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Finite element approximation of the Navier-Stokes Equation

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Abstract

In this paper we formulate the variational principle of the problem of stationary flow of a viscous fluid in a pipe with transversal section in the L-form and analyze the finite element approximation (Ritz algorithm on finite elements).

The coefficients and the solutions of the Ritz system and determined with a Turbo-Pascal program.

Numerical results demonstrating these bounds are also presented.

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The Navier-Stokes equation [4] that describes the stationary flow of a viscous fluid in a pipe with an arbitrary transversal section Ω is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \cdot \frac{dp}{dz},\ (x, y) \in \Omega$$

where u is the velocity, μ is the coefficient of viscosity and $\frac{dp}{dz}$ is the pressure fall on the length of the pipe. The problem is to determine the repartition of the velocity in the section Ω.

We consider the boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv -\nabla^2 u = f \text{ in } \Omega \subset \mathbb{R}^2 \\ u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

By using the Gauss formula in $\mathrm{C}^2(\Omega)$, the following integral identity is verified by classical solution u:

$$\begin{gathered} \int\limits_{\Omega} \nabla^T u \cdot \nabla v d\Omega = \int\limits_{\Omega} f v d\Omega, \ \forall v \in C_0^1(\Omega) \\ (C_0(\Omega) = \{u \in C^1(\Omega) | u = 0 \text{ on } \partial\Omega\}). \end{gathered} \tag{2}$$

Let us introduce the fundamental Hilbert space and its norm as

$$H_0^1(\Omega) = \{u \in H^2(\Omega), \ u = 0 \text{ on } \partial\Omega\}$$
$$||u||^2_{H_0^1(\Omega)} = \int\limits_{\Omega} |\nabla u|^2 d\Omega \ \ (\equiv ||u||^2_{1,0}).$$

where $H^n(\Omega)$ is the Sobolev space on Ω.

The triplet (H, a, φ), where H is the Hilbert space, can now be introduced as follows

$$H = H_0'(\Omega)$$
$$a(u, v) = \int\limits_{\Omega} \nabla^T u \cdot \nabla v dt\Omega, \quad \forall u, v \in H_0^1(\Omega)$$
$$\varphi(v) = \int\limits_{\Omega} f v d\Omega, \quad \forall v \in H_0^1(\Omega).$$

It is easy to prove that the form $a(u, v)$ is a bilinear symmetrical functional, boundary, coercive, and the functional $\varphi(v)$ is a linear and bounded form.

In this conditions, (2) can be represented as

$$a(u,v) = \varphi(v),\ \forall v \in H_0^1(\Omega). \tag{3}$$

Definition 1.1. *The integral identity (2) is normed weak equation for the boundary value problem 1 and the function $u \in H_0^1(\Omega)$ for which (2) hold is named weak solution.*

From the Lax-Milgram theorem the problem (3) has a solution u and it is unique.

Theorem 1.1. *The weak solution of u is the unique point of minimum of the functional*

$$F(u) = \frac{1}{2}a(u,u) - (f,u).$$

Thus the solving of the problem (1.3) is equivalent to the following minimization problem [2]:

$$(P_v) \qquad \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ F(u) \le F(v),\ \forall v \in H_0'(\Omega) \end{array}$$

The purpose of this paper is to analyze the finite element approximation of (P_v). Let Ω^h be a polynomial approximation of Ω defined by $\Omega^h \equiv \bigcup_{\tau \in T^h} \overline{\tau}$, where T^h is a partition of Ω^h into a finite number of disjoint open regular triangles τ, each of maximum diameter bounded above by h. In addition, for any two distinct triangle, their closure are either disjoint, or have a common vertex, or a common side. Let $\{P_j\}_{j=1}^N$ be the vertices associated with the triangulation T^h, where P_j has coordinates (x_j, y_j). Throughout we assume that $P_j \in \partial\Omega^h$ implies $P_j \in \partial\Omega$ and that $\Omega^h \subseteq \Omega$. The following finite dimensional space is associated to T^h:

$$S^h = \left\{ v \in C\overline{(\Omega^h)}, v|_r \text{ is linear } \forall \tau \in T^h \right\} \subset H'(\Omega^h).$$

Let $\prod_h : C(\overline{\Omega^h}) \to S^h$ denote the interpolation operator such that for any $v \in C(\overline{\Omega^h})$, the interpolant $\prod_h v \in S^h$ satisfies $\prod_h v(P_v) = v(P_j), j = 1, 2, ..., N$.

The finite element approximation of (P_v) that we shall consider is

$$(P_v^h) \qquad \begin{array}{c} \text{Find } u^h \in S_0^h \text{ such that} \\ F(u^h) \leq F(v^h),\ \forall v^h \in S_0^h \end{array}$$

where $S_0^h = \{v \in S^h : v = 0 \text{ on } \partial\Omega^h\}$.

The solution of the variational problem (P_v^h) is determined using the Ritz method with finite elements through the procedure of local approximation and assembly.

The approximate solution is chosen for the finite element τ as follows:

$$u_r^h = \{N(x, y)\}_\tau^T \{U\}_\tau^h \tag{4}$$

where $\{N\}$ and $\{U\}$ represent the column vectors of the local linear basis for the element τ and of the nodal values of the approximate solution:

$$\{N\}_\tau^T = (N_1 N_2 N_3); \quad \{U\}_\tau^h = (U_1 U_2 U_3)^T$$

where

$$N_r = \frac{1}{2\Delta_r}(a_r + b_r x + c_r y); \quad U_r = u_\tau^h(x_r, y_r),\ r = 1, 2, 3$$

$a_i = x_j y_k - x_k j; b_i = y_j - y_k; c_i = -(x_j - x_k)$ with permutation $i \to j \to k$, Δ_τ being the area of the finite element τ.

Now we invoke the principle of stationary functional energy $F^\tau = F(u_\tau^h)$:

$$\frac{\partial F^\tau}{\partial u_r} = 0,\ r = 1, 2, 3.$$

We obtain the matrix equation on the τ element (Ritz system) in the form:

$$[R]_\tau \{U\}_\tau^h = \{P\}_\tau. \tag{5}$$

In this case we have

$$[K]_r = \frac{1}{4\Delta_r}\begin{bmatrix} b_1^2+c_1^2 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_1b_2+c_1c_2 & b_2^2c_2^2 & b_2b_3+c_2c_3 \\ b_1b_3+c_1c_3 & b_2b_3+c_2c_3 & b_3^2+c_3^2 \end{bmatrix}.$$

Remark 1.1. *We note that the matrix $[K]_\tau$ is the same for all the elements if the following local counting is used (fig.1).*

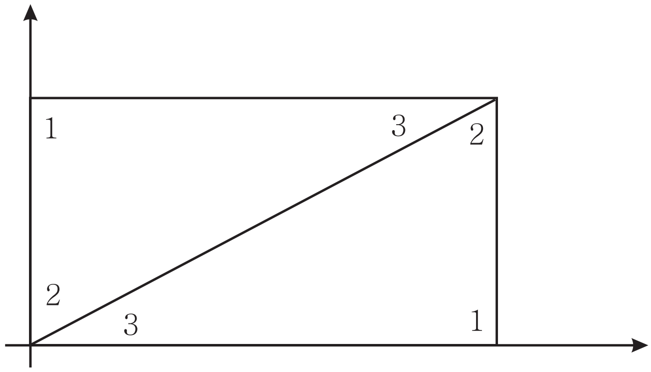

Fig. 1

The column vector $\{P\}_\tau$ is $\{P\}_\tau = f\frac{\Delta}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$. The coefficients of the matrix $\{P\}_\tau$ are determined by using the local coordinates (L_1, L_2, L_3) of the point $P(x, y)$ and the formula

$$\frac{1}{\Delta_\tau} I_{\alpha\beta\gamma} = \frac{a}{b}$$

where

$$I_{\alpha\beta\gamma} \equiv \int_\tau L_1^\alpha L_2^\beta L_3^\gamma d\tau = 2\Delta_c \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma)!}.$$

An equation of the type (5) is written for each element. The column vector $\{U\}_\tau^h$ is extended to the N number of nodes in the mesh by the introduction of all the nodal values. Taking into account the correspondence between the local counting and the overall counting the matrices $[K]_\tau$ and $\{P\}_\tau$ are also extended at dimensions $N \times N$ and $N \times 1$. We obtain the matrix of the mesh

$$[K] \cdot \{U\} = \{P\} \tag{6}$$

to which we attach conditions on main boundary.

The coefficients k_{ij} and p_i of the matrices $[K]$ and $\{P\}$ and the solutions of the Ritz system (by means of the Gauss elimination method) are determined with a Turbo-Pascal program. The program has been applied for the following numerical example:

$$\mu = 1,5 \cdot 10^{-4} Ns/m^2;$$

$$\frac{dp}{dz} = -5000 N/m^3;$$

$$\Omega \text{ in } L- \text{ form (fig.2)};$$

$$a = 0,1m.$$

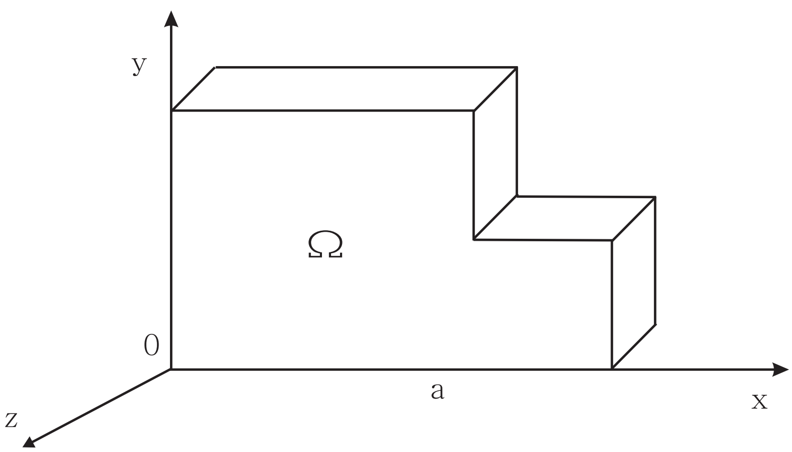

Fig. 2

The values of velocity at nodes are listed in Table 1, in the case $N = 51$.

0.00	0.00	0.00	0.00			
0.00	4866.45	4884.50	0.00			
0.00	7253.47	7308.98	0.00			
0.00	8481.01	8670.66	0.00			
0.00	9164.11	9807.93	0.00	0.00	0.00	0.00
0.00	9431.55	11615.97	8368.64	7008.61	5013.84	0.00
0.00	8567.15	11271.05	10467.33	9165.95	6517.56	0.00
0.00	5792.50	7690.83	7637.64	6872.63	4989.36	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 1. Numerical results for velocity.

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