*

$$M_2(f)(x) = (1-x)^2 f(0) + 5x(1-x)f\Big(\frac{1}{2}\Big) + 4x^2 f(1).$$

The function $f:[0,1]\to\mathbb{R}$, $f(x)=2x-2$ is increasing but has also negative values in $[0,1]$. It is transformed into $M_2(f)(x) = 3x^2 - x - 2$, which is not an increasing function on $[0,1]$.

Concerning monotonicity preserving for functions with arbitrary values, we have

Theorem 6 *The Campiti-Metafune operator M_n transforms any increasing function $f : [0,1] \to \mathbb{R}$ into an increasing function if and only if there exists $\alpha \geq 0$ such that*

$$M_n(f)(x) = \sum_{k=0}^{n} \alpha \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Proof. The constant functions $\pm\mathbf{1}$ are increasing, so that $\alpha := M_n(\mathbf{1})$ and $-\alpha := M_n(-\mathbf{1})$ are increasing functions. It follows that α is a constant function. Then

$$\sum_{k=0}^{n} \alpha_{n,k} x^k (1-x)^{n-k} = M_n(\mathbf{1})(x) = \alpha = \sum_{k=0}^{n} \alpha \binom{n}{k} x^k (1-x)^{n-k},$$

for $x \in [0,1]$. From the linear independence of the functions $\{x^k(1-x)^{n-k} \ : \ k = 0,\ldots,n\}$ we get $\alpha_{n,k} = \alpha\binom{n}{k}$, for $k = 0,\ldots,n$.

II. Another operator that can be described using matrices is an operator of Stancu type.

Let $\gamma \geq 0$ be fixed. For $x \in [0,1]$, $k \in \{1,\ldots,n\}$ let

$$A_k(x) = \frac{1}{1+(k-1)\gamma} \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & \ldots & 0 & 0 \\ 0 & b_{22} & b_{23} & 0 & \ldots & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & b_{kk} & b_{k,k+1} \end{pmatrix}$$

with $b_{jj} = 1 - x + (k-j)\gamma$ and $b_{j,j+1} = x + (j-1)\gamma$.

It is easy to see that, for any function $f \in C[0,1]$,

$$A_1(x)\ldots A_n(x)\left(f(0), f\left(\frac{1}{n}\right), \ldots, f(1)\right)^t = \sum_{k=0}^{n} u_{n,k}(x) f\left(\frac{k}{n}\right) =: S_n(f)(x),$$

where

$$u_{n,k}(x) = \binom{n}{k} \frac{x(x+\gamma)\ldots(x+(k-1)\gamma)(1-x)(1-x+\gamma)\ldots(1-x+(n-k-1)\gamma)}{(1+\gamma)(1+2\gamma)\ldots(1+(n-1)\gamma)}.$$

For $\gamma = 0$ the classical Bernstein operator is obtained.

Theorem 7 *Let $\gamma \geq 0$ be fixed. If the function $f : [0,1] \to \mathbb{R}$ is increasing, then also $S_n(f) : [0,1] \to \mathbb{R}$ is increasing.*

Proof. To check hypothesis (H1) of theorem 1 we will use Theorem 4. Let $k \in \{1,\ldots,n\}$.

We have $\sum_{j=1}^{k+1} a_{1j} = \sum_{j=1}^{k+1} a_{2j} = \cdots = \sum_{j=1}^{k+1} a_{kj} = 1$,

$$\Big(\sum_{j=2}^{k+1} a_{1j}, \dots, \sum_{j=2}^{k+1} a_{kj}\Big) = \Big(\frac{x}{1+(k-1)\gamma}, 1, \dots, 1, 1\Big), \ \dots\dots,$$
$$\Big(\sum_{j=l}^{k+1} a_{1j}, \dots, \sum_{j=l}^{k+1} a_{kj}\Big) = \Big(0, 0, \dots, \frac{x+(l-2)\gamma}{1+(k-1)\gamma}, \dots, 1, 1\Big), \ \dots\dots,$$
$$\Big(\sum_{j=k+1}^{k+1} a_{1j}, \dots, \sum_{j=k+1}^{k+1} a_{kj}\Big) = \Big(0, 0, \dots, 0, \frac{x+(k-1)\gamma}{1-(k-1)\gamma}\Big),$$

which belong to the set V_k^+.

We verify now hypothesis (H2) of Theorem 2, for each $k \in \{1, \dots, n\}$. Let $x, y \in [0,1]$, $x < y$, $u, v \in V_{k+1}$ with $u \leq v$. The inequality $A_k(x)u \leq A_k(y)u$ is equivalent to

$$\frac{1-x(k-j)\gamma}{1+(k-1)\gamma}u_j + \frac{x+(j-1)\gamma}{1+(k-1)\gamma}u_{j+1} \leq \frac{1-y(k-j)\gamma}{1+(k-1)\gamma}u_j + \frac{y+(j-1)\gamma}{1+(k-1)\gamma}u_{j+1},$$

for all $j \in \{1, \dots, k\}$. This means $(y-x)(u_j - u_{j+1} \leq 0)$, obviously true. For $u \leq v$ it is clear that $A_k(y)u \leq A_k(y)v$ holds. So we have finally $A_k(x)u \leq A_k(y)v$ as requested in Theorem 1.

III. Let τ be a strictly increasing function in $C[0,1]$ and for each $k \in \{1, \dots, n\}$ define the matrix

$$A_k(x) = \begin{pmatrix} 1-\tau(x) & \tau(x) & 0 & \dots & 0 & 0 \\ 0 & 1-\tau(x) & \tau(x) & \dots & 0 & 0 \\ 0 & 0 & 1-\tau(x) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1-\tau(x) & \tau(x) \end{pmatrix}$$

Using these matrices, the following operator $L_n : C[0,1] \to C[0,1]$ can be defined

$$\begin{aligned} L_n(f)(x) &:= A_1(x) \dots A_n(x)\Big(f(\tau^{-1}(0)), f\big(\tau^{-1}(\tfrac{1}{n})\big) \dots f(\tau^{-1}(1))\Big)^t \\ &= \sum_{k=0}^{n} \binom{n}{k} \tau^k(x)(1-\tau(x))^{n-k} f\Big(\tau^{-1}\Big(\frac{k}{n}\Big)\Big) \end{aligned}$$

for any $f \in C[0,1]$. This is another generalization of the classical Bernstein operator.

Theorem 8 *Let $\tau : [0,1] \to [0,1]$ be a strictly increasing function. If the function $f : [0,1] \to \mathbb{R}$ is increasing, then also $L_n(f) : [0,1] \to \mathbb{R}$ is increasing.*

Proof. If f is increasing, then so is $g := f \circ \tau^{-1}$ and we apply Theorem 1 for the function g.

We have $\sum_{j=1}^{k+1} a_{1j} = \sum_{j=1}^{k+1} a_{2j} = \cdots = \sum_{j=1}^{k+1} a_{kj} = 1$,

$$\Big(\sum_{j=2}^{k+1} a_{1j}, \dots, \sum_{j=2}^{k+1} a_{kj}\Big) = \Big(\tau(x), 1, \dots, 1, 1\Big) \in V_k^+, \ \ldots\ldots,$$
$$\Big(\sum_{j=k+1}^{k+1} a_{1j}, \dots, \sum_{j=k+1}^{k+1} a_{kj}\Big) = \Big(0, 0, \dots, 0, \tau(x)\Big) \in V_k^+,$$

so by Theorem 4, we have (H1) fulfiled.

To check (H2), let $x, y \in [0,1]$, $x < y$ and $u, v \in V_{k+1}$, $u \leq v$. The inequality $A_k(x)u \leq A_k(y)u$ is equivalent to

$$(1-\tau(x))u_j + \tau(x)u_{j+1} \leq (1-\tau(y))u_j + \tau(y)u_{j+1},$$

i.e., $(\tau(y) - \tau(x))(u_j - u_{j+1}) \leq 0$ which is obviously true. Finally, since $A_k(y)u \leq A_k(y)v$ it follows that $A_k(x)u \leq A_k(y)v$.

Remark 2 *The operators discussed in this section are of King type. Details concerning them, as well as the Campiti-Metafune operators, can be found in [3] and the references therein.*

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Two reverse inequalities of Bullen's inequality [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this paper we provide two reverse inequalities of Bullen′s inequality and several applications concerning the arithmetic mean and the logarithmic mean.

2010 Mathematics Subject Classification: 26A51, 26D15.
Key words and phrases: Bullen′s inequality, convex function.

1 Introduction

Given a convex function $f : [a, b] \longrightarrow R$, one has Bullen′s inequality (see e.g. [2,3])

$$(1) \qquad \frac{2}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right).$$

For certain constraints of f, in [1] Dragomir and Pearce found an improvement of Bullen′s inequalit given by the following:

Theorem 1 *Let $f : [a, b] \longrightarrow R$ be a twice differentiable function for wich there exist real constants m and M such that:*

$$m \leq f''(x) \leq M, \quad for \;\; all \;\; x \in [a, b].$$

Then

$$(2) \qquad m\frac{(b-a)^2}{24} \leq \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)dx \leq M\frac{(b-a)^2}{24}.$$

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2 Main results

Lemma 1 *Whenever $f : [a, b] \longrightarrow R$ is a twice differentiable function, we have the following equality:*
(3)
$$\frac{f(a)+f(b)}{2}+f(\frac{a+b}{2})-\frac{2}{b-a}\int_a^b f(x)dx=\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)q(x)f''(x)dx$$
where
$$q(x):=\begin{cases} a-x & , \ x\in\left[a,\frac{a+b}{2}\right) \\ b-x & , \ x\in\left[\frac{a+b}{2},b\right]\end{cases}$$

Proof. It is easy to see that
$$\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)q(x)f''(x)dx=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)(a-x)f''(x)dx$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)(b-x)f''(x)dx$$
$$=\frac{1}{b-a}\left[\left(x-\frac{a+b}{2}\right)(a-x)f'(x)\Big|_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}}f'(x)\left(-2x+\frac{3a+b}{2}\right)dx\right.$$
$$\left.+\left(x-\frac{a+b}{2}\right)(b-x)f'(x)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b f'(x)\left(-2x+\frac{a+3b}{2}\right)dx\right]$$
$$=\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}f'(x)\left(2x-\frac{3a+b}{2}\right)dx+\int_{\frac{a+b}{2}}^b f'(x)\left(2x-\frac{a+3b}{2}\right)dx\right]$$
$$=\frac{1}{b-a}\left[f(x)\left(2x-\frac{3a+b}{2}\right)\Big|_a^{\frac{a+b}{2}}-2\int_a^{\frac{a+b}{2}}f(x)dx+f(x)\left(2x-\frac{a+3b}{2}\right)\Big|_{\frac{a+b}{2}}^b\right.$$
$$\left.-2\int_{\frac{a+b}{2}}^b f(x)dx\right]=\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)dx.$$

Remark 1 *1. a) Clearly for $x\in[a,b]$, one has*
$$\left(x-\frac{a+b}{2}\right)q(x)\geq 0.$$
By some elementary computations one obtains:
$$\int_a^b\left(x-\frac{a+b}{2}\right)q(x)=\frac{(b-a)^3}{24}.$$
Therefore, for every $x\in[a,b]$, we can write
$$m\left(x-\frac{a+b}{2}\right)q(x)\leq\left(x-\frac{a+b}{2}\right)q(x)f''(x)\leq M\left(x-\frac{a+b}{2}\right)q(x).$$

Integrating from a to b, multiplying by $\frac{1}{b-a}$ and using relation (3), we obtain the inequalities from (2).

b) Inequalities (2) can also be obtained from inequality (1) by applying the Bullen′s inequality for the convex function $f(x) - m\frac{x^2}{2}$ and $M\frac{x^2}{2} - f(x)$.

In the following we give a reverse inequality of Bullen′s inequality.

Theorem 2 *Let $f : [a, b] \longrightarrow R$ be a twice differentiable and convex function. Then the following inequality holds*

$$\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{b-a}\int_a^b f(x)dx \le \frac{(b-a)[f'(b)-f'(a)]}{16} \tag{4}$$

Proof. Since f is a convex function, it follows that $f''(x) \geq 0$, for every $x \in [a, b]$. Because

$$0 \le \left(x - \frac{a+b}{2}\right)q(x) \le \frac{(b-a)^2}{16}$$

for any $x \in [a, b]$ we deduce the inequality

$$\left(x - \frac{a+b}{2}\right)q(x)f''(x) \le \frac{(b-a)^2}{16}f''(x).$$

Therefore, by integrating, the last inequality from a to b, we obtain:

$$\int_a^b \left(x - \frac{a+b}{2}\right)q(x)f''(x)dx \le \frac{(b-a)^2}{16}[f'(b) - f'(a)].$$

Using equality (3) in the previous inequality we find the inequality from statement.

Theorem 3 *Let $f : [a, b] \longrightarrow R$ be a twice differentiable function and assume there exist real constants m and M such that:*

$$m \le f''(x) \le M \quad for \quad all \quad x \in [a, b].$$

Then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{b-a}\int_a^b f(x)dx - \frac{(b-a)[f'(b)-f'(a)]}{24}\right| \\ &\qquad\le \frac{(M-m)(b-a)^2}{64}.\end{aligned} \tag{5}$$

Proof. Taking into account that $0 \le \left(x - \frac{a+b}{2}\right)q(x) \le \frac{(b-a)^2}{16}$ and $m \le f''(x) \le M$, for every $x \in [a, b]$, and applying the inequality of Grüss (see [2,3]), we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)q(x)f''(x)dx - \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)q(x)dx \frac{1}{b-a}\int_a^b f''(x)dx\right|$$
$$\le \frac{1}{64}(M-m)(b-a)^2.$$

This is equivalent to the inequality of the statement.

3 Applications

Application 1 *We consider $f(x) = x^p$, $p > 1$. Obviously f is a convex function. According to Theorem 2 one has:*

$$\frac{a^p + b^p}{2} + \left(\frac{a+b}{2}\right)^p - \frac{2}{b-a}\frac{b^{p+1} - a^{p+1}}{p+1} \leq \frac{p(b-a)(b^{p-1} - a^{p-1})}{16}$$

so

$$A(a^p, b^p) + A^p(a,b) - 2L_p^p(a,b) \leq \frac{p(b-a)(b^{p-1} - a^{p-1})}{16} L_{p-2}^{p-2}, \tag{6}$$

where $A(a,b) = \dfrac{a+b}{2}$ is the arithmetic mean and $L_p(a,b) = \left[\dfrac{a^p - b^p}{p(a-b)}\right]^{\frac{1}{p-1}}$ is the Stolarsky/ s mean.

Application 2 *For $f(x) = \dfrac{1}{x}$, $x > 0$, we have $f'(x) = -\dfrac{1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$, hence f is a convex function. From Theorem 2 we obtain:*

$$\frac{\frac{1}{a} + \frac{1}{b}}{2} + \frac{2}{a+b} - \frac{2(lnb - lna)}{b-a} \leq \frac{(b-a)}{16}\left(\frac{1}{a^2} - \frac{1}{b^2}\right),$$

so

$$\frac{1}{H(a,b)} + \frac{1}{A(a,b)} - \frac{2}{L(a,b)} \leq \frac{(b-a)^2}{8a^2b^2} A(a,b), \tag{7}$$

where $H(a,b) = \dfrac{2ab}{a+b}$ is the harmonic mean and $L(a,b)$ is the logarithmic mean.

Application 3 *For $f(x) = -lnx$, $x > 0$, we have $f'(x) = -\dfrac{1}{x}$, $f''(x) = \dfrac{1}{x^2} > 0$, hence f is a convex function. Applying Theorem 2 for f, we find the inequality*

$$A(a,b)G(a,b)e^{\frac{(b-a)^2}{16ab}} \geq I^2(a,b), \tag{8}$$

where $G(a,b) = \sqrt{ab}$ is the geometric mean and $I(a,b) = \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

Indeed, we have successively:

$$\frac{-lna - lnb}{2} - ln\left(\frac{a+b}{2}\right) - \frac{2}{b-a}[-b\ lnb + a\ lna + b - a] \leq \frac{(b-a)[-\frac{1}{b} + \frac{1}{a}]}{16}$$

$$\Leftrightarrow -\left[ln\ \sqrt{ab}\left(\frac{a+b}{2}\right)\right] + 2\ ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} - 2 \leq \frac{(b-a)^2}{16ab}$$

$$\Leftrightarrow ln\frac{\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}}{\sqrt{ab}(\frac{a+b}{2})} \leq \frac{(b-a)^2}{16ab} \Leftrightarrow \left(\frac{a+b}{2}\right)\sqrt{ab}\ e^{\frac{(b-a)^2}{16ab}} \geq \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

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The volume of the unit ball. A review [1]

Valentin Gabriel Cristea

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

The aim of this survey is to present recent research on the problem of estimating of the volume of unit n-dimensional ball. Some results from the theory of approximating of the gamma function are used. Finally, some inequalities on the area of the unit n-dimensional ball are given.

2010 Mathematics Subject Classification: 33B15, 51M16, 51N20.
Key words and phrases: volume of the unit n-dimensional ball, surface area of the unit n-dimensional ball, Gamma function, monotonicity, inequalities.

1 Introduction

Inequalities on Euler's gamma function Γ have requested the attention of many authors. In particular, several researchers have proven interesting monotonicity properties of the volume of the unit ball in $\mathbb{R}^n$,

$$(1) \qquad \Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad n = 1, 2, 3, \dots .$$

The volume $\Omega_n(R)$ of the n-dimensional ball of radius R can be expressed using polar coordinates (ρ, θ) by integrating $\Omega_{n-2}\left(\sqrt{R^2-\rho^2}\right)$ over the intersection disk of (x_1, x_2)-plane with n-dimensional ball. Hence

$$\Omega_n(R) = \frac{2\pi R^2}{n}\Omega_{n-2}(R), \quad n \geq 3$$

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and consequently

$$\Omega_n(R) = \begin{cases} \frac{2^{\frac{n}{2}}\pi^{\frac{n}{2}}R^n}{2\cdot 4\cdot\ldots\cdot n} & \text{if } n \text{ is even} \\ \frac{2^{\frac{n-1}{2}}\pi^{\frac{n-1}{2}}R^n}{1\cdot 3\cdot\ldots\cdot n} & \text{if } n \text{ is odd} \end{cases} = \frac{\pi^{\frac{n}{2}}R^n}{\Gamma(\frac{n}{2}+1)}.$$

As $\Omega_n(R) = \Omega_n R^n$, we get formula (1) for $R = 1$.

2 Estimates of $\frac{\Omega_{n-1}}{\Omega_n}$

Anderson et al. [5] and Klain and Rota [13] have presented the following inequalities for every $n \geq 1$

$$\sqrt{\frac{n}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+1}{2\pi}},$$

then Alzer [2, Theorem 2] and Qi and Guo [28, Theorem 2] have exploited the monotonicity of the function

$$p(x) = \left[\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\right]^2 - x$$

to establish the following estimates

$$\sqrt{\frac{n+A}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+B}{2\pi}}, \tag{2}$$

with the best possible constants $A = \frac{1}{2}$ and $B = \frac{\pi}{2} - 1$. Inequalities of type (2) were also discussed by Borgwardt [7, p. 253], Qiu and Vuorinen [34], or Karayannakis [12, p. 2-4].

Mortici [27, Theorem 3] has improved the previous results by introducing the following estimates

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi n}}, \tag{3}$$

as a result of the fact that $g(x) < 0$ on $(0, \infty)$, where

$$g(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}\right).$$

3 Estimates of $\Omega_{n+1}^{n/(n+1)}$

There are presented in Anderson et al. [5] and Klain and Rota [13] the following inequality for every $n \geq 1$

$$\Omega_{n+1}^{n/(n+1)} < \Omega_n.$$

Alzer [2, Theorem 1] has used the monotonicity of the sequence

$$x_n = \ln \Omega_n - \frac{n}{n+1} \ln \Omega_{n+1}$$

to establish the following double inequality:

$$a\Omega_{n+1}^{n/(n+1)} < \Omega_n < b\Omega_{n+1}^{n/(n+1)}, \quad n \geq 1, \tag{4}$$

with the best possible constants $a = 2/\sqrt{\pi} = 1,1283...$ and $b = \sqrt{e} = 1,6487...$.

F. Qi and B.-N. Guo [29], [30], [32] have rediscovered and refined Alzer's inequality (4) as a consequence of the logarithmically completely monotonicity on $(-2, \infty)$ of the function

$$Q(x) = \left[\frac{\pi^{x/2}}{\Gamma(1+x/2)}\right]^{1/x}.$$

Later Mortici [27, Theorem 2] improved (4) as

$$\frac{k}{\sqrt[2n]{2\pi}} \leq \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} \leq \frac{\sqrt{e}}{\sqrt[2n]{2\pi}}, \quad n \geq 4, \tag{5}$$

where $k = \dfrac{64 \cdot 720^{11/22} \cdot 2^{1/22}}{10395 \cdot \pi^{5/11}} = 1,5714...$. Inequality in the left-hand side of (5) holds with equality if and only if $n = 11$. Yin [35, Theorem 3.3] has proposed

$$\frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}} \frac{\left(\sqrt{n+\frac{4}{3}}\right)^{\frac{2n+1}{n+1}}}{\sqrt{(n+1)\left(n+1+\frac{\beta}{2}\right)}} < \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}} \frac{\left(\sqrt{n+1+\beta}\right)^{\frac{2n+1}{n+1}}}{\sqrt{(n+1)\left(n+\frac{7}{6}\right)}}, \quad n \geq 1,$$

where $\beta = \sqrt[3]{\frac{301}{30}} - 2 = 0,3533...$.

4 Estimates of $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$

Anderson et al. [5] and Klain and Rota [13] have presented the following inequalities for every $n \geq 1$

$$1 < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n}.$$

Alzer [2, Theorem 3] has found

$$\left(1+\frac{1}{n}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{\beta} \tag{6}$$

with $\alpha = 2 - \log_2 \pi = 0,34850...$ and $\beta = 1/2$, as a consequence of the monotonicity of the sequence

$$z_n = \frac{2\ln \Omega_n - \ln \Omega_{n-1} - \ln \Omega_{n+1}}{\ln\left(\frac{n}{2}+1\right)}.$$

Inequalities of type (6) were also studied by Karayannakis [12, p. 5-6], or Yin who proposed the following estimates for every integer $n \geq 1$:

$$\frac{(n+1)\left(n+\frac{1}{6}\right)}{(n+\beta)^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{(n+1)\left(n+\frac{\beta}{2}\right)}{\left(n+\frac{1}{3}\right)^2}$$

and

$$\frac{n+\frac{1}{6}}{n+\beta} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{n+1}{n+\frac{1}{3}},$$

where $\beta = \sqrt[3]{\frac{301}{30}} - 2 = 0,3533...$.

Mortici [27, Theorem 4] has improved (6), showing that for every integer $n \geq 4$,

$$\left(1+\frac{1}{n}\right)^{\frac{1}{2}-\frac{1}{4n}} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{\frac{1}{2}}. \tag{7}$$

5 Estimates of $\frac{\Omega_{n+1}^{1/(n+1)}}{\Omega_n^{1/n}}$ and $\frac{\Omega_{n+2}^{1/(n+2)}}{\Omega_n^{1/n}}$

Qi and Guo [29] and Qi and Guo [31] have proved the following double inequalities

$$\sqrt{\frac{n+2}{n+3}} < \frac{\Omega_{n+1}^{1/(n+1)}}{\Omega_n^{1/n}} < \sqrt[4]{\frac{n+2}{n+3}}$$

$$\sqrt{\frac{n+2}{n+4}} < \frac{\Omega_{n+2}^{1/(n+2)}}{\Omega_n^{1/n}} < \sqrt[4]{\frac{n+2}{n+4}},$$

respective

$$\frac{1}{\pi^{2/(n-2)n}}\sqrt{\frac{n+2}{n+4}} < \frac{\Omega_{n+2}^{1/(n+2)}}{\Omega_n^{1/n}} < \frac{1}{\pi^{2/(n-2)n}}\sqrt[8]{\frac{n+2}{n+4}}.$$

Qi and Guo [29, Remark 9] have proposed as an open problem the finding of the best positive constants $a \geq 3, b \leq 3, \lambda \leq 1, \mu \geq 1, \alpha \geq 2$ and $\beta \leq 4$ such that the double inequality holds true for every $n \geq 1$:

$$\sqrt[\alpha]{1-\frac{\lambda}{n+a}} < \frac{\Omega_{n+1}^{1/(n+1)}}{\Omega_n^{1/n}} < \sqrt[\beta]{1-\frac{\mu}{n+b}}.$$

Yin [35, Theorem 3.1] has proved that

$$\Omega_n \leq (\Omega_1\Omega_2...\Omega_n)^{\frac{1}{n-1}}, \quad n = 1, 2, 3, ...$$

and

$$(\Omega_1\Omega_2...\Omega_n)^{\frac{1}{n}} \leq \Omega_{(n+1)/2}, \quad n = 1, 3, 5, ...$$

Let H_n be the n-th harmonic number. In [35, Theorem 3.4], Yin has proved that the sequence $\{(\Omega_n)^{\frac{1}{H_n}}\}_{n\geq 1}$ attains its maximum at $n = 3$, while the sequence $\{(\Omega_n)^{\frac{1}{H_n}}\}_{n\geq 3}$ is monotonically decreasing to zero. Further similar results were obtained by Qi and Guo in [33, Theorem 1-2] by exploiting the properties of the following functions defined on $(0,\infty)\setminus\{1\}$:

$$F(x) = \frac{\ln\Gamma(x+1)}{\ln(x^2+1) - \ln(x+1)} \tag{8}$$

and

$$G(x) = \left[\frac{\pi^x}{\Gamma(x+1)}\right]^{1/\left[\ln(x^2+1)-\ln(x+1)\right]}. \tag{9}$$

As a consequence of the fact that (8)-(9) are monotone, the sequence $\Omega_n^{1/\left[\ln(n^2/4+1)-\ln(n/2+1)\right]}$ is strictly decreasing for $n \geq 3$ and logarithmically convex for $n \geq 1$.

Bhayo [6, Theorem 1.4] has proved the following inequalities for every integers $k, n \geq 1$:

$$\sqrt{\Omega_{2n}\Omega_{2(n-1)}} \leq \Omega_{(2n-1)}$$

and

$$\left(\frac{\Omega_{n-1}}{\Omega_n}\right)^k \leq \frac{\Omega_{k(n-1)}}{\Omega_{kn}}.$$

Anderson and Qiu [4, Conjecture 3.3] have conjectured that the function

$$\frac{\ln\Gamma(x+1)}{x\ln(x)} \tag{10}$$

is concave on $(0,\infty)$. Chen, Qi and Li [9], [14] have proven that function (10) is strictly increasing on $(0,\infty)$, while Elbert and Laforgia [10, Section 3] have proven that (10) is concave on $x \in (1,\infty)$. Alzer has demonstrated in [2]-[3] that the function

$$F(x) = \frac{\ln\Gamma(x+1)}{x\ln(2x)} \tag{11}$$

is strictly increasing on $[1,\infty)$ and strictly concave on $[46,\infty)$. As a consequence, he has proposed the following double inequality

$$\exp\left(\frac{a}{n(\ln n)^2}\right) \leq \frac{\Omega_n^{1/(n\ln n)}}{\Omega_{n+1}^{1/[(n+1)\ln(n+1)]}} < \exp\left(\frac{b}{n(\ln n)^2}\right),$$

valid for every $n \geq 2$ if and only if $a \leq \ln 2\ln\pi - \dfrac{2(\ln 2)^2\ln(4\pi/3)}{3\ln 3} = 0{,}3...$ and $b \geq \dfrac{1+\ln(2\pi)}{2} = 1{,}4....$ Qi and Guo have proved in [29] that function (11) is strictly increasing and concave on $(\frac{1}{2},\infty)$, then they have concluded that the sequence $(\Omega_n^{1/(n\ln n)})_{n\geq 2}$ is strictly logarithmically convex. These results solve and generalize the conjecture in [5, Remark 2.41] and the result in [4, Corrolary 3.1] 1997.

6 A new estimate for ratio of Ω_n

In this final section we show a general method that can be used to improve above results. In order to illustrate this method we present the following refinement of inequalities (3).

Theorem 1. *The following inequalities are valid for every integer $n \geq 2$:*

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi n}-\frac{1}{32\pi n^2}-\frac{5}{256\pi n^3}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+\frac{1}{2}}{2\pi}+\frac{1}{16\pi n}-\frac{1}{32\pi n^2}}. \tag{12}$$

In the proof we use the following well-known inequalities

$$u(x) < \Gamma(x+1) < v(x) \tag{13}$$

where

$$u(x) = \ln\sqrt{2\pi} + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3}$$

and

$$v(x) = \ln\sqrt{2\pi} + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}.$$

We need the following lemma:

Lemma 1. *The following inequalities are valid for every $x \geq 3$ in the left-hand side and $x \geq 5$ in the right-hand side:*

$$\sqrt{\frac{1}{4}+\frac{1}{2}x+\frac{1}{16x}-\frac{1}{32x^2}-\frac{5}{256x^3}} < \frac{\Gamma\left(\frac{x}{2}+1\right)}{\Gamma\left(\frac{x-1}{2}+1\right)} < \sqrt{\frac{1}{4}+\frac{1}{2}x+\frac{1}{16x}-\frac{1}{32x^2}} \tag{14}$$

Proof. For the right-hand side inequality (14), we have to prove that $g(x) < 0$, where

$$g(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}\right).$$

Using (13), we obtain $g(x) < r(x)$,where

$$r(x) = v(x) - u\left(x-\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}\right)$$

and it suffices to show that $r(x) < 0$. We have

$$r''(x) = -\frac{S(x)}{210x^7\left(2x-1\right)^5\left(4x+32x^2+128x^3-1\right)^2}$$

where $S(x) = 2688\,000x^{12} + ...$ is a polynomial of degree 12. As the polynomial $S\left(x-5\right)$ has all positive coefficients, it results that $S(x) > 0$ on $(5,\infty)$.

Now r is strictly concave on $(5,\infty)$ with $r\left(\infty\right) = 0$, so $r(x) < 0$. Hence $g(x) < 0$ and the first part of the proof is finished.

For left-hand side in (14), we have to prove that $h(x) > 0$, where

$$h(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}\right).$$

Using (13), we get $h(x) > t(x)$, where

$$t(x) = u(x) - v\left(x-\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}+\frac{1}{32x}-\frac{1}{128x^2}-\frac{5}{2048x^3}\right).$$

and it suffices to prove that $t(x) > 0$. We have

$$t''(x) = \frac{T(x)}{210x^5\,(2x-1)^7\,(64x^2-16x+512x^3+2048x^4-5)^2}$$

where $T(x) = 2063\,728\,640x^{13} + ...$ is a polynomial of degree 13. As the polynomial $T(x-3)$ has all coefficients positive, we deduce that $T(x) > 0$ on $(3,\infty)$.

Now t is strictly convex on $(3,\infty)$ with $t(\infty) = 0$, so $t(x) > 0$ and the proof is completed.

The proof of Theorem 1. By direct computation, inequalities (12) hold true for $n = 2, 3, 4$. Finally, (14) follows by multiplying (12) by $\sqrt{\pi}$.

The surface area of the unit ball in n dimensions can be expressed as

$$A_n = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad n = 1, 2, 3, ...,$$

or $A_n = n\Omega_n$. Mortici [27, Theorems 2–4] has obtained the following inequalities on A_n:

$$\frac{nk}{\sqrt[2n]{2\pi}\,(n+1)^{\frac{n}{n+1}}} \le \frac{A_n}{A_{n+1}^{n/(n+1)}} \le \frac{n\sqrt{e}}{\sqrt[2n]{2\pi}\,(n+1)^{\frac{n}{n+1}}} \qquad (n \ge 4)$$

$$(n-1)\sqrt{\frac{1}{2\pi n}+\frac{1}{4\pi n^2}} \le \frac{A_{n-1}}{A_n} \le (n-1)\sqrt{\frac{1}{2\pi n}+\frac{1}{4\pi n^2}+\frac{1}{16\pi n^3}} \qquad (n \ge 1)$$

$$\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{\frac{1}{2}} < \frac{A_{n-1}A_{n+1}}{A_n^2} < \left(1+\frac{1}{n}\right)^{\frac{1}{2}-\frac{1}{4n}} \qquad (n \ge 4).$$

Using Theorem 1, we can state the following estimates on the surface area A_n of the unit n-dimensional ball.

Corollary 1. *For every $n \ge 2$ we have the following inequalities:*

$$(n-1)\sqrt{\frac{1}{2\pi n}+\frac{1}{4\pi n^2}+\frac{1}{16\pi n^3}-\frac{1}{32\pi n^4}-\frac{5}{256\pi n^5}} \le \frac{A_{n-1}}{A_n}$$

$$\le (n-1)\sqrt{\frac{1}{2\pi n}+\frac{1}{4\pi n^2}+\frac{1}{16\pi n^3}-\frac{1}{32\pi n^4}}.$$

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Ramanujan type formulas for approximating the gamma function. A survey [1]

Sorinel Dumitrescu

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this survey we discuss Ramanujan formula and related formulas for approximating the gamma function as many improvements were presented in the recent past. In the final part some new inequalities are presented.

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Key words and phrases: Gamma function, approximations, Ramanujan formula.

1 Recent improvements on Ramanujan's formula

In mathematics and science in general, we usually face with situations in which we are forced to approximate large factorials. One of the most known formula used to estimate big factorials and gamma function is the Stirling formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x.$$

However, in pure mathematics more accurate approximations are required. A better estimate of the factorial function is given by Gosper's formula

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt{x+\frac{1}{6}},$$

or Ramanujan's formula

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{x^3+\frac{1}{2}x^2+\frac{1}{8}x+\frac{1}{240}}.$$

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which is a consequence of the formula

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{8x^3+4x^2+x+\frac{\vartheta_x}{30}} \quad \text{with } \lim_{x\to\infty} \vartheta_x = 1.$$

See [10], [1]-[9] and all references therein. Also, in [10] there have been established the following inequalities for every $x \geq 1$:
(1)

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{8x^3+4x^2+x+\frac{1}{100}} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{8x^3+4x^2+x+\frac{1}{30}}.$$

Motivated by these inequalities, Anderson, Vamanamurthy and Vuorinen [1] defined the function

$$h(x) = \left\{\frac{\Gamma(x+1)}{\sqrt{\pi}}\left(\frac{e}{x}\right)^x\right\}^6 - \left(8x^3+4x^2+x\right)$$

and asked whether $h(x)$ is increasing from $(1,\infty)$ into $\left(\frac{1}{100},\frac{1}{30}\right)$. Karatsuba [4] answered affirmatively to this question then she presented the following improvement of (1):

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{8x^3+4x^2+x+\frac{e^6}{\pi^3}-13} \leq \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{8x^3+4x^2+x+\frac{1}{30}}.$$

These inequalities are sharp, as $\lim_{x\to\infty} h(x) = \frac{1}{30}$ and $h(1) = \frac{e^6}{\pi^3} - 13$.

Mortici [5] introduced the new approximation formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[4]{4x^2+\frac{4}{3}x+\frac{2}{9}}$$

and proved that the associated function

$$H(x) = \left\{\frac{\Gamma(x+1)}{\sqrt{\pi}}\left(\frac{e}{x}\right)^x\right\}^4 - \left(4x^2+\frac{4}{3}x\right)$$

is strictly increasing on $[1,\infty)$ from $H(1) = \alpha := \frac{e^4}{\pi^2} - \frac{16}{3} = 0.198\cdots$ to $H(\infty) = \beta := \frac{2}{9} = 0.222\cdots$. The double inequality $H(1) \leq H(x) < H(\infty)$ can be written as

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[4]{4x^2+\frac{4}{3}x+\alpha} \leq \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \sqrt[4]{4x^2+\frac{4}{3}x+\beta}.$$

The following new formulas are also presented in [5]:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[8]{x^4+\frac{2}{3}x^3+\frac{2}{9}x^2+\frac{11}{405}x-\frac{8}{1215}}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[10]{x^5+\frac{5}{6}x^4+\frac{25}{72}x^3+\frac{89}{1296}x^2-\frac{95}{31104}x+\frac{2143}{1306368}}$$

that are much stronger than Ramanujan formula (3).

The general method for establishing increasingly more accurate formulas of order $2k$

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2k]{x^k+\cdots}$$

is also presented by Mortici in [6]. The idea is to start from the standard series

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\cdots\right)\\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{\exp\left[2k\left\{\frac{1}{12x}-\frac{1}{360x^3}+\cdots\right\}\right]\right\}^{\frac{1}{2k}}\end{aligned}$$

and to use the transformation $\exp t = 1+t/1!+t^2/2!+\cdots$.

This result was sistematically proven by Chen and Lin in [2].

We also mention the following class of approximations presented very recent by Dumitrescu and Mortici in [3]

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt[6]{1+\frac{1}{2\left(x+2a-\frac{1}{4}\right)}+\frac{a}{\left(x+2a-\frac{1}{4}\right)^2}+\frac{2a^2-\frac{13}{480}}{\left(x+2a-\frac{1}{4}\right)^3}}$$

that gives better results than Ramanujan's formula.

Other improvement was presented by Mortici in [6], who proposed the following formula of Ramanujan type, starting from Burnside's formula:

$$\Gamma(x+1) \sim \sqrt{\frac{2\pi}{e}}\left(\frac{x+\frac{1}{2}}{e}\right)^x \sqrt[6]{x^3+\frac{5}{4}x^2+\frac{17}{32}x+\frac{173}{1920}}. \tag{2}$$

Moreover, he proved that the following inequalities are valid for every $x \geq 13$:

$$\begin{aligned}&\sqrt{\frac{2\pi}{e}}\left(\frac{x+\frac{1}{2}}{e}\right)^x \sqrt[6]{x^3+\frac{5}{4}x^2+\frac{17}{32}x+\frac{172}{1920}}\\ <\ &\Gamma(x+1)\\ <\ &\sqrt{\frac{2\pi}{e}}\left(\frac{x+\frac{1}{2}}{e}\right)^x \sqrt[6]{x^3+\frac{5}{4}x^2+\frac{17}{32}x+\frac{173}{1920}}.\end{aligned}$$

2 New Results

Motivated by the Ramanujan-Burnside formula (2), we propose in this section the following family of approximations as $x \to \infty$:

$$\Gamma(x+1) \sim \sigma(x) := \sqrt{2\pi x\left(1+\frac{a}{x^4}\right)}\left(\frac{x}{e}\right)^x \sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}, \tag{3}$$

where a is any real number.

By using a method first introduced by Mortici in [7], we define the relative error sequence $\Gamma(n+1)=\sigma(n)\cdot\exp\{w_n\}$. As

$$w_n-w_{n+1}=-\left(2a+\frac{11}{2880}\right)\frac{1}{n^5}+\left(5a+\frac{29}{2016}\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right),$$

the best estimate is obtained when $2a+\frac{11}{2880}=0$, that is $a=-\frac{11}{5760}$.

In the following table we prove the superiority of our new formula (3) over Ramanujan's formula.

n	$\rho(n)-\Gamma(n+1)$	$\Gamma(n+1)-\sigma(n)$
10	$0.311\,61$	$3.488\,7\times10^{-2}$
50	$4.552\,5\times10^{54}$	$9.411\,8\times10^{52}$
100	$8.821\,1\times10^{146}$	$9.026\,7\times10^{144}$
500	$1.860\,3\times10^{1120}$	$3.776\,6\times10^{1117}$
1000	$3.838\,3\times10^{2552}$	$3.892\,1\times10^{2549}$

Moreover, further computations we made show us that formula (3) is much better than Burnside-Ramanujan formula (2).

Finally we give the following estimates associated to our new formula (3).

Theorem 1 *The following inequalities are valid for every $x\geq 2$:*

$$\sqrt{2\pi x\left(1-\frac{11}{5760x^4}\right)}\left(\frac{x}{e}\right)^x\sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}<\Gamma(x+1)$$
$$<\sqrt{2\pi x\left(1-\frac{11}{5760x^4}\right)}\left(\frac{x}{e}\right)^x\sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}\cdot\exp\left\{\frac{13}{13\,44x^5}\right\}$$

We prove that $F>0$ and $G<0$ on $[2,\infty)$, where

$$F(x)=\ln\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{\frac{-11}{5760}}{x^4}\right)}\left(\frac{x}{e}\right)^x\sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}}$$

and

$$G(x)=\ln\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{\frac{-11}{5760}}{x^4}\right)}\left(\frac{x}{e}\right)^x\sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}\cdot\exp\left\{\frac{13}{13\,44x^5}\right\}}.$$

We have

$$F'(x)=\psi(x)-\ln x+\frac{1}{2x}+\frac{11}{2880x^5\left(\frac{11}{5760x^4}-1\right)}+\frac{1}{6}\frac{\frac{1}{2x^2}+\frac{1}{4x^3}+\frac{1}{80x^4}}{\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}+1}$$

and

$$G'(x) = \psi(x) - \ln x + \frac{1}{2x} + \frac{11}{2880x^5\left(\frac{11}{5760x^4} - 1\right)} + \frac{1}{6}\frac{\frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{80x^4}}{\frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} + 1} + \frac{65}{1344x^6},$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. By using the standard inequalities

$$\begin{aligned} & -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{5}{660x^{10}} < \psi(x) - \ln x + \frac{1}{2x} \\ < \; & -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}, \end{aligned}$$

we get $F' < u$ and $G' > v$, where

$$u(x) = -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} + \frac{11}{2880x^5\left(\frac{11}{5760x^4} - 1\right)} + \frac{1}{6}\frac{\frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{80x^4}}{\frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} + 1}$$

and

$$\begin{aligned} v(x) \;=\; & -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{5}{660x^{10}} + \frac{11}{2880x^5\left(\frac{11}{5760x^4} - 1\right)} \\ & + \frac{1}{6}\frac{\frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{80x^4}}{\frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} + 1} + \frac{65}{1344x^6}. \end{aligned}$$

As rational functions, simple computations using Maple software show us that $u < 0$ and $v > 0$ on $[2, \infty)$. In consequence, $F' < 0$ and $G' > 0$ on $[2, \infty)$. Finally, taking into account the obtained monotonicity of F and G with $F(\infty) = G(\infty) = 0$, we deduce that $F > 0$ and $G < 0$ and the proof is now completed.

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Approximation of fractional derivatives by Bernstein operators [1]

Radu Păltănea

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

We show the property of uniform approximation of fractional derivatives in Caputo sense by using Bernstein operators and give estimates with the first and the second order moduli for the degree of approximation.

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1 1. Introduction

The fractional differential calculus is a classical subject of Analysis with many practical applications in different areas like mechanics, biochemistry, electrical engineering, medicine etc., see [3]. The theoretical study of fractional derivatives is connected with the theory of fractional differential and integral equations. In approximation theory there exist several directions of investigation. We mention only the recent results concerning the approximation by linear positive operators of Anastassiou [1], [2].

In our paper we consider another problem, namely the approximation of fractional derivatives by the classical Bernstein operators. In this section we give notations and we recall some basic facts of fractional calculus. In Section 2 we obtain certain auxiliary results about Bernstein operators and in Section 3 we give the main results.

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Let I be an interval. For $\alpha > 0$ and $a \in I$, the Riemann-Liouville operator is given by

$$J_a^\alpha(f,x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \tag{1}$$

where f is integrable on I and $x \in I,\ x > a$. For $\alpha = 0$, denote $J_a^0 = I$, identity operator. If $a = 0 \in I$ denote J^α instead of J_0^α. We have

$$J^\alpha \circ J^\beta = J^{\alpha+\beta},\ \alpha, \beta \geq 0.$$

For $p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, denote also by D^p the usual derivative operator of order p.

The two main extensions of the derivative operator for a non-integer order $\alpha \geq 0$ are the derivatives in Liouville-Riemann sense and the derivative in Caputo sense. These are defined as follows. Denote by $\lceil \alpha \rceil$ the smallest integer that is greater than α.

a. Fractional derivative in Riemann-Liouvile sense

For $\alpha > 0$, let $p \in \mathbb{N}$, $p \geq \lceil \alpha \rceil$ and $a \in I$, define

$$\tilde{D}_a^\alpha = D^p \circ J_a^{p-\alpha}.$$

For $\alpha = 0$ define $\tilde{D}_a^0 = I$, the identity operator. For $a = 0$ denote simple $\tilde{D}^\alpha$. The definition of $\tilde{D}_a^\alpha$ does not depend on the choice of p.

b. Fractional derivative in Caputo sense

For $\alpha \geq 0$, let $p \in \mathbb{N}$, $p = \lceil \alpha \rceil$ and $a \in I$, define

$$\hat{D}_a^\alpha = J_a^{p-\alpha} \circ D^p.$$

For $a = 0$ denote simple $\hat{D}^\alpha$.

Remark 1 Let $a \in I$. We have

$$\tilde{D}_a^\alpha = D^\alpha = \hat{D}_a^\alpha, \text{ if } \alpha \in \mathbb{N}.$$

Between the two definitions there is the following connection:

Theorem A *Let $\alpha \geq 0$ and let $p = \lceil \alpha \rceil$. Then, for $a \in I$ and any function $f \in AC^{p-1}(I)$ we have almost everywhere*

$$\hat{D}_a^\alpha f = \tilde{D}_a^\alpha (f - T_{p-1}[f; a]),$$

where $T_{p-1}[f; a]$ denotes the Taylor polynomial of degree $p-1$ of function f, centered in a.

If I is an interval we denote by $\mathcal{F}(I)$, the linear space of real functions defined on I. For $s \geq 0$ define $e_s(t) = t^s$, $t \in [0, 1]$. Also, for $n \in \mathbb{N}$ and $p \in \mathbb{N}$ denote $(n)_p := n(n-1)\ldots(n-p+1)$.

2 Auxiliary results for Bernstein operators

The Bernstein operators are given by:

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ f \in \mathbb{R}^{[0,1]},\ x \in [0,1],\ n \in \mathbb{N},$$

where

$$p_{n,k}(x) = \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}.$$

It is well known that the usual derivatives of Bernstein operators may be expressed in the following form:
Theorem B *For $n \in \mathbb{N}$, $p \in \mathbb{N}$, $f \in C^p[0,1]$ and $x \in [0,1]$ we have,*

$$B_n^{(p)}(f,x) = (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f^{(p)}(\frac{k}{n} + u_1 + \dots + u_p) d\overline{u},$$

where $d\overline{u} := du_1 \dots du_p$

For $n, p \in \mathbb{N}$ consider the linear positive operator $L_{n,p} : C[0,1] \to C[0,1]$, given by:

$$L_{n,p}(g,x) = (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} g(\frac{k}{n} + u_1 + \dots + u_p) d\overline{u},$$

$g \in C[0,1]$, $x \in [0,1]$.

Lemma 1 *For any $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $x \in [0,1]$ we have:*

i) $L_{n,p}(e_0,x) = \frac{(n)_p}{n^p}$,

ii) $L_{n,p}(e_1,x) = \frac{(n)_p}{n^p}\left[\frac{n-p}{n}x + \frac{p}{2n}\right]$,

iii) $L_{n,p}(e_2,x) = \frac{(n)_p}{n^p}\left[\frac{(n-p)(n-p-1)}{n^2} \cdot x^2 + \frac{(n-p)(p+1)}{n^2} \cdot x + \frac{3p^2+p}{12n^2}\right]$.

Proof.

i) Relation $L_{n,p}(e_0,x) = \frac{(n)_p}{n^p}$ is immediate.

ii) We have

$$\begin{aligned} L_{n,p}(e_1,x) &= (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} \Big(\frac{k}{n} + u_1 + \dots + u_p\Big) d\overline{u} \\ &= (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \Big[\frac{k}{n} \cdot \frac{1}{n^p} + \frac{p}{2}\Big(\frac{1}{n}\Big)^{p+1}\Big] \\ &= \frac{(n)_p}{n^p}\Big[\frac{n-p}{n}x + \frac{p}{2n}\Big]. \end{aligned}$$

iii) We have

$$\begin{aligned}
L_{n,p}(e_2,x) &= (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} \Big(\frac{k}{n} + u_1 + \dots + u_p\Big)^2 d\overline{u} \\
&= (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} \Big[\Big(\frac{k}{n}\Big)^2 + 2\sum_{j=1}^{p} \frac{k}{n} u_j \\
&\qquad + \sum_{j=1}^{p} (u_j)^2 + 2 \sum_{1\le i<j\le p} u_i u_j\Big] d\overline{u} \\
&= (n)_p \sum_{k=0}^{n-p} p_{n-p,k}(x) \Big[\Big(\frac{k}{n}\Big)^2 \frac{1}{n^p} + 2p \cdot \frac{k}{n} \cdot \frac{1}{2n^{n+1}} + \\
&\qquad + \frac{p}{3} \cdot \frac{1}{n^{p+2}} + 2 \cdot \frac{1}{4}\binom{p}{2} \frac{1}{n^{p+2}}\Big] \\
&= \frac{(n)_p}{n^p} \Big[\frac{(n-p)(n-p-1)}{n^2} \cdot x^2 + \frac{(n-p)(p+1)}{n^2} \cdot x + \frac{3p^2+p}{12n^2}\Big].
\end{aligned}$$

Corollary 1 *For any $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $x \in [0,1]$ we have:*

i) $L_{n,p}(e_1 - xe_0, x) = \frac{(n)_p}{n^p} \cdot \frac{p}{2n}(1-2x)$,

ii) $L_{n,p}((e_1 - xe_0)^2, x) = \frac{(n)_p}{n^p} \left[\frac{n-p^2-p}{n^2} x(1-x) + \frac{3p^2+p}{12n^2}\right]$.

Corollary 2 *For any $n \in \mathbb{N}$ and $p \in \mathbb{N}$ we have:*

i) $\|L(e_0) - e_0\| \le \frac{p(p-1)}{2n}$,

ii) $\|L_{n,p}(e_1 - \bullet)\| = \frac{p}{2n}$,

iii) $\|L_{n,p}((e_1 - \bullet)^2)\| \le \frac{1}{4n}$.

3 Approximation of fractional derivatives in Caputo's sense

We need the following simple result

Lemma 2 *For each $s \ge 0$, $\alpha > 0$ and $x \in [0,1]$ we have*

$$J^\alpha(e_s, x) = \frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)} \cdot x^{s+\alpha}. \tag{2}$$

Proof. From relation (1), for $a = 0$ and using the change of variable $t = xu$ we have

$$\begin{aligned} J^\alpha(e_s, x) &= \frac{1}{\Gamma(\alpha)} \int_0^x t^s (x-t)^{\alpha-1} dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{s+\alpha} u^s (1-u)^{\alpha-1} du \\ &= \frac{x^{\alpha+s}}{\Gamma(\alpha)} \cdot B(s+1, \alpha) \\ &= \frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)} \cdot x^{s+\alpha}. \end{aligned}$$

First we obtain estimates with the first order modulus of continuity, ω_1. We use the following general result of Mond [4], which is a variant of the estimate of Shisha and Mond.

Theorem C *Let I be an arbitrary interval and let $V \subset C(I)$ be a linear subspace such that $e_j \in V$, for $j = 0, 1, 2$. If $L : V \to \mathcal{F}(I)$ is a positive linear operator, then we have*

$$\begin{aligned} |L(f,x) - f(x)| \leq\ & |f(x)| \cdot |L(e_0, x) - 1| \\ & + \Big(L(e_0, x) + \frac{1}{h^2} L((e_1 - xe_0)^2, x)\Big) \omega_1(f, h), \end{aligned}$$

for any $f \in V$, any $x \in I$ and any $h > 0$.

Theorem 1 *For $\alpha \geq 0$, $p = \lceil \alpha \rceil$, $f \in C^p[0,1]$, $h > 0$, $n \in \mathbb{N}$ we have*

$$\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \leq \frac{p(p-1)}{n} \|f^{(p)}\| + \left(1 + \frac{1}{4nh^2}\right) \omega_1(f^{(p)}, h).$$

For $h = \frac{1}{\sqrt{n}}$ we have

$$\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \leq \frac{p(p-1)}{n} \|f^{(p)}\| + \frac{5}{4} \omega_1 \left(f^{(p)}, \frac{1}{\sqrt{n}}\right). \tag{3}$$

For $h = \frac{1}{\sqrt{n}}$ and $0 \leq \alpha < 1$ we have:

$$\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \leq \frac{5}{4} \omega_1 \left(f', \frac{1}{\sqrt{n}}\right). \tag{4}$$

Proof. Since $(B_n)^{(p)}(f) = L_{n,p}(f^{(p)})$, using Theorem C and Corollary 2 we obtain

$$\begin{aligned} |(B_n)^{(p)}(f,t) - f^{(p)}(t)| &= |L_{n,p}(f^{(p)}, t) - f^{(p)}(t)| \\ &\leq \frac{p(p-1)}{n} \|f^{(p)}\| + \left(1 + \frac{1}{4nh^2}\right) \omega_1(f^{(p)}, h). \end{aligned}$$

Since $J^{p-\alpha}$ is positive operator, using Lemma 2 we obtain for $x \in [0,1]$:

$$\begin{aligned}|\hat{D}^\alpha B_n(f,x) - \hat{D}^\alpha f(x)| &= |J^{p-\alpha}((B_n)^{(p)}(f) - f^{(p)}, x)| \\ &\leq J^{p-\alpha}(|(B_n)^{(p)}(f) - f^{(p)}|, x) \\ &\leq J^{p-\alpha}(\|(B_n(f))^{(p)} - f^{(p)}\| e_0, x) \\ &\leq \frac{p(p-1)}{n}\|f^{(p)}\| + \left(1 + \frac{1}{4nh^2}\right)\omega_1(f^{(p)}, h).\end{aligned}$$

The particular cases follows immediately.

Corollary 3 *Let $\alpha \geq 0$, and $f \in C^p[0,1]$, where $p = \lceil \alpha \rceil$. Then we have*

$$\lim_{n\to\infty} \|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| = 0. \tag{5}$$

A more refined estimate can be given using a combination of first and second order moduli of continuity. For this we shall use the following general estimate with optimal constants, given in Păltănea [5] or [6]. If I is an interval and $h > 0$ we write $h \preceq \frac{1}{2}length(I)$ if $h \leq \frac{1}{2}length(I)$ and I is closed of if $h < \frac{1}{2}length(I)$ and I is not closed.

Theorem D *Let I be an arbitrary interval and let $V \subset C(I)$ be a linear subspace such that $e_j \in V$, for $j = 0,1,2$. If $L : V \to \mathcal{F}(I)$ be a positive linear operator, then we have*

$$\begin{aligned}|L(f,x) - f(x)| \leq\ & |f(x)|\cdot|L(e_0,x) - 1| + |L(e_1 - xe_0, x)|h^{-1}\omega_1(f,h) \\ & + \Big(L(e_0,x) + \frac{1}{2h^2}F((e_1 - xe_0)^2, x)\Big)\omega_2(f,h),\end{aligned} \tag{6}$$

for any $f \in V$, any $x \in I$ and any $0 < h \preceq \frac{1}{2}length(I)$. First we give a separate estimate for the case of derivatives of integer orders

Theorem 2 *For $n \in \mathbb{N}$, $p \in \mathbb{N}$, $f \in C^p[0,1]$, $x \in [0,1]$, $0 < h \leq \frac{1}{2}$, we have*

$$\begin{aligned}\left|B_n^{(p)}(f,x) - f^{(p)}(x)\right| \leq\ & \frac{p(p-1)}{n}|f^{(p)}(x)| + \frac{p}{2n}\frac{|1-2x|}{h}\cdot\omega_1(f^{(p)},h) \\ & + \left[1 + \frac{1}{2h^2}\left(\frac{n-p-p^2}{n^2}\cdot x(1-x) + \frac{3p^2+p}{12n^2}\right)\right]\omega_2(f^{(p)},h).\end{aligned} \tag{7}$$

Also, for $n \in \mathbb{N}$, $n \geq 4$, $p \in \mathbb{N}$ and $f \in C^p[0,1]$ we have

$$\begin{aligned}\|B_n^{(p)}(f) - f^{(p)}\| \leq\ & \frac{p(p-1)}{n}\|f^{(p)}\| + \frac{p}{2\sqrt{n}}\cdot\omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right) \\ & + \frac{9}{8}\cdot\omega_2\left(f^{(p)}, \frac{1}{\sqrt{n}}\right).\end{aligned} \tag{8}$$

Proof. We apply Theorem D for the operator $L = L_{n,p}$ and take into account $(B_n)^{(p)}(f) = L_{n,p}(f^{(p)})$ and Corollaries 1, 2. Relation (8) follows from relation (7) choosing $h = \frac{1}{\sqrt{n}}$.

Theorem 3 *For $\alpha \geq 0$, $p = \lceil \alpha \rceil$, $f \in C^p[0,1]$, $n \in \mathbb{N}$, $n \geq 4$ we have*

$$\begin{aligned}\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \;&\leq\; \frac{p(p-1)}{n}\|f^{(p)}\| + \frac{p}{2\sqrt{n}} \cdot \omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right)\\ &+\frac{9}{8}\omega_2\left(f^{(p)}, \frac{1}{\sqrt{n}}\right).\end{aligned} \tag{9}$$

For $\alpha \in [0,1)$ we obtain

$$\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \leq \frac{1}{2\sqrt{n}}\omega_1\left(f', \frac{1}{\sqrt{n}}\right) + \frac{9}{8}\omega_2\left(f', \frac{1}{\sqrt{n}}\right). \tag{10}$$

Proof. Similarly as in the proof of Theorem 1 using (8) we have successively, for $x \in [0,1]$.

$$\begin{aligned}&|\hat{D}^\alpha B_n(f)(x) - \hat{D}^\alpha f(x)| \leq J^{p-\alpha}(\|(B_n(f))^{(p)} - f^{(p)}\|e_0, x)\\ &\leq \frac{p(p-1)}{n}\|f^{(p)}\| + \frac{p}{2\sqrt{n}} \cdot \omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right) + \frac{9}{8} \cdot \omega_2\left(f^{(p)}, \frac{1}{\sqrt{n}}\right).\end{aligned}$$

The particular case $0 \leq \alpha < 1$ follows immediately.

In the last part we obtain estimates using a combination of moduli $\omega_1\left(f^{(p)}, \cdot\right)$ and $\omega_1\left(f^{(p+1)}, \cdot\right)$. For this we use the following general estimate with optimal constants, given in [5] or [6].

Theorem E *Let I be an arbitrary interval and let $V \subset C(I)$ be a linear subspace such that $e_j \in V$, for $j = 0,1,2$. If $L : V \to \mathcal{F}(I)$ is a positive linear operator, then we have*

$$\begin{aligned}|L(f,x) - f(x)| \;&\leq\; |f(x)| \cdot |L(e_0,x) - 1| + |L(e_1 - xe_0, x)|h^{-1}\omega_1(f,h)\\ &+\Big(\frac{1}{6}L(e_0,x) + \frac{1}{2h^2}L((e_1 - xe_0)^2, x)\Big)h\omega_1(f',h),\end{aligned} \tag{11}$$

for any $f \in V \cap C^1(I)$, any $x \in I$ and any $0 < h \preceq \frac{1}{2}length(I)$.

For the usual derivative we have

Theorem 4 *For $n \in \mathbb{N}$, $p \in \mathbb{N}$, $f \in C^{p+1}[0,1]$, $x \in [0,1]$, $0 < h \leq \frac{1}{2}$, we have*

$$\begin{aligned}&\left|B_n^{(p)}(f,x) - f^{(p)}(x)\right| \leq \frac{p(p-1)}{n}|f^{(p)}(x)| + \frac{p}{2n}\frac{|1-2x|}{h} \cdot \omega_1(f^{(p)},h)\\ &+\left[\frac{1}{6} + \frac{1}{2h^2}\left(\frac{n-p-p^2}{n^2} \cdot x(1-x) + \frac{3p^2+p}{12n^2}\right)\right]h\omega_1(f^{(p+1)},h).\end{aligned} \tag{12}$$

Also, for $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $f \in C^{p+1}[0,1]$ we have

$$\begin{aligned}\|B_n^{(p)}(f) - f^{(p)}\| \;&\leq\; \frac{p(p-1)}{n}\|f^{(p)}\| + \frac{p}{2\sqrt{n}} \cdot \omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right)\\ &+\frac{7}{24} \cdot \omega_2\left(f^{(p+1)}, \frac{1}{\sqrt{n}}\right).\end{aligned} \tag{13}$$

Proof. Theorem 4 is a consequence of Theorem E applied to operator $L = L_{n,p}$ and function $f^{(p)}$ by take into account $(B_n)^{(p)}(f) = L_{n,p}(f^{(p)})$ and Corollaries 1, 2.

Theorem 5 *For $\alpha \geq 0$, $p = \lceil \alpha \rceil$, $f \in C^p[0,1]$, $n \in \mathbb{N}$, $n \geq 4$ we have*

$$\begin{aligned} \|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| &\leq \frac{p(p-1)}{n}\|f^{(p)}\| + \frac{p}{2\sqrt{n}} \cdot \omega_1\left(f^{(p)}, \frac{1}{\sqrt{n}}\right) \\ &\quad + \frac{7}{24}\omega_1\left(f^{(p+1)}, \frac{1}{\sqrt{n}}\right). \end{aligned} \tag{14}$$

For $\alpha \in [0,1)$ we obtain

$$\|\hat{D}^\alpha B_n(f) - \hat{D}^\alpha f\| \leq \frac{1}{2\sqrt{n}}\omega_1\left(f', \frac{1}{\sqrt{n}}\right) + \frac{7}{24}\omega_1\left(f'', \frac{1}{\sqrt{n}}\right). \tag{15}$$

Proof. The proof is based on relation (13) and is similar with the proof of Theorem 3.

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Uniform approximation of functions by Baskakov-Kantorovich operators [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this article we give a generalization of the Baskakov- Durmeyer operator using Kantorovich method and we prove a theorem of uniform convergence of these operators.

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1 Introduction

The Baskakov operators $V_n : C_2([0,\infty)) \to C([0,\infty))$ are given by

$$V_n(f)(x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), x \in [0,\infty),\ n \in \mathbb{N}\ , \tag{1}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k \cdot (1+x)^{-(n+k)}\ . \tag{2}$$

These operators were introduced by Baskakov [1] in 1957 and intensely studied by Mastroianni [4], Schurer [7], Totik [8]. and Della Vecchia [2].

In 1985 Sahai and Prasad [6] introduced modified Baskakov operators in the following integral form

$$V_n^{**}(f)(x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} v_{n,k}(t) f(t)\, dt, x \in [0,\infty),\ n \in \mathbb{N}\ , \tag{3}$$

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so that the integral exists and the series is convergent.

The purpose of this paper is to construct a Kantorovich type operator of higher order and obtain a uniform convergence theorem for this operator. Definition and some properties of these operators Kantorovich Baskakov are presented in Sections 2 and 3, and the main results on the uniform convergence are given in Section 4.

Uniform convergence theorems for various operators were given by Holhos in [3] and Păltănea [5].

2 Basic notations. Generalized Kantorovich operators

We consider a fixed $N \in \mathbb{N}$, and define the function space

$$C_N := \left\{ f \in C\left([0,\infty)\right) : \exists \lim_{x\to\infty} \tfrac{f(x)}{1+x^N} \text{ finite} \right\}, \tag{4}$$

endowed with the norm

$$\|f\|_N^* := \sup_{x\geq 0} \tfrac{|f(x)|}{1+x^N}. \tag{5}$$

Remark 1 *We mention that* $(C_N, \|\bullet\|_N^*)$ *is a Banach space.*

For $r \in \mathbb{N}$ we consider

(i) D^r is the differentiation operator of r order,

$$D^r(f)(x) = f^{(r)}(x),\ f \in C^r([0,\infty)), x \in [0,\infty),$$

(ii) I^r is the antiderivative operator of r order,

$$I^r(f)(x) = \int_0^x \frac{(x-u)^{r-1}}{(r-1)!} f(u)\,du,\ f \in C([0,\infty)), x \in [0,\infty).$$

Lemma 1 *For* $r \in \mathbb{N}$ *we have*

i) $(D^r \circ I^r)(f) = f$, *for all* $f \in C([0,\infty))$,

ii) $(I^r \circ D^r)(f) = f$, *for all* $f \in C^r([0,\infty))$, *such that* $f^{(i)}(0) = 0$ *for any* $i = 0, 1, ...r-1$.

Proof. It is clear.

Further note $e_n(x) = x^n,\ x \in [0,\infty),\ n \in \mathbb{N}$ and $(n)_i = n\cdot(n-1)\cdot...\cdot(n-i+1)$, $i \in \mathbb{N}$.

Definition 1 *Let* $r, n \in \mathbb{N}$. *Define* $V_n^r : C_N \to C([0,\infty))$

$$V_n^r := \tfrac{(n-1)_{r+1}}{(n+r-1)_{r+1}} \cdot D^r \circ V_n^{**} \circ I^r,\ n, r \in \mathbb{N}. \tag{6}$$

Theorem 1 *Let* $r, n \in \mathbb{N}$. *For* $f \in C_N$ *we have*

$$V_n^r(f)(x) = (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x) \int_0^{\infty} v_{n-r,k+r}(t) f(t)\,dt, x \in [0,\infty). \tag{7}$$

Proof. Let $f \in C_N$. We have

$$\begin{aligned}\left(V_n^{**} \circ I^r\right)(f)(x) &= (n-1)\sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} v_{n,k}(t) \int_0^{t} \frac{(t-u)^{r-1}}{(r-1)!} f(u)\,dudt \\ &= (n-1)\sum_{k=0}^{\infty} v_{n,k}(x) \int_0^{\infty} f(u) \int_u^{\infty} v_{n,k}(t) \frac{(t-u)^{r-1}}{(r-1)!}\,dtdu.\end{aligned}$$

In the last equality, we used the change of the domain

$$\begin{cases} t \in [0,\infty) \\ u \in [0,t] \end{cases} \quad \text{in} \quad \begin{cases} u \in [0,\infty) \\ t \in [u,\infty) \end{cases}$$

By induction we prove with regard to s, $0 \le s \le r-1$ the following relation

$$\begin{aligned}&D^s\left(V_n^{**} \circ I^r\right)(f)(x) \\ (8)\qquad &= \frac{(n+s-1)_{s+1}}{(n-1)_s}\sum_{k=0}^{\infty} v_{n+s,k}(x) \int_0^{\infty} f(u) \int_u^{\infty} v_{n-s,k+s}(t) \frac{(t-u)^{r-s-1}}{(r-s-1)!}\,dtdu.\end{aligned}$$

For $s=0$ this relation is clear. Supposing that it is true for $s<r-1$ and prove it for $s+1$. By a simple computation we obtain

$$D\left(v_{n,k}\right)(x) = n\left[v_{n+1,k-1}(x) - v_{n+1,k}(x)\right],$$

where we consider $v_{n,k}(x) = 0$, for $k<0$ and for any $x \in [0,\infty)$. Also, for any $n,k,s \in \mathbb{N}$ we have

$$\lim_{a\to\infty}\left[v_{n-k,k+s+1}(a) \frac{(a-u)^{r-s-1}}{(r-s-1)!}\right] = 0$$

Denote

$$G_{n,k}^{s}(u) = \int_u^{\infty} v_{n,k}(t) \frac{(t-u)^s}{s!}\,dt.$$

It follows

(9)

$$\begin{aligned}&D^{s+1}\left(V_n^{**} \circ I^r\right)(f)(x) \\ &= \frac{(n+s-1)_{s+1}}{(n-1)_s}\sum_{k=0}^{\infty} D\left(v_{n+s,k}\right)(x) \int_0^{\infty} f(u)\, G_{n-s,k+s}^{r-s-1}(u)\,du \\ &= \frac{(n+s-1)_{s+1}}{(n-1)_s}\sum_{k=0}^{\infty}(n+s)\left[v_{n+s+1,k-1}(x) - v_{n+s+1,k}(x)\right] \int_0^{\infty} f(u)\, G_{n-s,k+s}^{r-s-1}(u)\,du \\ &= \frac{(n+s)_{s+2}}{(n-1)_s}\sum_{k=0}^{\infty} v_{n+s+1,k}(x) \int_0^{\infty} f(u)\left[G_{n-s,k+s+1}^{r-s-1}(u) - G_{n-s,k+s}^{r-s-1}(u)\right] du.\end{aligned}$$

We have

$$
\begin{aligned}
& G^{r-s-1}_{n-s,k+s+1}(u) - G^{r-s-1}_{n-s,k+s}(u) \\
= & \int_u^\infty \left[v_{n-s,k+s+1}(t) - v_{n-s,k+s}(t)\right] \frac{(t-u)^{r-s-1}}{(r-s-1)!} dt \\
= & \int_u^\infty -\frac{1}{n-s-1} D\left(v_{n-s-1,k+s+1}\right)(t) \frac{(t-u)^{r-s-1}}{(r-s-1)!} dt \\
= & -\frac{1}{n-s-1} \lim_{a\to\infty} \left[v_{n-s-1,k+s+1}(a) \frac{(a-u)^{r-s-1}}{(r-s-1)!} \right] \\
& + \frac{1}{n-s-1} \int_u^\infty v_{n-s-1,k+s+1}(t) \frac{(t-u)^{r-s-2}}{(r-s-2)!} dt \\
= & \frac{1}{n-s-1} G^{r-s-2}_{n-s-1,k+s+1}(u).
\end{aligned}
$$

Replacing in (9) we obtain

$$
\begin{aligned}
& D^{s+1}\left(V_n^{**} \circ I^r\right)(f)(x) \\
= & \frac{(n+s)_{s+2}}{(n-1)_{s+1}} \sum_{k=0}^{\infty} v_{n+s+1,k}(x) \int_0^\infty f(u)\, G^{r-s-2}_{n-s-1,k+s+1}(u)\, dt du
\end{aligned}
$$

which is the analogous of relation (8) for $s+1$ instead of s. For $s = r-1$ we obtain

$$
\begin{aligned}
& D^{r-1}\left(V_n^{**} \circ I^r\right)(f)(x) \\
= & \frac{(n+r-2)_r}{(n-1)_{r-1}} \sum_{k=0}^{\infty} v_{n+r-1,k}(x) \int_0^\infty f(u) \int_u^\infty v_{n-r+1,k+r-1}(t)\, dt du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
D^r\left(V_n^{**} \circ I^r\right)(f)(x) & = \frac{(n+r-2)_r}{(n-1)_{r-1}} \sum_{k=0}^{\infty} (n+r-1)\left[v_{n+r,k-1}(x) - v_{n+r,k}(x)\right] \\
& \cdot \int_0^\infty f(u) \int_u^\infty v_{n-r+1,k+r-1}(t)\, dt du \\
& = \frac{(n+r-1)_{r+1}}{(n-1)_{r-1}} \sum_{k=0}^{\infty} v_{n+r,k}(x) \\
& \cdot \int_0^\infty f(u) \int_u^\infty \left[v_{n-r+1,k+r}(t) - v_{n-r+1,k+r-1}(t)\right] dt du \\
& = \frac{(n+r-1)_{r+1}}{(n-1)_r} \sum_{k=0}^{\infty} v_{n+r,k}(x) \int_0^\infty f(u)\, v_{n-r,k+r}(u)\, du.
\end{aligned}
$$

From (6) we obtain (7).

Remark 2 *This operator is linear and positive.*

Remark 3 *For $f \in C_N$ the integral exists and the series is convergent.*

3 Moments and their recursion

Lemma 2 *For any $n, r \in \mathbb{N}$, n sufficiently high and $x \in [0, \infty)$ we have*
i) $V_n^r(e_0)(x) = 1$.
ii) $V_n^r(e_1)(x) = x + \frac{r+1}{n-r-2}(2x+1)$.

Proof. It is clear.

Lemma 3 *For any $n, r, s \in \mathbb{N}$, n sufficiently high and $x \in [0, \infty)$ we have*
(10)
$$V_n^r(e_{s+1})(x) = \frac{1}{n-r-s-2}\left\{x(x+1) \cdot D(V_n^r(e_s))(x) + [(n+r)x + r + s + 1] V_n^r(e_s)(x)\right\}.$$

Proof. From (7) we have

(11)
$$\begin{aligned}
V_n^r(e_s)(x) &= (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty} v_{n-r,k+r}(t) \cdot t^s dt \\
&= (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty}\binom{n+k-1}{k+r} t^{k+r+s} \cdot (1+t)^{-(n+k)} dt \\
&= (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x)\frac{(n+k-1)!}{(k+r)!\,(n-r-1)!} \cdot \frac{(k+r+s)!\,(n-r-s-2)!}{(n+k-1)!} \\
&= \sum_{k=0}^{\infty} v_{n+r,k}(x) \cdot \frac{(k+r+s)_s}{(n-r-2)_s} \\
&= (1+x)^{-(n+r)}\sum_{k=0}^{\infty}\binom{n+r+k-1}{k}\left(\frac{x}{1+x}\right)^k \cdot \frac{(k+r+s)_s}{(n-r-2)_s}.
\end{aligned}$$

and

$$(12) \qquad V_n^r(e_{s+1})(x) = (1+x)^{-(n+r)}\sum_{k=0}^{\infty}\binom{n+r+k-1}{k}\left(\frac{x}{1+x}\right)^k \cdot \frac{(k+r+s+1)_{s+1}}{(n-r-2)_{s+1}}.$$

We have
$$D(V_n^r(e_s))(x) = T_1 + T_2$$
where
$$T_1 = -\frac{n+r}{1+x} V_n^r(e_s)(x)$$

and

$$
\begin{aligned}
T_2 &= (1+x)^{-(n+r)} \sum_{k=0}^{\infty} \binom{n+r+k-1}{k} \cdot k \cdot \left(\frac{x}{1+x}\right)^{k-1} \\
&\quad \cdot \frac{1}{(1+x)^2} \cdot \frac{(k+r+s)_s}{(n-r-2)_s} \\
&= (1+x)^{-(n+r)} \sum_{k=0}^{\infty} \binom{n+r+k-1}{k} \cdot \left(\frac{x}{1+x}\right)^{k} \cdot \frac{1}{x(1+x)} \\
&\quad \cdot \left[\frac{(k+r+s+1)_{s+1}}{(n-r-2)_s} - (r+s+1)\frac{(k+r+s)_s}{(n-r-2)_s}\right] \\
&= \frac{n-r-s-2}{x(1+x)} \cdot V_n^r(e_{s+1})(x) - \frac{r+s+1}{x(1+x)} V_n^r(e_s)(x).
\end{aligned}
$$

Hence

$$
D(V_n^r(e_s))(x) = \frac{n-r-s-2}{x(1+x)} \cdot V_n^r(e_{s+1})(x) - \frac{(n+r)x+r+s+1}{x(1+x)} V_n^r(e_s)(x),
$$

which implies (10).

Corollary 1 *For any $n, r, s \in \mathbb{N}$, n sufficiently high and $x \in [0,\infty)$ we have*

$$
V_n^r(e_s)(x) = x^s + R_{n,r,s}(x) \ , \tag{13}
$$

where

$$
|R_{n,r,s}(x)| \le (1+x^s)\frac{M}{n},
$$

and M is a constant which depends on s and r.

Lemma 4 *For any $n, r, s \in \mathbb{N}$, n sufficiently high and $x \in [0,\infty)$ we have*

$$
V_n^r(|t^s - x^s|)(x) \le \sqrt{\tfrac{(1+x^{2s})\cdot M}{n}}, \ , \tag{14}
$$

where M is a constant which depends on s and r.

Proof. Using Corollary (1) we have

$$
\begin{aligned}
V_n^r(|t^s - x^s|)(x) &\le \sqrt{V_n^r\left((e_s - x^s)^2\right)(x) \cdot V_n^r(e_0)(x)} \\
&\le \sqrt{V_n^r(e_{2s})(x) - 2x^s V_n^r(e_s)(x) + x^{2s}} \\
&\le \sqrt{(1+x^{2s})\frac{M}{n}}.
\end{aligned}
$$

Lemma 5 *For any $n, r \in \mathbb{N}$, n sufficiently high and $x \in [0,\infty)$ we have*

$$
V_n^r\left(\frac{1}{1+t}\right)(x) \le \frac{1}{1+x} \cdot \frac{n+r}{n+r-1}.
$$

Proof. Using Theorem (1) we have

$$\begin{aligned} V_n^r\left(\frac{1}{1+t}\right)(x) &= (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x)\int_0^{\infty} v_{n-r,k+r}(t)\cdot\frac{1}{1+t}dt \\ &= (n-r-1)\sum_{k=0}^{\infty} v_{n+r,k}(x)\cdot\frac{1}{n+k} \\ &= \frac{1}{1+x}\sum_{k=0}^{\infty} v_{n+r-1,k}(x)\cdot\frac{(n-r-1)(n+r+k-1)}{(n+k)(n+r-1)} \\ &\le \frac{1}{1+x}\cdot\frac{n+r}{n+r-1}. \end{aligned}$$

Lemma 6 *For any $n,r\in\mathbb{N}$, n sufficiently high and $x\in[0,\infty)$ we have*

$$V_n^r\left(\left|\ln(t+1)-\ln(x+1)\right|\right)(x)\le\frac{\sqrt{2r+3}}{\sqrt{n+r-2}}.$$

Proof. From the geometric-logaritmic-arithmetic mean inequality we have

$$\left|\ln(t+1)-\ln(x+1)\right|\le\frac{|t-x|}{\sqrt{(1+t)(1+x)}}=\left|\sqrt{\frac{1+x}{1+t}}-\sqrt{\frac{1+t}{1+x}}\right|.$$

Whence, using Lemma (5) we have

$$\begin{aligned} & V_n^r\left(\left|\ln(t+1)-\ln(x+1)\right|\right)(x) \\ \le\ & \sqrt{V_n^r\left(\left[\left|\ln(t+1)-\ln(x+1)\right|\right]^2\right)(x)} \\ \le\ & \sqrt{V_n^r\left(\frac{1+x}{1+t}\right)(x)-2+V_n^r\left(\frac{1+t}{1+x}\right)(x)} \\ \le\ & \sqrt{(1+x)\cdot\frac{1}{1+x}\cdot\frac{n+r}{n+r-1}-2+\frac{1}{1+x}\left(1+x+\frac{r+1}{n-r-2}(2x+1)\right)} \\ \le\ & \frac{\sqrt{2r+3}}{\sqrt{n+r-2}}. \end{aligned}$$

4 Main results

Theorem 2 *For any $r\in\mathbb{N}$ the Baskakov-Kantorovich operators $V_n^r: C_N\to C_N$ have the property that if $f(e^x-1)\cdot\frac{1}{1+(e^x-1)^N}$ is uniformly continuous on $[0,\infty)$ then*

$$\left\|V_n^r(f)-f\right\|_N^*\to 0 \text{ when } n\to\infty.$$

Moreover, for every $f\in C_N$, $r\in\mathbb{N}$ and n sufficiently high one has

$$\left\|V_n^r(f)-f\right\|_N^*\le\|f\|_N^*\cdot\frac{C}{\sqrt{n}}+2\omega\left(f(e^x-1)\cdot\frac{1}{1+(e^x-1)^N},\frac{\sqrt{2r+3}}{\sqrt{n+r-2}}\right),$$

where $C \in \mathbb{R}$ is a constant depending on N and r.

Proof. Let $\widetilde{f}(x) = \frac{1}{1+(e^x-1)^N} \cdot f(e^x - 1)$ and $g(x) = \ln(x+1)$.We notice that $\left(\widetilde{f} \circ g\right)(x) = f(x) \cdot \frac{1}{1+x^N}$.

We consider the usual first order continuity modulus of the function $\widetilde{f}$ given by

$$\omega\left(\widetilde{f}, \delta\right) = \sup\left\{\left|\widetilde{f}(x) - \widetilde{f}(y)\right| : x, y \in [0, \infty), |x-y|\right\}.$$

Taking into account the properties of this modulus and the uniform continuity of $\widetilde{f}$ we have

$$\left|\frac{1}{1+t^N} f(t) \quad - \frac{1}{1+x^N} \ f(x)\right| = \left|\widetilde{f}(g(t)) - \widetilde{f}(g(x))\right| \le \left(1 + \frac{|g(t) - g(x)|}{\delta}\right) \omega\left(\widetilde{f}, \delta\right).$$

Further we obtain

$$\begin{aligned} |f(t) - f(x)| &= \left|f(t) - \frac{1+x^N}{1+t^N} f(t) + \frac{1+x^N}{1+t^N} f(t) - f(x)\right| \\ &\le \frac{1}{1+t^N} \cdot |f(t)| \cdot \left|t^N - x^N\right| + \left(1+x^N\right) \cdot \left|\frac{1}{1+t^N} f(t) - \frac{1}{1+x^N} f(x)\right| \\ &\le \|f\|_N^* \cdot \left|t^N - x^N\right| + \left(1+x^N\right) \cdot \left(1 + \frac{|g(t) - g(x)|}{\delta}\right) \omega\left(\widetilde{f}, \delta\right). \end{aligned}$$

Applying the operators V_n^r and using the Lemmas (4) and (6) we obtain

$$\begin{aligned} & \left|V_n^r(f)(x) - f(x)\right| \\ \le\ & V_n^r(|f(t) - f(x)|)(x) \\ \le\ & \|f\|_N^* V_n^r\left(\left|t^N - x^N\right|\right)(x) + \left(1+x^N\right)\left(1 + \frac{1}{\delta} V_n^r(|g(t) - g(x)|)(x)\right) \omega\left(\widetilde{f}, \delta\right) \\ \le\ & \|f\|_N^* \cdot \sqrt{\frac{\left(1+x^{2N}\right) \cdot M}{n}} + \left(1+x^N\right) \cdot \left(1 + \frac{1}{\delta} \cdot \frac{\sqrt{2r+3}}{\sqrt{n+r-2}}\right) \omega\left(\widetilde{f}, \delta\right). \end{aligned}$$

Whence we obtain

$$\frac{1}{1+x^N} \cdot \left|V_n^r(f)(x) - f(x)\right| \le \|f\|_N^* \cdot \sqrt{\frac{M}{n}} + \left(1 + \frac{1}{\delta} \cdot \frac{\sqrt{2r+3}}{\sqrt{n+r-2}}\right) \omega\left(\widetilde{f}, \delta\right).$$

Choosing $\delta = \frac{\sqrt{2r+3}}{\sqrt{n+r-2}}$ the Theorem is proved.

Theorem 3 *For any $r \in \mathbb{N}$ the Baskakov integral operators $V_n^{**} : C_N \to C_N$ have the property that if $D^r(f)(e^x - 1) \cdot \frac{1}{1+(e^x-1)^N}$ is uniformly continuous on $[0, \infty)$ then*

$$\left\|D^r\left(V_n^{**}\right)(f) - D^r(f)\right\|_N^* \to 0 \text{ when } n \to \infty.$$

Moreover, for every $f \in C_N$, $r \in \mathbb{N}$ and n sufficiently high one has

$$\begin{aligned} & \left\| D^r \left(V_n^{**} \right) (f) - D^r \left(f \right) \right\|_N^* \\ \leq \; & \left\| D^r \left(f \right) \right\|_N^* \cdot \frac{C}{\sqrt{n}} + 2\omega \left(D^r \left(f \right) \left(e^x - 1 \right) \cdot \frac{1}{1 + \left(e^x - 1 \right)^N}, \frac{\sqrt{2r+3}}{\sqrt{n+r-2}} \right), \end{aligned}$$

where $C \in \mathbb{R}$ is a constant depending on N and r.

Proof. It obviously follows from the previous Theorem (2).

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Properties of the intermediate point from a mean-value theorem [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this paper we consider a mean-value theorem. For this one we study the properties of the intermediate point and formulate some important results related to its derivatives.

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1 Introduction and Preliminaries

In mathematical analysis for the intermediate point from the mean-value theorems was given a special interest after these theorems were proved. We remember some authors in this field, like E. Abason [1, 2, 3], U. Abel and M. Ivan [4, 5], A. M. Acu [6, 7], D. Acu and P. Dicu [8], D.I. Duca [10, 11, 12], D.I. Duca and E. Duca [13], D.I. Duca and O. Pop [14, 15, 16], Al. Lupaş [17], D. Pompeiu [18], E.C. Popa [19, 20], T. Tchakaloff [21, 22], L. Teodoriu [23, 24], T. Trif [25].

In this paper we study the properties related to the intermediate point from a mean-value theorem less known. This theorem follows.

Theorem 1 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and derivable on $]a, b[$. Then there exists a point $c \in]a, b[$ such that*

$$f(a) - f(c) = f'(c)(c - b). \tag{1}$$

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Proof. Let us consider the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = (x - b)(f(x) - f(a)), \ \forall x \in [a, b].$$

From the hypothesis of the theorem follows that the function g is continuous on $[a, b]$ and derivable on $]a, b[$. Also, one has that $g(a) = g(b) = 0$. Consequently, we can apply Rolle's theorem and follows that there exists at least one point $c \in]a, b[$ such that $g'(c) = 0$. This means $f(c) - f(a) + (c - b)f'(c) = 0$ and consequently

$$f(a) - f(c) = f'(c)(c - b).$$

2 Main results

Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$ and $f : I \to \mathbb{R}$ a differentiable function on I. Then for each $x \in I \setminus \{a\}$, there exists $c_x \in I$ such that

$$f(a) - f(c_x) = f'(c_x)(c_x - x).$$

If, for each $x \in I \setminus \{a\}$, we choose one c_x from the interval with the extremities x and a which satisfies this relation, then we can define the function $c : I \setminus \{a\} \to I \setminus \{a\}$ by

$$c(x) = c_x, \text{ for all } x \in I \setminus \{a\}.$$

This function c satisfies relation

$$f(a) - f(c(x)) = f'(c(x))(c(x) - x), \quad \text{for all } x \in I \setminus \{a\}. \tag{2}$$

If $x \in I \setminus \{a\}$ tends to a, because

$$|c(x) - a| \leq |x - a|,$$

we have

$$\lim_{x \to a} c(x) = a.$$

Then the function $\overline{c} : I \to I$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in I \setminus \{a\} \\ a, & \text{if } x = a \end{cases} \tag{3}$$

is continuous at $x = a$.

One of the purpose of this paper is to establish under which circumstances the function $\overline{c}$ is differentiable at the point $x = a$ and to compute its derivative $\overline{c}'(a)$. If there exist several functions $\overline{c}$ which satisfies (2), does the derivative of the function $\overline{c}$ at $x = a$ depend upon the function $\overline{c}$ we choose?

Since for $x \in I \setminus \{a\}$

$$\frac{\overline{c}(x) - \overline{c}(a)}{x - a} = \frac{c(x) - a}{x - a}$$

if we denote by

$$\theta(x) = \frac{c(x) - a}{x - a},$$

then

$$\theta(x) \in]0, 1[, \ c(x) = a + (x - a)\theta(x)$$

and hence

$$f(a + (x - a)\theta(x)) - f(a) = (x - a)[1 - \theta(x)]f'(a + (x - a)\theta(x)). \tag{4}$$

Obviously, the function $\overline{c} : I \to I$ defined by (3) is differentiable at $x = a$ if and only if the function $\theta : I \setminus \{a\} \to]0, 1[$ defined by

$$\theta(x) = \frac{\overline{c}(x) - \overline{c}(a)}{x - a} = \frac{c(x) - a}{x - a}, \text{ for all } x \in I \setminus \{a\}$$

has limit at the point $x = a$. Moreover, if the function $\overline{c}$ is differentiable at $x = a$, then

$$\overline{c}'(a) = \lim_{x \to a} \theta(x).$$

In what follows, we assume that f is two times differentiable on I, the function f'' is continuous on int$I$ and $f'(a) \neq 0$. In this hypotheses, passing to limit with $x \to a$ in relation (2) written in the form

$$\frac{f(a + (x - a)\theta(x)) - f(a)}{x - a} = [1 - \theta(x)]f'(a + (x - a)\theta(x)),$$

for all $x \in I \setminus \{a\}$, we obtain

$$f'(a) \lim_{x \to a} \theta(x) = (1 - \lim_{x \to a} \theta(x))f'(a),$$

hence

$$\lim_{x \to a} \theta(x) = \frac{1}{2}.$$

Let's consider the function $\overline{\theta} : I \to]0, 1[$ defined by

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \text{if } x \neq a \\ \frac{1}{2}, & \text{if } x = a. \end{cases}$$

Obviously, the function $\overline{\theta}$ is continuous at a. Next, we study the problem of the differentiability of the function $\overline{\theta}$ at $x = a$. For this we remark that, by using Taylor's formula, we have that

$$f(a+(x-a)\theta(x)) = f(a) + \frac{f'(a)}{1!}(x-a)\theta(x) + \frac{f''(a + (x - a)\theta(x)\widetilde{\theta})}{2!}(x-a)^2\theta^2(x) \tag{5}$$

and

$$f'(a + (x - a)\theta(x)) = f'(a) + \frac{f''(a + (x - a)\theta(x)\widehat{\theta})}{1!}(x - a)\theta(x), \tag{6}$$

where $\widetilde{\theta}, \widehat{\theta} \in]0,1[$.

Taking into account the relations (5) and (6), the relation (4) becomes

$$\begin{aligned} &\frac{f'(a)}{1!}\theta(x) + \frac{f''(a+(x-a)\theta(x)\widetilde{\theta})}{2!}(x-a)\theta^2(x) \\ &= [1-\theta(x)][f'(a) + \frac{f''(a+(x-a)\theta(x)\widehat{\theta})}{1!}(x-a)\theta(x)], \end{aligned} \tag{7}$$

or equivalently,

$$2f'(a)\frac{\theta(x)-\frac{1}{2}}{x-a} = -\frac{f''(a+(x-a)\theta(x)\widetilde{\theta})}{2!}\theta^2(x) + \frac{f''(a+(x-a)\theta(x)\widehat{\theta})}{1!}\theta(x)[1-\theta(x)].$$

But $\overline{\theta}(a) = \dfrac{1}{2}$ and so, for every $x \in I \setminus \{a\}$, we have

$$\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a} = \frac{1}{2f'(a)}\left\{-\frac{f''(a+(x-a)\overline{\theta}(x)\widetilde{\theta})}{2!}\overline{\theta}^2(x) + \frac{f''(a+(x-a)\overline{\theta}(x)\widehat{\theta})}{1!}\overline{\theta}(x)[1-\overline{\theta}(x)]\right\}.$$

Here we pass to limit with $x \to a$ and obtain

$$\begin{aligned} \lim_{x\to a}\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a} &= -\frac{1}{2f'(a)}\lim_{x\to a}\frac{f''(a+(x-a)\overline{\theta}(x)\widetilde{\theta})}{2!}\overline{\theta}^2(x) \\ &+\frac{1}{2f'(a)}\lim_{x\to a}\frac{f''(a+(x-a)\overline{\theta}(x)\widehat{\theta})}{1!}\overline{\theta}(x)[1-\overline{\theta}(x)] \end{aligned}$$

which means

$$\overline{\theta}'(a) = -\frac{1}{2f'(a)}\frac{f''(a)}{2}\overline{\theta}^2(a) + \frac{1}{2f'(a)}f''(a)\overline{\theta}(a)[1-\overline{\theta}(a)],$$

hence

$$\overline{\theta}'(a) = \frac{f''(a)}{16f'(a)}.$$

Consequently, we have the following theorem.

Theorem 2 *Let $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ a function satisfying the conditions*

(a) the function f is two times differentiable on I,

(b) the function f'' is continuous on int$I$,

(c) $f'(a) \neq 0$.

Then

1^0 There exists a function $c : I \setminus \{a\} \to I \setminus \{a\}$ such that (2) is true.

2^0 There exists a function $\theta : I \setminus \{a\} \to]0,1[$ such that (4) is true.

3^0 The function $\overline{\theta} : I \to [0,1]$ defined by

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \text{if } x \in I \setminus \{a\} \\ \frac{1}{2}, & \text{if } x = a \end{cases}$$

is differentiable at $x = a$ *and*

$$\overline{\theta}'(a) = \frac{f''(a)}{16f'(a)}.$$

4^0 *The function* $\overline{c} : I \to I$ *defined by*

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in I \setminus \{a\} \\ a, & \text{if } x = a \end{cases}$$

is two times differentiable at $x = a$ *and*

$$\overline{c}'(a) = \frac{1}{2} \text{ and } \overline{c}''(a) = \frac{f''(a)}{8f'(a)}.$$

For studying the differentiability of order two of the function $\overline{\theta}$ at $x = a$ we assume that f is three times differentiable on I, the function f''' is continuous on int$I$ and $f'(a) \neq 0$. Let's consider the following Taylor's formula

$$\begin{aligned} f(a + (x - a)\theta(x)) = f(a) + \frac{f'(a)}{1!}(x - a)\theta(x) + \frac{f''(a)}{2!}(x - a)^2\theta^2(x) \\ + \frac{f'''(a + (x - a)\theta(x)\widetilde{\theta})}{3!}(x - a)^3\theta^3(x), \end{aligned}$$

and

$$\begin{aligned} f'(a + (x - a)\theta(x)) = f'(a) + f''(a)(x - a)\theta(x) \\ + \frac{f'''(a + (x - a)\theta(x)\widehat{\theta})}{2!}(x - a)^2\theta^2(x), \end{aligned}$$

where $\widetilde{\theta}, \widehat{\theta} \in]0, 1[$. Then relation (4) becomes

$$\begin{aligned} &\frac{f'(a)}{1!}\theta(x) + \frac{f''(a)}{2!}(x - a)\theta^2(x) + \frac{f'''(a + (x - a)\theta(x)\widetilde{\theta})}{3!}(x - a)^2\theta^3(x) \\ &= [1 - \theta(x)][f'(a) + f''(a)(x - a)\theta(x) + \frac{f'''(a + (x - a)\theta(x)\widehat{\theta})}{2!}(x - a)^2\theta^2(x)] \end{aligned}$$

or equivalently,

$$\begin{aligned} f'(a)[2\theta(x)-1] = \theta(x)\left[1-\frac{3}{2}\theta(x)\right] f''(a)(x-a) + \frac{1}{2}\theta^2(x) \\ \times\left\{-\frac{f'''(a+(x-a)\theta(x)\widetilde{\theta})}{3}\theta(x) + [1-\theta(x)]f'''(a+(x-a)\theta(x)\widehat{\theta})\right\}(x-a)^2 \end{aligned}$$

where we divide by $x - a$ and by the fact that $\overline{\theta}(a) = \frac{1}{2}$ we obtain

(8)

$$\begin{aligned} 2f'(a)\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a} &= \overline{\theta}(x)\left[1-\frac{3}{2}\overline{\theta}(x)\right] f''(a) + \frac{1}{2}\overline{\theta}^2(x) \\ &\times\left\{-\frac{f'''(a+(x-a)\overline{\theta}(x)\widetilde{\theta})}{3}\overline{\theta}(x) + [1-\overline{\theta}(x)]f'''(a+(x-a)\overline{\theta}(x)\widehat{\theta})\right\}(x-a). \end{aligned}$$

Passing to limit with $x \to a$ in the previous relation we get

$$2f'(a)\overline{\theta}'(a) = \overline{\theta}(a)\left[1 - \frac{3}{2}\overline{\theta}(a)\right] f''(a).$$

Consequently

$$\overline{\theta}'(a) = \frac{f''(a)}{16 f'(a)}.$$

Now, the relation (8) can be rewritten as follows

$$\frac{\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a} - \frac{f''(a)}{16f'(a)}}{x-a} = -\frac{3}{4}\frac{f''(a)}{f'(a)}\frac{\overline{\theta}(x)-\frac{1}{2}}{x-a}\left[\overline{\theta}(x) - \frac{1}{6}\right]$$
$$+\frac{\overline{\theta}^2(x)}{4f'(a)}\left\{f'''(a+(x-a)\overline{\theta}(x)\widehat{\theta})[1-\overline{\theta}(x)] - \frac{f'''(a+(x-a)\overline{\theta}(x)\widetilde{\theta})}{3}\overline{\theta}(x)\right\},$$

for all $x \in I \setminus \{a\}$.

If, in this relation, we pass to limit with $x \to a$ we obtain

$$\frac{1}{2}\overline{\theta}''(a) = -\frac{3}{4}\frac{f''(a)}{f'(a)}\overline{\theta}'(a)\left[\overline{\theta}(a) - \frac{1}{6}\right] + \frac{\overline{\theta}^2(a)}{4f'(a)}\left\{f'''(a)[1-\overline{\theta}(a)] - \frac{f'''(a)}{3}\overline{\theta}(a)\right\}.$$

Using that $\overline{\theta}(a) = \dfrac{1}{2}$ and $\overline{\theta}'(a) = \dfrac{f''(a)}{16f'(a)}$ we obtain

$$\overline{\theta}''(a) = \frac{4f'(a)f'''(a) - 3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2}.$$

Hence, we have the following result.

Theorem 3 *Let $I \subseteq \mathbb{R}$ be an interval, a an interior point of I and $f : I \to \mathbb{R}$ a function satisfying the conditions*

(a) the function f is three times differentiable on I,

(b) the function f''' is continuous on $\mathrm{int} I$,

(c) $f'(a) \neq 0$.

Then the function $\overline{\theta}$ is two times differentiable at $x = a$,

$$\overline{\theta}'(a) = \frac{f''(a)}{16f'(a)}$$

and

$$\overline{\theta}''(a) = \frac{4f'(a)f'''(a) - 3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2}.$$

For studying the differentiability of order three of the function $\overline{\theta}$ at $x = a$ we assume that f is four times differentiable, $f^{(4)}$ is continuous on int$I$ and $f'(a) \neq 0$. Let's consider the following Taylor's formula

$$f(a+(x-a)\theta(x)) = f(a) + \frac{f'(a)}{1!}(x-a)\theta(x) + \frac{f''(a)}{2!}(x-a)^2\theta^2(x)$$
$$+\frac{f'''(a)}{3!}(x-a)^3\theta^3(x) + \frac{f^{(4)}(a+(x-a)\theta(x)\widetilde{\theta})}{4!}(x-a)^4\theta^4(x),$$

and

$$f'(a+(x-a)\theta(x)) = f'(a) + f''(a)(x-a)\theta(x) + \frac{f'''(a)}{2!}(x-a)^2\theta^2(x)$$
$$+\frac{f^{(4)}(a+(x-a)\theta(x)\widehat{\theta})}{3!}(x-a)^3\theta^3(x),$$

where $\widetilde{\theta}, \widehat{\theta} \in]0,1[$. Then relation (4) becomes, after the calculus,

(9)
$$2f'(a)\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a} = \overline{\theta}(x)\left[1-\frac{3}{2}\overline{\theta}(x)\right]f''(a)$$
$$+\frac{\overline{\theta}^2(x)}{2}\left[1-\frac{4}{3}\overline{\theta}(x)\right]f'''(a)(x-a)$$
$$+\frac{\overline{\theta}^3(x)}{6}\left\{-\frac{f^{(4)}(A(x))}{4}\overline{\theta}(x)+[1-\overline{\theta}(x)]f^{(4)}(B(x))\right\}(x-a)^2,$$

where $A(x) = a + (x-a)\overline{\theta}(x)\widetilde{\theta}$ and $B(x) = a + (x-a)\overline{\theta}(x)\widehat{\theta}$.

Passing to limit with $x \to a$ in the previous relation we get

$$\overline{\theta}'(a) = \frac{f''(a)}{16f'(a)}.$$

Now, the relation (9) can be rewritten as follows

$$\frac{\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a}-\frac{f''(a)}{16f'(a)}}{x-a} = -\frac{3}{4}\frac{f''(a)}{f'(a)}\frac{\overline{\theta}(x)-\frac{1}{2}}{x-a}\left[\overline{\theta}(x)-\frac{1}{6}\right]+\frac{\overline{\theta}^2(x)}{4}\left[1-\frac{4}{3}\overline{\theta}(x)\right]\frac{f'''(a)}{f'(a)} \tag{10}$$
$$+\frac{\overline{\theta}^3(x)}{12}\left\{-\frac{f^{(4)}(A(x))}{4}\overline{\theta}(x)+[1-\overline{\theta}(x)]f^{(4)}(B(x))\right\}\frac{(x-a)}{f'(a)}.$$

If, in this relation, we pass to limit with $x \to a$ we obtain

$$\overline{\theta}'(a) = \frac{f''(a)}{16f'(a)}$$

and

$$\overline{\theta}''(a) = \frac{4f'(a)f'''(a) - 3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2}.$$

Then relation (10) can be rewritten as follows

$$\frac{\dfrac{\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a}-\frac{f''(a)}{16f'(a)}}{x-a}-\dfrac{4f'(a)f'''(a)-3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2}}{x-a} \tag{11}$$
$$=\frac{[-36\overline{\theta}^2(x)+24\overline{\theta}(x)-3]f''(a)+[12\overline{\theta}^2(x)-16\overline{\theta}^3(x)]f'''(a)(x-a)}{48f'(a)(x-a)^2}$$
$$-\frac{4f'(a)f'''(a)-3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2(x-a)}$$
$$+\frac{\overline{\theta}^3(x)}{12f'(a)}\left\{-\frac{f^{(4)}(A(x))}{4}\overline{\theta}(x)+[1-\overline{\theta}(x)]f^{(4)}(B(x))\right\}.$$

Passing to limit with $x\to a$ we get

$$\frac{1}{3!}\overline{\theta}'''(a)=\lim_{x\to a}\frac{[-72\overline{\theta}(x)\overline{\theta}'(x)+24\overline{\theta}'(x)]f''(a)+[12\overline{\theta}^2(x)-16\overline{\theta}^3(x)]f'''(a)}{96f'(a)(x-a)} \tag{12}$$
$$+\lim_{x\to a}\frac{[24\overline{\theta}'(x)-48\overline{\theta}^2(x)\overline{\theta}'(x)]f'''(a)(x-a)-48\overline{\theta}''(a)f'(a)}{96f'(a)(x-a)}-\frac{f^{(4)}(a)}{256f'(a)}$$

and so

$$\overline{\theta}'''(a)=-\frac{3}{128}\frac{\left[f''(a)\right]^3+\left[f'(a)\right]^2f^{(4)}(a)}{\left[f'(a)\right]^3}-\frac{3}{4}\frac{f''(a)}{f'(a)}\overline{\theta}''(a).$$

Hence, we have the following result.

Theorem 4 *Let $I\subseteq\mathbb{R}$ be an interval, a an interior point of I and $f:I\to\mathbb{R}$ a function satisfying the conditions*

(a) the function f is four times differentiable on I,

(b) the function $f^{(4)}$ is continuous on int$I$,

(c) $f'(a)\neq 0$.

Then the function $\overline{\theta}$ is three times differentiable at $x=a$,

$$\overline{\theta}'(a)=\frac{f''(a)}{16f'(a)},\ \overline{\theta}''(a)=\frac{4f'(a)f'''(a)-3\left[f''(a)\right]^2}{96\left[f'(a)\right]^2}$$

and

$$\overline{\theta}'''(a)=-\frac{3}{128}\frac{\left[f''(a)\right]^3+\left[f'(a)\right]^2f^{(4)}(a)}{\left[f'(a)\right]^3}-\frac{3}{4}\frac{f''(a)}{f'(a)}\overline{\theta}''(a).$$

3 Conclusions and further challenges

For the function intermediate point given by formula (2) we gave sufficient differentiability conditions of superior order. The differentiability of order $n\geq 4$ needs some complicated calculus that will be given in a following paper.

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Characterization theorems of Jacobi and Laguerre polynomials [1]

Ioan Ţincu

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this paper we prove a property of the Jacobi polynomials using the interpolation polynomial of Hermite and prove a property of the Laguerre polynomials.

2010 Mathematics Subject Classification: 33C45, 33C52.

Key words and phrases: Jacobi, Laguerre, Hermite interpolation formula.

1 Introduction

For $\alpha > -1$, let $R_n^{(\alpha)}(x) = {}_2F_1\left(-n, n+2\alpha+1; \alpha+1 = \dfrac{1-x}{2}\right), x \in [-1,1]$, be the Jacobi polynomials of degree n normalized by $R_n^{(\alpha)}(1) = 1$ and $L_n^{(\alpha)}(x) = \dfrac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} e^x x^{-\alpha}(e^{-x}x^{n+\alpha})^{(n)}, x \geq 0$ be the polynomials Laguerre of degree n normalized $L_n^{(\alpha)}(0) = 1$.

The following formulas are known:

(1) $\quad (1-x^2)y''(x) - (2\alpha+2)xy'(x) + n(n+2\alpha+1)y(x) = 0, y(x) = R_n^{(\alpha)}(x),$

(2) $\quad R_{n+1}^{(\alpha)}(x) = \dfrac{2n+2\alpha+1}{n+2\alpha+1} x R_n^{(\alpha)}(x) - \dfrac{n}{n+2\alpha+1} R_{n-1}^{(\alpha)}(x),$

(3) $\quad (1-x^2)\dfrac{d}{dx} R_n^{(\alpha)}(x) = -nxR_n^{(\alpha)}(x) + nR_{n-1}^{(\alpha)}(x),$

(4) $\quad xy''(x) + (1+\alpha-x)y'(x) + ny(x) = 0,\ y(x) = L_n^{(\alpha)}(x),$

(5) $\quad (n+\alpha+1)L_{n+1}^{(\alpha)}(x) + (x-\alpha-2n-1)L_n^{(\alpha)}(x) + nL_{n-1}^{(\alpha)}(x) = 0,$

(6) $\quad x\dfrac{d}{dx} L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - nL_{n-1}^{(\alpha)}(x)$

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and the Christoffel-Darboux formula's

$$\sum_{k=0}^{n} w_k R_k^{(\alpha)}(x) R_k^{(\alpha)}(y) = \lambda_n \frac{R_n^{(\alpha)}(y) R_{n+1}^{(\alpha)}(x) - R_n^{(\alpha)}(x) R_{n+1}^{(\alpha)}(y)}{x-y}, x \neq y \tag{7}$$

where

$$\frac{1}{w_k} = \int_{-1}^{1} [R_k^{(\alpha)}(x)]^2 (1-x^2)^\alpha dx, \ k = \overline{0,n},$$

$$\lambda_n = \frac{\Gamma(n+2\alpha+2)}{2^{2\alpha+1}\Gamma^2(\alpha+1) n!}$$

From (7), for $y \to x$ we obtain

$$\sum_{k=0}^{n} w_k [R_k^{(\alpha)}(x)]^2 = \lambda_n [R_n^{(\alpha)}(x)[R_{n+1}^{(\alpha)}(x)]' - R_{n+1}^{(\alpha)}(x)[R_n^{(\alpha)}(x)]']. \tag{8}$$

2 Main Results

Let $f : [-1,1] \to \mathbb{R}, f(x) = (1-x^2)\{R_n^\alpha(x)[R_{n+1}^{(\alpha)}(x)]' - R_{n+1}^{(\alpha)}(x)[R_n^{(\alpha)}(x)]'\}$.
Using (2), (3) and (8) one finds

$$f(x) = (n+1)[R_n^{(\alpha)}(x)]^2 - xR_n^{(\alpha)}(x)R_{n+1}^{(\alpha)}(x) - nR_{n-1}^{(\alpha)}(x)R_{n+1}^{(\alpha)}(x), f \in \Pi_{2n+2}. \tag{9}$$

Notation: $R_n^{(\alpha)}(x) = R_n(x), n \in \mathbb{N}$.
According to Hermite interpolation formula

$$\begin{aligned} f(x) &= H_{2n+2}(x_1, x_1, x_2, x_2, ..., x_{n+1}, x_{n+1}, c; f/x) \\ &= \left[\frac{R_{n+1}(x)}{R_{n+1}(x)}\right]^2 f(c) + (x-c) \sum_{k=1}^{n+1} \frac{\varphi_k(x)}{x_k - x} B_k(f;x), \end{aligned} \tag{10}$$

where

$x_1, x_2, ..., x_{n+1}$ are the roots of $R_{n+1}(x)$,
$c \in [-1,1], c \neq x_k, \ k = \overline{1, n+1}$
$\varphi_k(x) = \left[\dfrac{R_{n+1}(x)}{(x-x_k)R'_{n+1}(x_k)}\right]^2$
$B_k(f;x) = f(x_k) + (x-x_k)\left[f'(x_k) - \dfrac{R''_{n+1}(x_k)}{R'_{n+1}(x_k)} f(x_k) - \dfrac{f(x_k)}{x_k - c}\right]$.
Further, we investigate $B_k(f;x)$.
From (9) we obtain

$$f(x_k) = (n+1)R_n^2(x_k). \tag{11}$$

We have

$$\begin{aligned} f'(x) = 2(n+1)R_n(x)R'_n(x) - R_n(x)R_{n+1}(x) - xR'_n(x)R_{n+1}(x) - xR_n(x)R'_{(n+1)}(x) \\ - nR'_{n-1}(x)R_{n+1}(x) - nR_{n-1}(x)R'_{(n+1)}(x), \end{aligned}$$

$$f'(x_k) = 2(n+1)R_n(x_k)R'_n(x_k) - x_kR_n(x_k)R'_{n+1}(x_k) - nR_{n-1}(x_k)R'_{n+1}(x_k).$$

Using (2), (3) we find

$$
\begin{aligned}
R_{n-1}(x_k) &= \frac{2n+2\alpha+1}{n} x_k R_n(x_k),\\
R_n'(x_k) &= (n+2\alpha+1)\frac{x_k}{1-x_k^2} R_n(x_k),\\
f'(x_k) &= 2\alpha(n+1)\frac{x_k}{1-x_k^2} R_n^2(x_k).
\end{aligned}
$$

For $c=1$, observe that

$$B_k(f;x) = (n+1)\frac{1+x}{1+x_k} R_n^2(x_k), \; f(1)=0,$$

$$f(x) = \frac{(1-x^2)R_{n+1}^2(x)}{n+1}\sum_{k=1}^{n+1}\frac{1-x_k^2}{(x-x_k)^2}. \tag{12}$$

From (2), (3), (9) and (12) we obtain

$$
\begin{aligned}
&n\frac{R_{n-1}(x)}{R_{n+1}(x)} = (2n+2\alpha+1)x\frac{R_n(x)}{R_{n+1}(x)} - (n+2\alpha+1),\\
&(n+1)\left[\frac{R_n(x)}{R_{n+1}(x)}\right]^2 - x\frac{R_n(x)}{R_{n+1}(x)} - n\frac{R_{n-1}(x)}{R_{n+1}(x)} = \frac{1-x^2}{n+1}\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2},\\
&(n+1)\left[\frac{R_n(x)}{R_{n+1}(x)}\right]^2 - (2n+2\alpha+2)x\frac{R_n(x)}{R_{n+1}(x)} + (n+2\alpha+1) = \frac{1-x^2}{n+1}\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2},\\
&\frac{R_n(x)}{R_{n+1}(x)} = x + \frac{1-x^2}{n+1}\cdot\frac{R_{n+1}'(x)}{R_{n+1}(x)},\\
&\frac{R_{n+1}'(x)}{R_{n+1}(x)} = \sum_{k=1}^{n+1}\frac{1}{x-x_k},\\
&(n+2\alpha+1)(1-x^2) - 2\alpha x\frac{1-x^2}{n+1}\cdot\sum_{k=1}^{n+1}\frac{1}{x-x_k} + \frac{(1-x^2)^2}{n+1}\sum_{k=1}^{n+1}\frac{1}{(x-x_k)^2}\\
&+2\frac{(1-x^2)^2}{n+1}\sum_{1\le i<k\le n+1}\frac{1}{(x-x_i)(x-x_k)^2} = \frac{(1-x^2)^2}{n+1}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\\
&+\frac{2x(1-x^2)}{n+1}\sum_{k=1}^{n}\frac{1}{x-x_k} - (1-x^2),\\
&n+2\alpha+2 - \frac{2(\alpha+1)x}{n+1}\sum_{k=1}^{n+1}\frac{1}{x-x_k} + \frac{2(1-x^2)}{n+1}\sum_{1\le i<k\le n+1}\frac{1}{(x-x_i)\cdot(x-x_k)} = 0.
\end{aligned}
$$

In conclusion, we proof the following theorems:

Theorem 1 *If* $\{x_1, x_2, ..., x_{n+1}\} \subset \mathbb{R},\ x_i \neq x_j$ *for* $i \neq j, i, j \in \{1, 2, ..., n+1\}$ *verifies*

$$\frac{1-x^2}{n+1} \sum_{1\leq i<k\leq n+1} \frac{1}{(x-x_i)(x-x_k)} - \frac{\alpha+1}{n+1} x \sum_{k=1}^{n+1} \frac{1}{x-x_k} + \frac{n+2\alpha+2}{2} = 0, \tag{13}$$

$x \in (-1, 1), x \neq x_k,\ k = \overline{1, n+1}$ *then* $x_k \in (-1, 1),\ k = \overline{1, n+1}$ *(or* $R_{n+1}^{(\alpha)}(x_k) = 0,\ k = \overline{1, n+1}$*).*

Theorem 2 *If* $\{x_1, x_2, ..., x_n\} \subset \mathbb{R},\ x_i \neq x_j$ *for* $i \neq j,\ i, j \in \{1, 2, ..., n\}$, $x_j \neq 0, j \in \{1, 2, ..., n\}$ *verifies*

$$\frac{x_j - \alpha - 1}{2x_j} = \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{x_j - x_k}, \tag{14}$$

for $\alpha > -1$ *then* $x_j > 0, (\forall) j = \overline{1, n}$.

Proof. Let $P(x) = \prod_{k=1}^{n} (x - x_k)$. We obtain

$$\frac{P'(x)}{P(x)} - \frac{1}{x - x_j} = \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{x - x_k},$$

$$\lim_{x\to x_j} \frac{(x - x_j)P'(x) - P(x)}{(x - x_j)P(x)} = \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{x_j - x_k},$$

$$\sum_{\substack{k=1\\k\neq l}} \frac{1}{x_j - x_k} = \frac{P''(x_j)}{2P'(x_j)}. \tag{15}$$

From (14) and (15) we have

$$\frac{P''(x_j)}{2P'(x_j)} = \frac{x_j - \alpha - 1}{2x_j},$$

$$x_j P''(x_j) = (x_j - \alpha - 1)P'(x_j), j \in \{1, 2, ..., n\}. \tag{16}$$

Let $h(x) = xP'' + (\alpha + 1 - x)P'(x)$. From (16), we observe $h(x_j) = 0,\ j = 1, 2, ..., n$. In conclusion exists $c_n \in \mathbb{R}^*$ such that $h(x) = c_n P(x)$, then

$$xP''(x) + (\alpha + 1 - x)P'(x) - c_n P(x) = 0. \tag{17}$$

Because $\{L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), ..., L_n^{(\alpha)}(x)\}$ is base in Π_n, exists $b_k \in \mathbb{R}, k \in \{0, 1, ..., n\}$ such that

$$P(x) = \sum_{k=0}^{n} b_k L_k^{(\alpha)}(x).$$

From (4) and (17) we obtain

$$b_k = 0,\ k \in \{0, 1, ..., n-1\},\ c_n = -n.$$

In conclusion, the polynomial P verifies following identity

$$xP''(x) + (\alpha + 1 - x)P'(x) + nP(x) = 0,$$

hence it exists $\lambda_n \in \mathbb{R}$ such that $P(x) = \lambda_n L_n^{(\alpha)}(x)$.

Therefore $x_j > 0,\ j - \overline{1, n}$.

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Interpolation operators on a triangle with one curved side [1]

Alina Baboş

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

We construct Hermite and Birkhoff-type operators, which interpolate a given function and some of its derivatives on some interior lines of a triangle with one curved side. We consider the case when the interior line is a median. We also consider some of their product and Boolean sum operators. We study the interpolation properties and the degree of exactness of the constructed operators.

2010 Mathematics Subject Classification: 65D30, 65D32, 26A15.
Key words and phrases: Triangle, curved edges, interpolation operators.

1 Introduction

There have been constructed interpolation operators of Lagrange, Hermite and Birkhoff type on a triangle with all straight sides, starting with the paper [5] of R.E. Barnhil, G. Birkhoff and W.J. Gordon, and in many others papers (see, e.g., [4], [6], [7], [10], [11]). Then, were considered interpolation operators on triangles with curved sides (one, two or all curved sides), many of them in connection with their applications in computer aided geometric design and in finite element analysis (see, e.g, [1], [2], [8],[9], [12]-[15]).

In [12] the authors consider a standard triangle,$\tilde{T}_h$, having the vertices $V_1 = (h, 0)$, $V_2 = (0, h)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e. $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$ and F a real-valued function defined on $\tilde{T}_h$ (see Figure 1).

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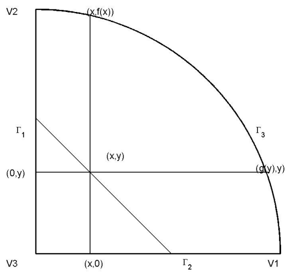

Figure 1.

They construct certain Lagrange, Hermite and Birkhoff type operators, which interpolate a given function and some of its derivatives on the border of this triangle with one curved side, as well as some of their product and Boolean sum operators.

In [1] we introduce an Lagrange operator which interpolate the function F on cathetus, on the curved side, but also on a interior line of the triangle $\tilde{T}_h$. We considered the case when the interior line is a median. Thus we have introduced the operator L_2^x which interpolate the function F in relation to x in points $(0, y)$, $\left(\frac{h-y}{2}, y\right)$ and $(g(y), y)$:

$$\begin{aligned} (L_2^x F)(x, y) &= \frac{(2x - h + y)[x - g(y)]}{(h-y)g(y)} F(0, y) + \frac{4x[x - g(y)]}{(h-y)[h- y - 2g(y)]} F\left(\frac{h - y}{2}, y\right) \\ &+ \frac{x(2x - h - y)}{g(y)[2g(y) - h + y]} F(g(y), y)). \end{aligned}$$

So, the operator L_2^x interpolate the function F on cathetus V_2V_3, on curved side and on median V_2M (see Figure 2).

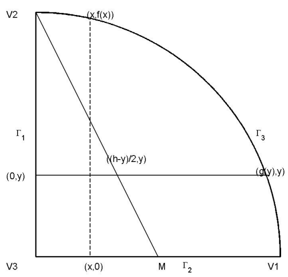

Figure 2.

In this paper we construct Hermite and Birkhoff-type operators, which interpolate a given function and some of its derivatives on the median. We also consider some of their product and Boolean sum operators. We study the interpolation properties and the degree of exactness of the constructed operators. Recall that $dex(P) = r$ (where P is an interpolation operators) if $Pf = f$, for $f \in \mathbb{P}_r$, and there exists $g \in \mathbb{P}_{r+1}$ such that $Pg \neq g$, where $\mathbb{P}_m$ denote the space of the polynomials in two variables of global degree at most m.

2 Hermite and Birkhoff-type operators

Let H_1 be the Hermite operator who interpolate the function F with respect to x in the points $(0, y)$, $\left(\frac{h-y}{2}, y\right)$, $(g(y), y)$:

$$
\begin{aligned}
(H_1F)(x,y) &= \frac{(2x-h+y)^2[x-g(y)]^2}{(h-y)^2g^2(y)}F(0,y)\\
&+ \frac{16x[x-g(y)]^2[2(h-y-g(y))(h-y-x)-x(h-y)]}{(h-y)^2[h-y-2g(y)]^3}F\Big(\frac{h-y}{2},y\Big)\\
&+ \frac{4x[x-g(y)]^2(2x-h+y)}{(h-y)[h-y-2g(y)]^2}F^{(1,0)}\Big(\frac{h-y}{2},y\Big) \qquad (1)\\
&+ \frac{x(2x-h+y)^2[2g(y)(4g(y)-h+y)+x(h-y-6g(y))]}{g^2(y)[2g(y)-h+y]^3}F(g(y),y)\\
&+ \frac{x[x-g(y)](2x-h+y)^2}{g(y)[2g(y)-h+y]^2}F^{(1,0)}(g(y),y).
\end{aligned}
$$

Theorem 1 *If $F : \tilde{T}_h \to R$, then we get*

1. *the interpolation properties:*

$$
\begin{aligned}
H_1F &= F, \ on\,\Gamma_1 \cup \Gamma_3 \cup V_2M,\\
H_1F^{(1,0)} &= F^{(1,0)}, \ on\,\Gamma_3 \cup V_2M,
\end{aligned}
$$

(see Figure 3).

2. *the degree of exactness: $dex(H_1) = 2$.*

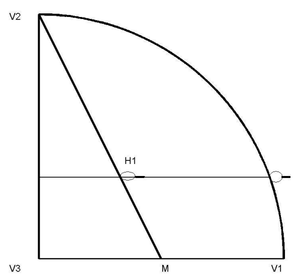

Figure 3.

Proof. 1.

$$(H_1F)(0,y) = F(0,y),$$

$$(H_1F)\Big(\frac{h-y}{2}, y\Big) = F\Big(\frac{h-y}{2}, y\Big),\ (H_1F)(g(y),y) = F(g(y),y),$$

$$(H_1F)^{(1,0)}\Big(\frac{h-y}{2}, y\Big) = F^{(1,0)}\Big(\frac{h-y}{2}, y\Big),\ (H_1F)^{(1,0)}(g(y),y) = F^{(1,0)}(g(y),y).$$

2. We obtain $H_1e_{ij} = e_{ij}$ for $i+j \leq 2$ and $H_1e_{30} \neq e_{30}$, where $e_{ij}(x,y) = x^iy^j$. So, it follows that $dex(H_1) = 2$.

Let H_2 be the interpolation operator defined in [12]:

$$\begin{aligned}(H_2F) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0) + \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))\\ &+ \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)).\end{aligned}$$

The product of the operators H_1 and H_2 is given by

$$\begin{aligned}(P_{21}F)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}\Big[\frac{(2x-h)^2(x-h)^2}{h^4}F(0,0) + \frac{16x^2(x-h)^2}{h^4}F\Big(\frac{h}{2},0\Big)\\ &+ \frac{4x(x-h)^2(2x-h)}{h^3}F^{(1,0)}\Big(\frac{h}{2},0\Big)\\ &+ \frac{x(2x-h)^2(6h-5x)}{h^4}F(h,0) + \frac{x(x-h)(2x-h)^2}{h^3}F^{(1,0)}(h,0)\Big]\\ &+ \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x)) + \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x))\end{aligned}$$

with the interpolation properties:

$$P_{21}F = F, on\, \Gamma_2 \cup \Gamma_3,$$

$$(P_{21}F)^{(1,0)} = F^{(1,0)}, on\, \Gamma_2,\ (P_{21}F)^{(1,0)} = F^{(0,1)}, on\, \Gamma_3,$$

(see Figure 4) and the degree of exactness: $dex(P_{21}) = 2$.

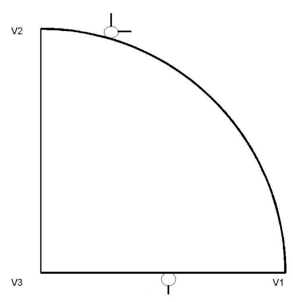

Figure 4.

The boolean sum of the operators H_1 and H_2 is given by

$$
\begin{aligned}
(S_{21}F)(x,y) &= H_2 \oplus H_1 = H_2 + H_1 - H_2 \cdot H_1 \\
&= \frac{[y-f(x)]^2}{f^2(x)} F(x,0) + \frac{(2x-h+y)^2[x-g(y)]^2}{(h-y)^2 g^2(y)} F(0,y) \\
&+ \frac{16x[x-g(y)]^2[2(h-y-g(y))(h-y-x)-x(h-y)]}{(h-y)^2[h-y-2g(y)]^3} F\Big(\frac{h-y}{2},y\Big) \\
&+ \frac{4x[x-g(y)]^2(2x-h+y)}{(h-y)[h-y-2g(y)]^2} F^{(1,0)}\Big(\frac{h-y}{2},y\Big) \\
&+ \frac{x(2x-h+y)^2[2g(y)(4g(y)-h+y)+x(h-y-6g(y))]}{g^2(y)[2g(y)-h+y]^3} F(g(y),y) \\
&+ \frac{x[x-g(y)](2x-h+y)^2}{g(y)[2g(y)-h+y]^2} F^{(1,0)}(g(y),y) \\
&- \frac{[y-f(x)]^2}{f^2(x)} \Big[\frac{(2x-h)^2(x-h)^2}{h^4} F(0,0) + \frac{16x^2(x-h)^2}{h^4} F\Big(\frac{h}{2},0\Big) \\
&+ \frac{4x(x-h)^2(2x-h)}{h^3} F^{(1,0)}\Big(\frac{h}{2},0\Big) \\
&+ \frac{x(2x-h)^2(6h-5x)}{h^4} F(h,0) + \frac{x(x-h)(2x-h)^2}{h^3} F^{(1,0)}(h,0) \Big]
\end{aligned}
$$

with the interpolation properties:

$$
\begin{gathered}
S_{21}F = F, on\, \partial T_h \cup V_2M, \\
(S_{21}F)^{(1,0)} = F^{(1,0)}, on\, \Gamma_3 \cup V_2M,
\end{gathered}
$$

and the degree of exactness: $dex(S_{21}^H) = 2$.

We suppose that the function $F : \tilde{T}_h \to R$ has the partial derivatives $F^{(1,0)}$ on Γ_3 and V_2M.

We consider the Birkhoff-type operator B_1 defined by

$$
\begin{aligned}
(2) \qquad (B_1F)(x,y) &= F(0,y) + \frac{x[2g(y)-x]}{2g(y)-h+y} F^{(1,0)}\Big(\frac{h-y}{2},y\Big) \\
&+ \frac{x(x-h+y)}{2g(y)-h+y} F^{(1,0)}(g(y),y).
\end{aligned}
$$

Theorem 2 *If $F : \tilde{T}_h \to R$, then we get*

1. *the interpolation properties:*

$$
\begin{aligned}
B_1F &= F, \; on\, \Gamma_1, \\
(B_1F)^{(1,0)} &= F^{(1,0)} on\, \Gamma_3 \cup V_2M,
\end{aligned}
$$

(see Figure 5).

2. the degree of exactness: $dex(B_1) = 2$.

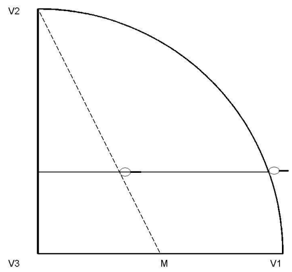

Figure 5.

Proof. 1.

$$(B_1 f)(0, y) = F(0, y)$$

$$(B_1 F)^{(1,0)}\Big(\frac{h-y}{2}, y\Big) = F^{(1,0)}\Big(\frac{h-y}{2}, y\Big), \ (B_1 F)^{(1,0)}(g(y), y) = F^{(1,0)}(g(y), y).$$

2. $B_1 e_{ij} = e_{ij}$ for $i+j \leq 2$ and $B_1 e_{30} \neq e_{30}$, where $e_{ij}(x, y) = x^i y^j$. So, it follows that $dex(B_1) = 2$.

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Some inequalities for the Landau constants [1]

Emil C. Popa

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

In this paper we obtain some new inequalities for the Landau constants.

2010 Mathematics Subject Classification: 26D15, 26D20, 33C20, 33B15.
Key words and phrases: Wallis inequality, Zhang inegualities, Landau constants, Riemann Zeta function.

1 Introduction and results

Let $A_n = \dfrac{(2n-1)!!}{(2n)!!}$, $n = 1, 2, \ldots$ directly related to the Wallis formula

$$\lim_{n\to\infty} \frac{1}{nA_n^2} = \pi \tag{1}$$

and the Wallis inequalities

$$\frac{2}{2n+1} < \pi A_n^2 < \frac{1}{n}, \quad n = 1, 2, \ldots.$$

G.M. Zhang [6] has proved the double inequality

$$\frac{32n+8}{32n^2+16n+3} < \pi A_n^2 < \frac{8n+3}{8n^2+5n+1}, \quad n = 1, 2, \ldots \tag{2}$$

Theorem 1 *For $n \in \mathbb{N}$, $n \geq 1$ we have*

$$\frac{32n^2+16n+3}{32n^3+24n^2+8n+1} < \pi A_n^2. \tag{3}$$

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Theorem 2 *For $n \in \mathbb{N}$, $n \geq 3$ we have*

$$\pi A_n^2 < \frac{8n^2+35n+8}{8n^3+37n^2+17n+3}. \tag{4}$$

Remark 1 *We observe that*

$$\frac{32n+8}{32n^2+16n+3} < \frac{32n^2+16n+3}{32n^3+24n^2+8n+1} < \pi A_n^2 <$$
$$\frac{8n^2+35n+8}{8n^3+37n^2+17n+3} < \frac{8n+3}{8n^2+5n+1}$$

Hence (3) is an improvement for the lower bound in Zhang's inequality (2) and the inequality (4) is an improvement for the upper bound in Zhang's inequality (2).

The sums

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k}\binom{2k}{k}^2, \quad n = 0, 1, 2, \ldots$$

are known as the Landau constants. It was proved by Landau in 1913, that, if a function $f(z)$ such that $|f(z)| < 1$ for $|z| < 1$, is analytic throughout the interior of the unit circle and its Taylor series is

$$f(z) = \sum_{i=0}^{\infty} a_i z^i, \quad \text{then} \quad \left|\sum_{i=0}^{n} a_i\right| \leq G_n.$$

Moreover, if $T_n(f)$ is the Taylor polynom associated to $f(z)$, then its norm is given by $||T_n|| = G_n$.

D. Zhao establishes in [7] several inequalities for G_n.

$$\frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} < G_n$$

$$G_n < \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3}, \tag{5}$$

where $n \geq 1$, $c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1,06627\ldots$, and $\gamma = 0,57721\ldots$ denotes Euler's constant.

Using the Riemann Zeta function, we obtained in [4] and [5] some improvements of inequalities (5) in the following evaluations

$$\frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{17}{256\pi}\left(\zeta(4) - \sum_{k=1}^{n+1}\frac{1}{k^4}\right) < G_n$$

$$G_n < \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} \tag{6}$$

$$+\frac{9}{128\pi}\left(\zeta(4) - \sum_{k=1}^{n}\frac{1}{k^4}\right) - \frac{2263}{61440\pi(n+1)^4},$$

where $\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$ is the Riemann - Zeta function.

In this paper we will establish some sharp inequalities for G_n wich improve the inequalities (6).

Theorem 3 *We have for* $n\geq 3$

$$\frac{1}{\pi}\ln(n+2)+c_0-\frac{1}{4\pi(n+2)}+\frac{5}{192\pi(n+2)^2}$$

(7)
$$+\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+2}\frac{1}{k^4}\right)-\frac{1}{\pi}\cdot\frac{8n^2+51n+51}{8n^3+61n^2+115n+65}<G_n,$$

and for $n\geq 1$
(8)
$$G_n<\frac{1}{\pi}\ln(n+2)+c_0-\frac{1}{4\pi(n+2)}+\frac{5}{192\pi(n+2)^2}$$

$$+\frac{9}{128\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right)-\frac{2263}{61440\pi(n+2)^4}-\frac{1}{\pi}\cdot\frac{32n^2+80n+51}{32n^3+120n^2+152n+65}.$$

2 Proofs of the theorems

We are now able to prove our theorems.
Proof of Theorem 1. We denote

$$\alpha_n=A_n^2\frac{32n^3+24n^2+8n+1}{32n^2+16n+3}.$$

From Wallis formula (1) it follows that

(9)
$$\lim_{n\to\infty}\alpha_n=\frac{1}{\pi}.$$

We have now

$$\alpha_{n+1}-\alpha_n=A_n^2\left(\frac{(2n+1)^2(32n^3+120n^2+152n+65)}{(2n+2)^2(32n^2+80n+51)}-\frac{32n^3+24n^2+8n+1}{32n^2+16n+3}\right)$$
$$=-A_n^2\frac{56n^2+84n+9}{(2n+2)^2(32n^2+80n+51)(32n^2+16n+3)}<0.$$

It is obvious that $\alpha_{n+1}<\alpha_n$ for any $n\in\mathbb{N}$, $n\geq 1$. Next, by using (9) we deduce that $\alpha_n>\frac{1}{\pi}$ and hence

$$\pi A_n^2>\frac{32n^2+16n+3}{32n^3+24n^2+8n+1},\quad n\geq 1.$$

Proof of Theorem 2. We denote

$$\beta_n = A_n^2 \frac{8n^3+37n^2+17n+3}{8n^2+35n+8}.$$

We have

$$\begin{aligned}\beta_{n-1}-\beta_n &= A_n^2\left(\frac{(2n+1)^2(8n^3+61n^2+115n+65)}{(2n+2)^2(8n^2+51n+51)} - \frac{8n^3+37n^2+17n+3}{8n^2+35n+8}\right)\\ &= A_n^2\frac{3n^3+9n^2-29n-92}{(2n+2)^2(8n^2+51n+51)(8n^2+35n+8)} > 0.\end{aligned}$$

and hence $\beta_{n+1} > \beta_n$ for any $n \geq 4$.

By using (9) we deduce that $\beta_n < \dfrac{1}{\pi}$, or

$$\pi A_n^2 < \frac{8n^2+35n+8}{8n^3+37n^2+17n+3}, \quad \text{for } n \geq 3.$$

Proof of Theorem 3. It is easy to verify that

$$G_n = G_{n+1} - A_{n+1}^2 \tag{10}$$

From (3) and (4) we obtain

$$\frac{1}{\pi}\frac{32n^2+80n+51}{32n^3+120n^2+152n+65} < A_{n+1}^2 < \frac{1}{\pi}\frac{8n^2+51n+51}{8n^3+61n^2+115n+65}. \tag{11}$$

Using (6) and (11), we obtain from (10) the inequalities (7) and (8).

Remark 2 *For $n \in \mathbb{N}$, $n \geq 1$ we have*

$$\begin{aligned}\frac{8n^2+51n+51}{8n^3+61n^2+115n+65} &< \ln\left(1+\frac{1}{n+1}\right)+\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\\ &-\frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{17}{256(n+2)^2}.\end{aligned} \tag{12}$$

Proof. From [4, p. 115-116] we have

$$\begin{aligned}\frac{8n+3}{8n^2+5n+1} &< \ln\left(1+\frac{1}{n}\right)+\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &-\frac{5}{192}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)-\frac{17}{256(n+1)^2}.\end{aligned}$$

But

$$\frac{8n^2+51n+51}{8n^3+61n^2+115n+65} < \frac{8n+11}{8n^2+21n+14}$$

$$< \ln\left(1+\frac{1}{n+1}\right)+\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{17}{256(n+2)^2},$$

and the inequality (12) follows.

Notice that, according to Remark 2 the first inequality (7) is an improvment of the first inequality from (6).

Remark 3 *For $n \geq 1$ we have*

$$\frac{32n^2+80n+51}{32n^3+120n^2+152n+65} > \ln\left(1+\frac{1}{n+1}\right)+\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$-\frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)-\frac{9}{128(n+1)^4}+\frac{2263}{61440}\left(\frac{1}{(n+1)^4}-\frac{1}{(n+2)^4}\right).$$

Proof. From [5, p. 1459] we obtain

$$\frac{32n+8}{32n^2+16n+8} > \ln\left(1+\frac{1}{n}\right)+\frac{1}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

(13)

$$-\frac{5}{192}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)-\frac{9}{128n^4}+\frac{2263}{61440}\left(\frac{1}{n^4}-\frac{1}{(n+1)^4}\right).$$

But,

$$\frac{32n^2+80n+51}{32n^3+120n^2+152n+65} > \frac{32n+40}{32n^2+80n+56}$$

$$> \ln\left(1+\frac{1}{n+1}\right)+\frac{1}{4}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\frac{5}{192}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}\right)$$

$$-\frac{9}{128(n+1)^4}+\frac{2263}{61440}\left(\frac{1}{(n+1)^4}-\frac{1}{(n+2)^4}\right),$$

and the inequality (13) follows.

Notice that, according to Remark 3 the second inequality (8) is an improvement of the second inequality (6).

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Error bounds for a class of quadrature formulas [1]

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Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

A class of optimal quadrature formulas in sense of minimal error bounds are obtained. The estimations of remainder term will be given in terms of variety of norms, from an inequality point of view. Some improvements and generalizations of some results from literature will be considered.

2010 Mathematics Subject Classification: 65D30, 65D32, 26A15.
Key words and phrases: optimal quadrature formula, corrected quadrature formula, remainder term.

1 Introduction

In the last years the problem of numerical integration attracted attention of many authors. The deduction of the optimal quadrature formulas, in terms of variety of norms, from an inequality point of view was considered by Ujević ([7]-[10]) who obtained optimal two-point and three-point quadrature formulas. In [12], F. Zafar and N.A. Mir found out an optimal quadrature formula of the form

$$(1)\qquad \int_{-1}^{1} f(t)dt - [hf(-1) + (1-h)f(x) + (1-h)f(y) + hf(1)] = \int_{-1}^{1} K(x,y,t)f''(t)dt, \text{ where } K(x,y,t) \text{ is defined by}$$

$$(2)\qquad K(x,y,t) = \begin{cases} \frac{1}{2}(t-\alpha)^2 + \alpha_1, \; t \in [-1,x], \\ \frac{1}{2}(t-\beta)^2 + \beta_1, \; t \in (x,y), \\ \frac{1}{2}(t-\gamma)^2 + \gamma_1, \; t \in [y,1], \end{cases}$$

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and $x, y \in [-1+2h, 1-2h]$ with $x < y$, $h \in \left[0, \frac{1}{2}\right]$.

The parameters $\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1 \in \mathbb{R}$ involved in $K(x,y,t)$ are required to be determined in a way such that the representation (1) is obtained.

The nodes x and y are obtained putting conditions that the remainder term which is evaluated in the following sense

$$\left| \int_{-1}^{1} K(x,y,t) f''(t) dt \right| \leq \max_{t \in [a,b]} |f''(t)| \int_{-1}^{1} |K(x,y,t)| dt$$

to be minimal, namely $\int_{-1}^{1} |K(x,y,t)| dt$ to attains the minimum value.

F. Zafar and N.A. Mir found the quadrature formula of type (1) such that the estimation of its error to be best possible in uniform norm. The main result obtained by F. Zafar and N.A. Mir in the above described procedure is formulated bellow

Theorem 1 *[12] Let $I \subset \mathbf{R}$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to \mathbf{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then,*

$$\begin{aligned} \int_{-1}^{1} f(t)dt = \Big[& hf(-1) + (1-h)f(-4+4h+2\sqrt{3-6h+4h^2}) \\ & + (1-h)f(4-4h-2\sqrt{3-6h+4h^2}) + hf(1) \Big] + \mathcal{R}[f], \end{aligned} \tag{3}$$

where $|\mathcal{R}[f]| \leq 2\Delta(h) \|f''\|_{\infty}$, $h \in \left[0, \frac{1}{2}\right]$, *and* $\Delta(h)$ *is defined as*

$$\begin{aligned} \Delta(h) = & \frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(1-h)^2\sqrt{4h^2-6h+3} \\ & + \frac{2}{3}\left[8h^2 - 14h + 7 - 4(1-h)\sqrt{4h^2-6h+3}\right]^{\frac{3}{2}}. \end{aligned}$$

In [1] using a similar way by F. Zafar and A. Mir is obtained the following optimal quadrature formula, where the estimation of the error is best possible in L_2-norm.

Theorem 2 *[1] Let $I \subset \mathbf{R}$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to \mathbf{R}$ be a twice differentiable function such that f'' is integrable. Then,*

$$\begin{aligned} \int_{-1}^{1} f(t)dt &= hf(-1) + (1-h)f(-3h+3-\sqrt{9h^2-12h+6}) \\ &+ (1-h)f(3h-3+\sqrt{9h^2-12h+6}) + hf(1) + \mathcal{R}[f], \end{aligned} \tag{4}$$

where $|\mathcal{R}[f]| \leq \Phi(h) \|f''\|_2$, $h \in \left(-\infty, \frac{1}{2}\right]$, *and* $\Phi(h)$ *is defined as*

$$\begin{aligned} \Phi(h) = \Big\{ & -36h^5 + 144h^4 - 240h^3 + \frac{632}{3}h^2 - 98h + \frac{98}{5} \\ & + \left[-12h^4 + 40h^3 - 52h^2 + 32h - 8\right]\sqrt{9h^2-12h+6} \Big\}^{\frac{1}{2}}. \end{aligned}$$

2 A class of optimal quadrature formulas

The quadrature formulas (3) and (4) can be obtained using Peano's theorem. In this section we will present this method. Also, an optimal quadrature formula, where the estimation of the error is best possible in L_1-norm will be obtained.

We consider the quadrature formula (1) has degree of exactness equal 1. Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}[e_i] = 0$, $e_i(x) = x^i$, $i = 0, 1$, and we obtain $y = -x$. We can choose $x \geq 0$.

Let $I \subset \mathbb{R}$ be an open interval such that $[-1, 1] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is integrable. Using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}[f] = \int_{-1}^{1} K(t) f''(t) dt, \text{ where} \tag{5}$$

$$K(t) = \mathcal{R}[(x-t)_+] = \begin{cases} \dfrac{t^2}{2} + t(1-h) - h + \dfrac{1}{2}, t \in [-1, -x], \\ \dfrac{t^2}{2} - (1-h)x - h + \dfrac{1}{2}, t \in (-x, x), \\ \dfrac{t^2}{2} - t(1-h) - h + \dfrac{1}{2}, t \in [x, 1]. \end{cases}$$

For the remainder term we have the evaluation

$$|\mathcal{R}[f]| \leq \left[\int_{-1}^{1} |f''(t)|^p dt\right]^{\frac{1}{p}} \left[\int_{-1}^{1} |K(t)|^q dt\right]^{\frac{1}{q}}, \ \frac{1}{p} + \frac{1}{q} = 1,$$

with the remark that in the cases $p = 1$ and $p = \infty$ this evaluation is

$$|\mathcal{R}[f]| \leq \int_{-1}^{1} |f''(t)| dt \cdot \sup_{t \in [-1,1]} |K(t)|, \tag{6}$$

$$|\mathcal{R}[f]| \leq \sup_{t \in [-1,1]} |f''(t)| \cdot \int_{-1}^{1} |K(t)| dt. \tag{7}$$

The function $\int_{-1}^{1} |K(t)| dt$ attains the minimum value if $K|_{[-x,x]}$ is the Chebyshev orthogonal polynomial of the second kind of degree 2, on the interval $[-1, 1]$ with the coefficient of t^2 equal to $\frac{1}{2}$, namely

$$\frac{t^2}{2} - (1-h)x - h + \frac{1}{2} = \frac{1}{2} x^2 \tilde{U}_2\left(\frac{t}{x}\right), \tag{8}$$

where $\tilde{U}_2(t) = t^2 - \frac{1}{4}$ is the Chebyshev polynomial of the second kind of degree 2, on the interval $[-1, 1]$. Ecuation (8) has the following solutions

$$x_1 = 4(1-h) - 2\sqrt{4h^2 - 6h + 3},$$
$$x_2 = 4(1-h) + 2\sqrt{4h^2 - 6h + 3}.$$

We find that x_1 is the global minima of $F(x) = \int_{-1}^{1} |K(t)|dt$ and the result of F.Zafar and N.A. Mir is obtained.

The function $\int_{-1}^{1} (K(t))^2\, dt$ attains the minimum value if $K|_{[-x,x]}$ is the Legendre orthogonal polynomial of degree 2, on the interval $[-x, x]$, with the coefficient of t^2 equal to $\frac{1}{2}$, namely

$$\frac{t^2}{2} - (1-h)x - h + \frac{1}{2} = \frac{1}{2}x^2 \tilde{X}_2\left(\frac{t}{x}\right), \tag{9}$$

where $\tilde{X}_2(t) = t^2 - \frac{1}{3}$ is the Legendre orthogonal polynomial of degree 2, on the interval $[-1, 1]$. Equation (9) has the following solutions

$$x_1 = 3(1-h) - \sqrt{9h^2 - 12h + 6},$$
$$x_2 = 3(1-h) + \sqrt{9h^2 - 12h + 6}.$$

We find that x_1 is the global minima of $G(x) = \int_{-1}^{1} (K(t))^2\, dt$ and Theorem 2 is obtained.

In the following Theorem we give a minimal estimation of the error bound in L_1-norm for the quadrature formula (1). In order to obtain this result we will consider the estimation (6).

Theorem 3 *Let $I \subset \mathbf{R}$ be an open interval such that $[-1, 1] \subset I$ and let $f : I \to \mathbf{R}$ be a twice differentiable function such that f'' is integrable. Then,*

$$\begin{aligned}\int_{-1}^{1} f(t)dt = hf(-1) + (1-h)f\left(2 - 2h - \sqrt{4h^2 - 4h + 2}\right)\\ + (1-h)f\left(-2 + 2h + \sqrt{4h^2 - 4h + 2}\right) + hf(1) + \mathcal{R}[f],\end{aligned} \tag{10}$$

where $|\mathcal{R}[f]| \leq \Psi(h)\|f''\|_1$, $h \in \left[0, \frac{1}{2}\right]$, and $\Psi(h)$ is defined as

$$\Psi(h) = \begin{cases} \frac{3}{2} - 3h + 2h^2 + (h-1)\sqrt{4h^2 - 4h + 2}, \; h \in \left[0, -\frac{1}{7} + \frac{2}{7}\sqrt{2}\right] \\ \frac{h^2}{2}, \; h \in \left(-\frac{1}{7} + \frac{2}{7}\sqrt{2}, \frac{1}{2}\right]. \end{cases}$$

Proof. The function $\sup_{t\in[-1,1]} |K(t)|$ attains the minimum value if $K|_{[-x,x]}$ is the Chebyshev orthogonal polynomial of the first kind of degree 2, on the interval $[-x, x]$, with the coefficient of t^2 equal to $\frac{1}{2}$, namely

$$\frac{t^2}{2} - (1-h)x - h + \frac{1}{2} = \frac{1}{2}x^2 \tilde{T}_2\left(\frac{t}{x}\right), \tag{11}$$

where $\tilde{T}_2(t) = t^2 - \frac{1}{2}$ is the Chebyshev orthogonal polynomial of the first kind of degree 2, on the interval $[-1,1]$. Equation (11) has the following solutions

$$x_1 = 2 - 2h - \sqrt{4h^2 - 4h + 2},$$
$$x_2 = 2 - 2h + \sqrt{4h^2 - 4h + 2}.$$

Denote $H(x) = \sup_{t\in[-1,1]} |K(t)|$. We find that $H(x_1) < H(x_2)$, therefore

$$x_1 = 2 - 2h - \sqrt{4h^2 - 4h + 2}$$

is the global minima of $H(x)$.

Remark 1 *For $h = -\frac{1}{7} + \frac{2}{7}\sqrt{2}$, $\Psi(h)$ attains its minimum value.*

Corollary 1 *Let the assumption of Theorem 3 holds. Then, one has following optimal quadrature rule*

$$\int_{-1}^{1} f(t)dt = \frac{1}{7}\left\{(2\sqrt{2}-1)f(-1) + (8-2\sqrt{2})f\left(\frac{4-\sqrt{2}}{7}\right)\right.$$
$$\left. + (8-2\sqrt{2})f\left(\frac{\sqrt{2}-4}{7}\right) + (2\sqrt{2}-1)f(1) + \mathcal{R}[f]\right\},$$

where $|\mathcal{R}[f]| \le \frac{(2\sqrt{2}-1)^2}{98}\|f''\|_1$.

Theorem 4 *For the remainder term of the quadrature formula (10) the following estimations can be established*

(i) $|\mathcal{R}[f]| \le \eta(h)\|f''\|_\infty$, where

$$\eta(h) = \frac{32}{3}h^3 - 20h^2 + \frac{33}{2}h - \frac{11}{2} + 4(h-1)^2\sqrt{4h^2-4h+2}$$
$$+ 4\left(\frac{4}{3}h^2 - 2h + 1 + \frac{2}{3}(h-1)\sqrt{4h^2-4h+2}\right)$$
$$\cdot \sqrt{3 - 6h + 4h^2 + 2(h-1)\sqrt{4h^2-4h+2}};$$

(ii) $|\mathcal{R}[f]| \le \nu(h)\|f''\|_2$, where

$$\nu(h) = \left\{-\frac{64}{3}h^5 + 80h^4 - \frac{376}{3}h^3 + \frac{314}{3}h^2 - \frac{283}{6}h + \frac{283}{30}\right.$$
$$\left. + \sqrt{4h^2-4h+2}\left(\frac{80}{3}h - 44h^2 + \frac{104}{3}h^3 - \frac{32}{3}h^4 - \frac{20}{3}\right)\right\}^{1/2}.$$

Proof. The inequality (i) follows considering the following estimation of remainder term

$$|\mathcal{R}[f]| \leq \|f''\|_\infty \int_{-1}^{1} |K(t)|dt.$$

Denote

$$\begin{aligned} K_1(t) &:= \frac{t^2}{2} + t(1-h) - h + \frac{1}{2}; \\ K_2(t) &:= \frac{t^2}{2} - (1-h)x - h + \frac{1}{2}; \\ K_3(t) &:= \frac{t^2}{2} - t(1-h) - h + \frac{1}{2}. \end{aligned}$$

We have

$$\begin{aligned} \int_{-1}^{1} |K(t)|dt &= \int_{0}^{2h-1} -K_1(t)dt + \int_{2h-1}^{-(2-2h-\sqrt{4h^2-4h+2})} K_1(t)dt \\ &+ \int_{-(2-2h-\sqrt{4h^2-4h+2})}^{-\sqrt{3-6h+4h^2+2(h-1)\sqrt{4h^2-4h+2}}} K_2(t)dt + \int_{-\sqrt{3-6h+4h^2+2(h-1)\sqrt{4h^2-4h+2}}}^{\sqrt{3-6h+4h^2+2(h-1)\sqrt{4h^2-4h+2}}} -K_2(t)dt \\ &+ \int_{\sqrt{3-6h+4h^2+2(h-1)\sqrt{4h^2-4h+2}}}^{2-2h-\sqrt{4h^2-4h+2}} K_2(t)dt + \int_{2-2h-\sqrt{4h^2-4h+2}}^{1-2h} K_3(t)dt \\ &+ \int_{1-2h}^{1} -K_3(t)dt = \eta(h). \end{aligned}$$

The inequality (ii) follows considering the following estimation of remainder term

$$|\mathcal{R}[f]| \leq \|f''\|_2 \left[\int_{-1}^{1} K(t)^2 dt\right]^{1/2}.$$

We have

$$\begin{aligned} \int_{-1}^{1} K(t)^2 dt = \int_{-1}^{-(2-2h-\sqrt{4h^2-4h+2})} K_1(t)^2 dt + \int_{-(2-2h-\sqrt{4h^2-4h+2})}^{2-2h-\sqrt{4h^2-4h+2}} K_2(t)^2 dt \\ + \int_{2-2h-\sqrt{4h^2-4h+2}}^{1} K_3(t)^2 dt = \nu(h)^2. \end{aligned}$$

Theorem 5 *Let $f : [-1,1] \to \mathbb{R}$ be a function that $f' \in L_1[-1,1]$. If there exists a real number γ, such that $\gamma \leq f'(t)$, $t \in [-1,1]$, then*

$$|\mathcal{R}[f]| \leq 2h(S-\gamma),$$

and if there exist a real number Γ such that $f'(t) \leq \Gamma$, $t \in [-1,1]$, then

$$|\mathcal{R}[f]| \leq 2h(\Gamma - S),$$

where $S = \dfrac{f(1) - f(-1)}{2}$ *and* $h \in \left[0, \dfrac{1}{2}\right]$. *If there exist real numbers* γ, Γ *such that* $\gamma \le f'(t) \le \Gamma$, $t \in [-1, 1]$, *then*

$$|\mathcal{R}[f]| \le \frac{1}{2}\varphi(h)(\Gamma - \gamma),$$

where $\varphi(h)$ *are defined as*

$$\varphi(h) = 4h^2 - 6h + 3 + 2(h-1)\sqrt{4h^2 - 4h + 2}.$$

Proof. Let us define

$$p(t) = \begin{cases} t + 1 - h, \ t \in [-1, -x], \\ t, \ t \in (-x, x), \\ t - 1 + h, \ t \in [x, 1], \end{cases}$$

where $x = 2 - 2h - \sqrt{4h^2 - 4h + 2}$.

Since $\int_{-1}^{1} p(t)dt = 0$ and $\int_{-1}^{1} p(t)f'(t)dt = -\mathcal{R}[f]$, it follows

$$|\mathcal{R}[f]| = \left|\int_{-1}^{1} p(t)f'(t)dt\right| = \left|\int_{-1}^{1} \left(f'(t) - \gamma\right) p(t)dt\right| \le \|p\|_\infty \cdot \|f' - \gamma\|_1 = 2h(S - \gamma)$$

and

$$|\mathcal{R}[f]| = \left|\int_{-1}^{1} p(t)f'(t)dt\right| = \left|\int_{-1}^{1} \left(\Gamma - f'(t)\right) p(t)dt\right| \le \|p\|_\infty \cdot \|\Gamma - f'\|_1 = 2h(\Gamma - S).$$

Since $\left|f'(t) - \dfrac{\gamma + \Gamma}{2}\right| \le \dfrac{\Gamma - \gamma}{2}$, we obtain

$$|\mathcal{R}[f]| = \left|\int_{-1}^{1} p(t)\left(f'(t) - \frac{\gamma + \Gamma}{2}\right) dt\right| \le \frac{\Gamma - \gamma}{2}\|p\|_1 = \varphi(h)\frac{\Gamma - \gamma}{2}.$$

3 The corrected quadrature formulas

In the last years many authors have considered so called corrected quadrature rules ([2]-[6],[11]). In this section we will construct the corrected quadrature formulas of the optimal quadrature formula (10) and we show that the estimations using different norms are better in the corrected formula than in the original one.

Let

$$A := \int_{-1}^{1} K(t)dt = 4h^3 - 10h^2 + \frac{17}{2}h - \frac{17}{6} + 2(h-1)^2\sqrt{4h^2 - 4h + 2}.$$

The corrected quadrature formula of (10) is defined bellow:

(12)
$$\int_{-1}^{1} f(x)dx = hf(-1) + (1-h)f\left(2-2h-\sqrt{4h^2-4h+2}\right)$$
$$+(1-h)f\left(-2+2h+\sqrt{4h^2-4h+2}\right) + hf(1) + A\left[f'(1)-f'(-1)\right] + \tilde{\mathcal{R}}[f],$$

where
$$\tilde{\mathcal{R}}[f] = \int_{-1}^{1} \tilde{K}(t) f''(t)dt \text{ and } \tilde{K}(t) = K(t) - A.$$

Theorem 6 *For the remainder term of the quadrature formula (12) the following estimations can be established*
$$\left|\tilde{\mathcal{R}}[f]\right| \le \Omega(h) \cdot \|f''\|_1,$$
where
$$\Omega(h) = -4h^3 + 12h^2 - \frac{23}{2}h + \frac{13}{3} + \left(-2h^2+5h-3\right)\sqrt{4h^2-4h+2}.$$

Remark 2 *From the below figure we remark that there are values of h such that the estimation in the corrected formula is better than in the original one.*

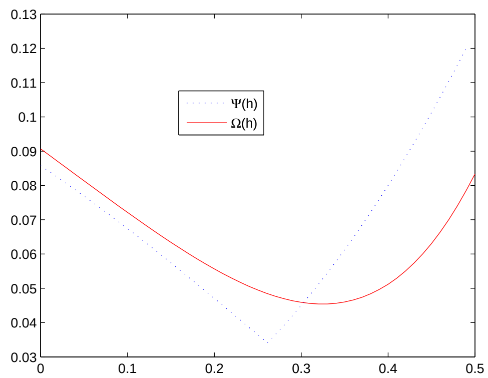

Figure 1.

Theorem 7 *For the remainder term of the quadrature formula (12) the following estimation can be established*
$$\left|\tilde{\mathcal{R}}[f]\right| \le \nu^*(h) \cdot \|f''\|_2,$$
where
$$\nu^*(h) = -64h^6 + \frac{896}{3}h^5 - 592h^4 + 644h^3 - \frac{2455}{6}h^2 + \frac{871}{6}h - \frac{1018}{45}$$
$$+\sqrt{4h^2-4h+2}\left(-32h^5 + \frac{400}{3}h^4 - \frac{676}{3}h^3 + \frac{584}{3}h^2 - \frac{260}{3}h + 16\right).$$

Remark 3 *In Figure 2 are represented the coefficients ν, ν^* which appear in estimations of the remainder term in L_2-norm for the original and corrected quadrature formulas. The estimation in the corrected formula is better than in the original one.*

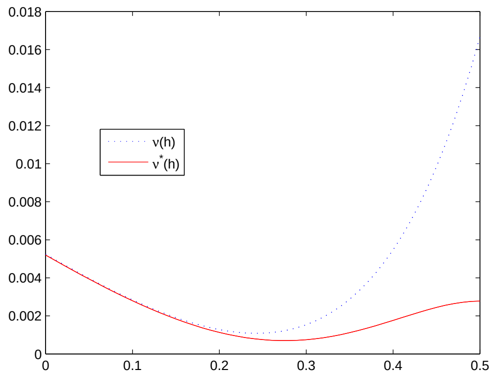

Figure 2.

Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt,$$

is well known in the literature as the Čebyšev functional and the inequality

$$|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$$

holds. Denote by $\sigma(f,a,b) = \sqrt{(b-a)T(f,f)}$.

Theorem 8 *Let $f : [-1,1] \to \mathbb{R}$ be a twice differentiable function such that f'' is integrable. Then,*

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \nu^*(h)\sigma(f'',-1,1).$$

Proof. We have

$$\begin{aligned}\tilde{\mathcal{R}}[f] &= \int_{-1}^1 \tilde{K}(t)f''(t)dt = \int_{-1}^1 \left[K(t) - \frac{1}{2}\int_{-1}^1 K(t)dt\right] f''(t)dt \\ &= \int_{-1}^1 K(t)f''(t)dt - \frac{1}{2}\int_{-1}^1 K(t)dt \int_{-1}^1 f''(t)dt = 2T(K,f'').\end{aligned}$$

Therefore

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \sqrt{2T(K,K)} \cdot \sqrt{2T(f'',f'')} = \nu^*(h)\sigma(f'',-1,1).$$

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