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Two diophantine equations [1]

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Dedicated to Associated Professor Silviu Crăciunaş on his 60^{th} birthday.

Abstract

In this paper we study the diophantine equations (1) and (2).

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The diophantine equation play an ever increasing rolled in mathematics and in modern applications(see [6]).

In this paper we study in positive integer numbers the following tho diophantine equations

$$3 + \frac{(x-1)x}{2} = 3^y \tag{1}$$

and

$$3^{x-1}(3 + \frac{(x-1)x}{2}) = y^2. \tag{2}$$

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1 The diophantine equation (1)

The equation (1) is equivalent with

$$x^2 - x + 6 - 2 \cdot 3^y = 0$$

where we obtain

$$x = \frac{1 + \sqrt{\Delta}}{2}$$

with

(3) $$\Delta = 8 \cdot 3^y - 23 = z^2,$$

z odd number.

If $y = 1$, then we have $\Delta = 1$ and $x = 1$.

If $y = 2$, then we obtain $\Delta = 7^2$ and $x = 4$.

If $y = 3$, then we find $\Delta = 193 \neq z^2$, $z \in \mathbb{N}$.

If $y = 4$, then we have $\Delta = 25^2$ and $x = 13$.

Now we consider $y \geq 5$. We write the equation (3) under the form

(4) $$z^2 + 23 = 8 \cdot 3^y.$$

We consider that $8 \cdot 3^y \equiv 0 (mod 243)$. As we know $x = 81k + r,\ k \in \mathbb{N}$ and $r \in \{0, \pm 1, \pm 2, \ldots, \pm 40\}$, where we find $x^2 \equiv r^2 (mod 81)$,with $r^2 \in \{0, 1, 4, 7, 8, 9, 10, 13, 16, 19, 22, 25, 28, 31, 33, 36, 37, 40, 46, 49, 52, 55, 58, 61, 63, 64, 67, 70, 76\}$.

From here,it results $x^2 + 23 \equiv 0 (mod 81)$ only if $x = 81k + 25$,$k \in \mathbb{N}$. From x is odd number, we deduce that $k = 2p$,$p \in \mathbb{N}$.Thus we have $x = 162p + 25$,$p \in \mathbb{N}$.We now have

$x^2 + 23 = 162^2 \cdot p^2 + 50 \cdot 162p + 625 = 243t + 81p + 139.$

The equations (4)has the solutions only if $81p + 139 = 243 \cdot u$,$u \in \mathbb{N}$. From here we obtain $139 = 81(3u - p)$, which it is not possible.It results that the equation (4) has not the solutions if $y \geq 5$.

Thus we have proved.

Theorem 1.1. *The diophantine equation (1) has in positive integer numbers only the solutions:* $(x, y) \in \{(1, 1), (4, 2), (13, 4)\}$.

2 The diophantine equation (2).

If $x = 1$, then from (2) we obtain $y^2 = 3$,which it is not possible in natural numbers.Let $x \geq 2$ be natural number.We consider two cases.

The cases 2.1. x **odd number.** Let x be odd number $x = 2t + 1$, $t \in \mathbb{N}^*$, $\mathbb{N}^* = \mathbb{N} - \{0\}$. Then the diophantine equation (2) takes the form

$$(3^t)^2(3 + 2t^2 + t) = y^2, \tag{5}$$

which it is possible only if $2t^2 + t + 3 = u^2$,$u \in \mathbb{N}^*$.

From here it results $t = \frac{-1+\sqrt{\Delta}}{4}$ with $\Delta = 8u^2 - 23 = v^2$,$v \in \mathbb{N}^*$.

Therefore, we obtained that the v and u are the solutions of the diophantine equation:

$$v^2 - 8u^2 = -23. \tag{6}$$

But, as we know, this equation can be reduced to the equation of Pell([1],[3],[4],[5]).

Thus we consider the diophantine equation of Pell:

$$v^2 - 8u^2 = 1 \tag{7}$$

which it has the least solution $(v_0, u_0) = (3, 1)$. Now we find the solutions of the equation (6) $v + u\sqrt{8}$ for which we have $\mid v \mid \leq 1$ and $\mid u \mid \leq 4$. Thus we find for the equation (6) the solutions $(v, u) = (3, 2)$ and $(v, u) = (7, 3)$. Starting with $(v, u) = (3, 2)$ we find the solutions for (1) from the identities

$v_k + u_k\sqrt{8} = (3 + \sqrt{8})^k(3 + 2\sqrt{8})$ or

$v_k + u_k\sqrt{8} = (3 + \sqrt{8})^k(3 - 2\sqrt{8})$,$k \in \mathbb{N}^*$.

From here we obtain for the equation (6) the solutions

$$v_k = 3A_k + 16B_k\ ,\ u_k = 2A_k + 3B_k \tag{8}$$

or

$$v_k = -3A_k + 16B_k\ ,\ u_k = 2A_k - 3B_k \tag{9}$$

where

$$A_k = 3^k + \binom{k}{2}3^{k-2}8 + \binom{k}{4}3^{k-4}8^2 + \cdots \tag{10}$$

and

$$B_k = \binom{k}{1}3^{k-1} + \binom{k}{3}3^{k-3}8 + \binom{k}{5}3^{k-5}8^2 + \cdots\ , k \in \mathbb{N}^*. \tag{11}$$

For the solutions (8) we have

$t_k = \frac{-1+v_k}{4} = 4B_k + \frac{3A_k-1}{4}$

which is number natural only if $4 \mid 3A_k - 1$, that is $4 \mid 3^{k+1} - 1$, whence it results that k is odd number. Thus, if k is odd number, we have for the equation (2) the following solutions

$$x_k = \frac{1 + v_k}{2}\ ,\ y_k = 3^{\frac{-1+v_k}{4}} \cdot u_k, \tag{12}$$

where u_k and v_k are given by (8).

Example 1. For $k = 1$ we have $A_1 = 3$,$B_1 = 1$,$v_1 = 3A_1 + 16B_1 = 25$,$u_1 = 2A_1 + 3B_1 = 9$, $x_1 = 13, y_1 = 3^8$, that is the equation (2) has the solution $(x_1, y_1) = (13, 3^8)$. Similarly starting from the solutions(9) we obtain for the equation (2) the solutions:

$$x_k = \frac{1 + v_k}{2}\ ,\ y_k = 3^{\frac{-1+v_k}{4}} \cdot u_k, \tag{13}$$

k even number.

Example 2. For $k = 2$, we obtain $A_2 = 17$,$B_2 = 16$,$v_2 = -3A_2 + 16B_2 = 45$, $u_2 = 2A_12 - 3B_2 = 16$, $x_2 = 23, y_2 = 3^{11} \cdot 16$, that is the diophantine equation (2) has the solution $(x_2, y_2) = (23, 3^{11} \cdot 16)$.

Now , we start with the solutions $(u, v) = (7, 3)$ for the equation (6). We find for this equation the following solutions:

$$v_k = 7A_k + 24B_k\ ,\ u_k = 3A_k + 7B_k \tag{14}$$

or

$$v_k = -7A_k + 24B_k\ ,\ u_k = 3A_k - 7B_k \tag{15}$$

where A_k, B_k are given by (10) and (11),$k \in \mathbb{N}^*$.

From these we obtain the solutions

$$x_k = \frac{1 + v_k}{2}\ ,\ y_k = 3^{\frac{-1+v_k}{4}} \cdot u_k, \tag{16}$$

where u_k and v_k are given by (14) for k odd number and (15) for k even number.

The cases 2.2. x even number.

Let x be even number $x = 2t$, $t \in \mathbb{N}^*$. Then the diophantine equation (2) can be written in the form

$$(3^{t-1})^2(6t^2 - 3t + 9) = y^2. \tag{17}$$

This last diophantine equation is possible only if $6t^2-3t+9 = u^2$,$u \in \mathbb{N}^*$.

From here it results $t = \frac{3+\sqrt{\Delta}}{12}$, with $\Delta = 24u^2 - 2007 = v^2$,$v \in \mathbb{N}^*$.

Hence, we obtained that the (v, u) are the solutions of the diophantine equation:

$$v^2 - 24u^2 = -207. \tag{18}$$

Using the same method that to the case 2.1, we obtain that all the solutions of the equation (18) are

$$v_k = 3(C_k + 24D_k)\ ,\ u_k = 3(C_k + D_k) \tag{19}$$

or

$$v_k = 3(-C_k + 24D_k)\ ,\ u_k = 3(C_k - D_k) \tag{20}$$

with

$C_k = 5^k + \binom{k}{2}5^{k-2}24 + \binom{k}{4}5^{k-4}24^2 + \cdots$

and

$D_k = \binom{k}{1}5^{k-1} + \binom{k}{3}5^{k-3}24 + \binom{k}{5}5^{k-5}24^2 + \cdots\ ,\ k \in \mathbb{N}^*.$

For the solutions (19) we have

$t_k = \frac{3+3(C_k+24D_k)}{12} = 6D_k + \frac{1+C_k}{4}, k \in \mathbb{N}^*$.

The numbers $t_k \in \mathbb{N}^*$ only if $4 \mid 5^k + 1$ which is impossible since $1 + 5^k = 1 + (4+1)^k \equiv 2 \ (mod\ 4)$. For the solutions (20) we have $t_k \in \mathbb{N}^*, k \in \mathbb{N}^*$, and we find for the equation (2) the following solutions

$$x_k = \frac{3 + v_k}{6} \ , \ y_k = 3^{\frac{v_k - 9}{12}} \cdot u_k, \tag{21}$$

where v_k and u_k are given by (20) $k \in \mathbb{N}^*$.

Example 3. For $k = 1$ we find $C_1 = 5, D_1 = 1, u_1 = 3(C_1 - D_1) = 12$, $v_1 = 3(-C_1 + 24D_1) = 57$, $x_1 = 10, y_1 = 4 \cdot 3^5$, that is the diophantine equation (2) has the solution $(x_1, y_1) = (10, 4 \cdot 3^5)$.

Finally, we obtain:

Theorem 2.1. *All the solutions of the diophantine equation (2) are given of (12), (13), (16) and (21).*

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