General Mathematics Vol. 14, No. 3 (2006), 61–78

Integral Means and fractional calculus operators for comprehensive family of univalent functions with negative coefficients [1]

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Abstract

In this paper, we obtain the integral means inequality for the function $f(z)$ belongs to the class $UT(\Phi, \Psi, \gamma, k)$ of analytic and univalent functions with negative coefficients defined in [3] with the extremal functions of this class. And also we derive some distortion theorems using fractional calculus techniques for the class $UT(\Phi, \Psi, \gamma, k)$.

2000 Mathematics Subject Classification: 30C45.

Keywords: Univalent, starlike, convex, uniformly convex, uniformly starlike, Hadamard product, integral means, Fractional calculus

[1] *Received 28 August, 2006*

Accepted for publication (in revised form) 19 September, 2006

1 Introduction and definitions

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathcal{U} = \{z : z \in \mathcal{C},\ |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,\ z \in \mathcal{U} \tag{1.2}$$

introduced and studied by Silverman [16].

Following Gooodman [5, 6], Rønning [12, 13] introduced and studied the following subclasses

(i) A function $f \in A$ is said to be in the class $S_p(\gamma, k)$, $k-$uniformly starlike functions of order γ, if it satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathcal{U}, \tag{1.3}$$

$0 \leq \gamma < 1$ and $k \geq 0$.

(ii) A function $f \in A$ is said to be in the class $UCV(\gamma, k)$, $k-$uniformly convex functions of order γ, if it satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U}, \tag{1.4}$$

$0 \leq \gamma < 1$ and $k \geq 0$.

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\gamma, k) \Leftrightarrow zf' \in S_p(\gamma, k). \tag{1.5}$$

Definition 1.1 ([3]). Given $\gamma(-1 \leq \gamma < 1)$, $k(k \geq 0)$ and functions

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \text{ and } \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$$

analytic in U, such that $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$ for $n \geq 2$, we let $f \in A$ is in $U(\Phi, \Psi, \alpha, \beta)$ if $(f * \Psi)(z) \neq 0$ and

$$\text{Re}\ \left\{ \frac{(f * \Phi)(z)}{(f * \Psi)(z)} - \gamma \right\} \geq k \left| \frac{(f * \Phi)(z)}{(f * \Psi)(z)} - 1 \right|, \ \forall \ \ z \in \mathcal{U}.$$

where (*) stands for the Hadamard product.
Further let $UT(\Phi, \Psi, \alpha, \beta) = U(\Phi, \Psi, \alpha, \beta) \cap T$.

We note that, by taking suitable choice of Φ, Ψ, α and β we obtain the following subclasses studied in literature.

1. $UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ 1\right) = TS_p(\gamma)$ (Subrmanian et al., [22])

2. $UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ k\right) = S_pT(\gamma, k)$ (Bharati et al., [1])

3. $UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ 0,\ 1\right) = UCT$ (Subrmanian et al., [21])

4. $UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ 0,\ k\right) = UCT(k)$ (Subrmanian et al., [21])

5. $UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ 1\right) = UCT(\gamma)$ (Bharati et al., [1])

6. $UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ k\right) = UCT(\gamma, k)$ (Bharati et al., [1])

7. $UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ 0\right) = S_T^*(\gamma)$ (Silverman [16])

8. $UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ 0\right) = K_T(\gamma)$ (Silverman [16])

9. $UT(\Phi, \Psi, \gamma, 0) = E_T(\Phi, \Psi, \gamma)$ (Juneja et al.[7]).

10. $UT(\Phi, \Psi, \frac{1+\beta-2\alpha}{2(1-\alpha)}, 0) = B_T(\Phi, \Psi, \alpha, \beta)$ (Frasin [4]).

In fact many subclasses of T are defined and studied to investigate coefficient estimates, extreme points, convolution properties and closure properties etc. suitably choosing Φ, Ψ, γ and k.

In this paper, we obtain integral means inequalities for functions $f(z) \in UT(\Phi, \Psi, \gamma, k)$ and also we state integral means results for the classes studied in [21, 1, 22, 16, 4] as corollaries.

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ so that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$ and $g(z) = h(w(z))$, we denote this subordination by $g(z) \prec h(z)$.

To prove our main results, we need the following lemmas.

Lemma 1.1 ([3]). *A function $f(z) \in UT(\Phi, \Psi, \gamma, k)$ for $\gamma(-1 \leq \gamma < 1)$ and $k(k \geq 0)$ if and only if*

$$\sum_{n=2}^{\infty}[(1+k)\lambda_n - (\gamma+k)\mu_n]a_n \leq 1-\gamma. \tag{1.1}$$

The result is sharp with the extremal functions

$$f_n(z) = z - \frac{1-\gamma}{\sigma(\gamma, k, n)}z^n, \quad n \geq 2 \tag{1.2}$$

where $\sigma(\gamma, k, n) = (1 + k)\lambda_n - (\gamma + k)\mu_n$, $\gamma(-1 \leq \gamma < 1)$, $k(k \geq 0)$ *and* $n \geq 2$.

Lemma 1.2 ([8]). *If the functions* $f(z)$ *and* $g(z)$ *are analytic in* U *with* $g(z) \prec f(z)$ *then*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \ \eta > 0, \ z = re^{i\theta} \text{ and } 0 < r < 1. \tag{1.3}$$

2 Integral mean

Applying Lemma 1.1 and Lemma 1.2, we prove the following theorem.

Theorem 2.1. *Let* $\eta > 0$. *If* $f(z) \in UT(\Phi, \Psi, \gamma, k)$, $-1 \leq \gamma < 1$, $k \geq 0$ *and* $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ *is non-decreasing sequence, then for* $z = re^{i\theta}$ *and* $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right| \eta d\theta \tag{2.1}$$

where $f_2(z) = z - \frac{1-\gamma}{\sigma(\gamma,k,2)} z^2$.

Proof. Let $f(z)$ of the form (1.2) and $f_2(z) = z - \frac{(1-\gamma)}{\sigma(\gamma,k,2)} z^2$, then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right| \eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\gamma)}{\sigma(\gamma, k, 2)} z\right| \eta d\theta.$$

By Lemma 1.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\sigma(\gamma, k, 2)} z$$

Setting

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}=1-\frac{1-\gamma}{\sigma(\gamma,k,2)}w(z). \tag{2.2}$$

From (2.2) and (1.1), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\sigma(\gamma,k,2)}{1-\gamma}a_nz^{n-1}\right| \\ &\le |z|\sum_{n=2}^{\infty}\frac{\sigma(\gamma,k,n)}{1-\gamma}a_n \\ &\le |z|<1.\end{aligned}$$

This completes the proof of the Theorem 2.1.

By taking different choices of Φ, Ψ, γ and k in the above theorem, we can state the following integral means results for various subclasses studied earlier [21, 1, 22, 16, 4].

Corollary 2.2. *Let* $\eta>0$. *If* $f(z)\in UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ 0,\ 1\right)=UCT$, *then for* $z=re^{i\theta}$; $0<r<1$, *we have*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|\eta d\theta\le\int_0^{2\pi}\left|g_2(re^{i\theta})\right|\eta d\theta \tag{2.3}$$

where $g_2(z)=z-\frac{z^2}{6}$.

Corollary 2.3. *Let* $\eta>0$. *If* $f(z)\in UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ 0,\ k\right)=UCT(k)$ *and* $k\ge 0$, *then for* $z=re^{i\theta}$; $0<r<1$, *we have*

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|\eta d\theta\le\int_0^{2\pi}\left|g_2(re^{i\theta})\right|\eta d\theta \tag{2.4}$$

where $g_2(z)=z-\frac{z^2}{2(k+2)}$.

Corollary 2.4. *Let* $\eta > 0$*. If* $f(z) \in UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ 1\right) = UCT(\gamma)$ *and* $-1 \leq \gamma < 1$*, then for* $z = re^{i\theta}$; $0 < r < 1$*, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.5}$$

where $g_2(z) = z - \frac{(1-\gamma)}{2(3-\gamma)} z^2$.

Corollary 2.5. *Let* $\eta > 0$*. If* $f(z) \in UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ k\right) = UCT(\gamma, k)$, $-1 \leq \gamma < 1$ *and* $k \geq 0$*, then for* $z = re^{i\theta}$; $0 < r < 1$*, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.6}$$

where $g_2(z) = z - \frac{(1-\gamma)}{2(2-\gamma+k)} z^2$.

Corollary 2.6. *Let* $\eta > 0$*. If* $f(z) \in UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ 1\right) = TS_p(\gamma)$ *and* $-1 \leq \gamma < 1$*, then for* $z = re^{i\theta}$; $0 < r < 1$*, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.7}$$

where $g_2(z) = z - \frac{(1-\gamma)}{(3-\gamma)} z^2$.

Corollary 2.7. *Let* $\eta > 0$*. If* $f(z) \in UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ k\right) = S_pT(\gamma, k)$, $-1 \leq \gamma < 1$ *and* $k \geq 0$*, then for* $z = re^{i\theta}$; $0 < r < 1$*, we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \leq \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.8}$$

where $g_2(z) = z - \frac{(1-\gamma)}{(2-\gamma+k)} z^2$.

By taking $\gamma = \frac{1+\beta-2\alpha}{2(1-\alpha)}$ *and* $k = 0$ *in Theorem 2.1 we get the following result of Frasin and Darus [4].*

Corollary 2.8. *Let* $\eta > 0$. *If* $f(z) \in UT(\Phi, \Psi, \frac{1+\beta-2\alpha}{2(1-\alpha)}, 0) = B_T(\Phi, \Psi, \alpha, \beta)$, $0 \le \beta < 1$ *and* $0 \le \alpha < 1$, *then for* $z = re^{i\theta}$; $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \le \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.9}$$

where $g_2(z) = z - \frac{(1-\beta)}{\psi(\alpha,\beta,2)} z^2$ *and* $\psi(\alpha, \beta, 2) = 2(1-\alpha)\lambda_2 - (1+\beta-2\alpha)\mu_2$.

Corollary 2.9. *Let* $\eta > 0$. *If* $f(z) \in UT\left(\frac{z}{(1-z)^2},\ \frac{z}{1-z},\ \gamma,\ 0\right) = S_T^*(\gamma)$ *and* $\gamma \ge 0$, *then for* $z = re^{i\theta}$; $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \le \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.10}$$

where $g_2(z) = z - \frac{1-\gamma}{2-\gamma} z^2$.

Corollary 2.10. *Let* $\eta > 0$. *If* $f(z) \in UT\left(\frac{z+z^2}{(1-z)^3},\ \frac{z}{(1-z)^2},\ \gamma,\ 0\right) = K_T(\gamma)$, *and* $\gamma \ge 0$, *then for* $z = re^{i\theta}$; $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right| \eta d\theta \le \int_0^{2\pi} \left|g_2(re^{i\theta})\right| \eta d\theta \tag{2.11}$$

where $g_2(z) = z - \frac{1-\gamma}{2(2-\gamma)} z^2$.

Remark 2.11. *If we take* $\gamma = 0$ *in* $S_T^*(\gamma)$ *of Corollary 2.9 and* $K_T(\gamma)$ *of Corollary 2.10, we get the integral means results obtained by Silverman [17].*

3 Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [2],[9],[11], [14], [15], [18]and[19]). We find it to be convenient to recall here the following definitions which are used earlier by Owa [10](and, subsequently, by Srivastava and Owa [19]).

Definition 3.1. *The fractional integral of order* ξ *is defined, for a function* $f(z)$, *by*

$$D_z^{-\xi} f(z) = \frac{1}{\Gamma(\xi)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\xi}} d\zeta \qquad (\xi > 0), \tag{3.1}$$

where the function $f(z)$ *is analytic in a simply-connected region of the* z-*plane containing the origin and the multiplicity of the the function* $(z-\zeta)^{\xi-1}$ *is removed by requiring the function* $\log(z-\zeta)$ *to be real when* $z-\zeta > 0$.

Definition 3.2. *The fractional derivative of order* ξ is *defined, for a function* $f(z)$, *by*

$$D_z\xi f(z) = \frac{1}{\Gamma(-\xi)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\xi}} d\zeta \qquad (0 \leq \xi < 1), \tag{3.2}$$

where the function $f(z)$ *is constrained, and the multiplicity of the the function* $(z-\zeta)^{-\xi}$ *is removed as in Definition 3.1*

Definition 3.3. *Under the hypotheses of Definition 3.2, the fractional derivative of order* $n+\lambda$ *is defined by*

$$D_z^{m+\xi} f(z) = \frac{d^m}{dz^m} D_z\xi f(z) \qquad (0 \leq \xi < 1;\, m \in \mathbb{N}_0). \tag{3.3}$$

Remark 3.4. *From Definition 3.2, we have* $D_z^0 f(z) = f(z)$, *which in view of Definition 3.3 yields* $D_z^{m+0} f(z) = \frac{d^m}{dz^m} D_z^0 f(z) = f^{(m)}(z)$. *Thus,* $\lim\limits_{\xi\to 0} D_z^{-\xi} f(z) = f(z)$ *and* $\lim\limits_{\xi\to 0} D_z^{1-\xi} f(z) = f'(z)$.

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa[20].

Definition 3.5 *For real number $\eta > 0, \mu$ and δ, the fractional integral operator $I_{0,z}^{\eta,\mu,\delta}$ is defined by*

$$I_{0,z}^{\eta,\mu,\delta} f(z) = \frac{z^{-\eta-\mu}}{\Gamma(\eta)} \int_0^z (z-t)^{\eta-1} F(\eta+\mu, -\delta; \eta; 1-t/z) f(t) dt, \tag{3.4}$$

where a function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|\,\varepsilon) \qquad (z \to 0),$$

with $\varepsilon > \max\{0, \mu-\delta\} - 1$. Here $F(a,b;c;z)$ is the Gauss hypergeometric function defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{3.5}$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)(\nu+2)\cdots(\nu+n-1) & (n \in \mathbb{N}) \end{cases} \tag{3.6}$$

and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $\log\,(z-t)$ to be real when $z-t>0$.

Remark 3.4. *For $\mu = -\eta$, we note that*

$$I_{0,z}^{\eta,-\eta,\delta} f(z) = D_z^{-\eta} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [20].

Lemma 3.7. *If* $\eta > 0$ *and* $n > \mu - \delta - 1$, *then*

$$I_{0,z}^{\eta,\mu,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\mu+\delta+1)}{\Gamma(n-\mu+1)\Gamma(n+\eta+\delta+1)} z^{n-\mu}. \tag{3.7}$$

With aid of Lemma 3.7, we prove

Theorem 3.8. *Let* $\eta > 0$, $\mu < 2$, $\eta + \delta > -2$, $\mu - \delta < 2, \mu(\eta + \delta) \leq 3\eta$. *Let the function* $f(z)$ *defined by (1.2) be in the class* $UT(\Phi, \Psi, \gamma, k)$. *If* $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ *is a non-decreasing sequence. Then we have*

$$\left| I_{0,z}^{\eta,\mu,\delta} f(z) \right| \geq \frac{\Gamma(2-\mu+\delta)\,|z|^{1-\mu}}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} \left(1 - \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)} |z| \right) \tag{3.8}$$

and

$$\left| I_{0,z}^{\eta,\mu,\delta} f(z) \right| \leq \frac{\Gamma(2-\mu+\delta)\,|z|^{1-\mu}}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} \left(1 + \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)} |z| \right) \tag{3.9}$$

for $z \in \mathcal{U}_0$, *where*

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & (\mu \leq 1), \\ \mathcal{U} - \{0\} & (\mu > 1). \end{cases} \tag{3.10}$$

The equalities in (3.8) and(3.9) are attained for the function $f(z)$ *given by*

$$f(z) = z - \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)} z^2. \tag{3.11}$$

Proof. By using Lemma 3.7, we have

$$I_{0,z}^{\eta,\mu,\delta} f(z) = \frac{\Gamma(2-\mu+\delta)}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)} z^{1-\mu}$$

$$- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+\delta+1)}{\Gamma(n-\mu+1)\Gamma(n+\eta+\delta+1)} a_n z^{n-\mu} \qquad (z \in \mathcal{U}_0). \tag{3.12}$$

Letting

$$G(z) = \frac{\Gamma(2-\mu)\Gamma(2+\eta+\delta)}{\Gamma(2-\mu+\delta)} z\mu I_{0,z}^{\eta,\mu,\delta} f(z)$$

$$= z - \sum_{n=2}^{\infty} g(n) a_n z^n, \tag{3.13}$$

where

$$g(n) = \frac{(2-\mu+\delta)_{n-1}(1)_n}{(2-\mu)_{n-1}(2+\eta+\delta)_{n-1}} \qquad (n \geq 2), \tag{3.14}$$

we can see that the function $g(k)$ is non-increasing for integers $n(n \geq 2)$,and thus we have

$$0 < g(n) \leq g(2) = \frac{2(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)}. \tag{3.15}$$

From Lemma 1.1, we obtain

$$\sigma(\gamma, k, 2) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \sigma(\gamma, k, n) a_n \leq 1 - \gamma \tag{3.16}$$

Hence, using (3.15) and (3.16), we have

(3.17)
$$|G(z)| \geq |z| - g(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)}\,|z|^2\,,$$

and

(3.18)
$$|G(z)| \leq |z| + g(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{2(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)}\,|z|^2\,,$$

for $z \in \mathcal{U}_0$, where $\mathcal{U}_0$ is defined by (3.10). This completes the proof Theorem 3.8.

By using the same proof as in Theorem 3.8, we can prove

Theorem 3.9. *Let the function $f(z)$ be defined by (1.2) be in the class $UT(\Phi,\Psi,\gamma,k)$. If $\{\sigma(\gamma,k,n)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have*

(3.19)
$$\left|I_{0,z}^{\eta,\mu,\delta} f(z)\right| \geq \frac{\Gamma(2-\mu+\delta)\,|z|^{1-\mu}}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)}\left(1 - \frac{4(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)}\,|z|\right)$$

and

(3.20)
$$\left|I_{0,z}^{\eta,\mu,\delta} f(z)\right| \leq \frac{\Gamma(2-\mu+\delta)\,|z|^{1-\mu}}{\Gamma(2-\mu)\Gamma(2+\eta+\delta)}\left(1 + \frac{4(1-\gamma)(2-\mu+\delta)}{(2-\mu)(2+\eta+\delta)\sigma(\gamma,k,2)}\,|z|\right)$$

for $z \in \mathcal{U}_0$, where $\mathcal{U}_0$ is defined by (3.10). The equalities in (3.19) and (3.20) are attained for the function $f(z)$ given by (3.11).

Taking $\mu = -\eta = -\xi$ Theorem 3.8, we get

Corollary 3.10. *Let the function $f(z)$ defined by (1.2) be in the class $UT(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have*

$$\left|D_z^{-\xi} f(z)\right| \geq \frac{|z|^{1+\xi}}{\Gamma(2+\xi)} \left(1 - \frac{2(1-\gamma)}{(2+\xi)\sigma(\gamma, k, 2)} |z|\right) \tag{3.21}$$

and

$$\left|D_z^{-\xi} f(z)\right| \leq \frac{|z|^{1+\xi}}{\Gamma(2+\xi)} \left(1 + \frac{2(1-\gamma)}{(2+\xi)\sigma(\gamma, k, 2)} |z|\right) \tag{3.22}$$

for $\xi > 0$, $z \in \mathcal{U}$. The result is sharp for the function

$$D_z^{-\xi} f(z) = \frac{|z|^{1+\xi}}{\Gamma(2+\xi)} \left(1 - \frac{2(1-\gamma)}{(2+\xi)\sigma(\gamma, k, 2)} |z|\right). \tag{3.23}$$

Taking $\mu = -\eta = \xi$ in Theorem 3.9, we get

Corollary 3.11. *Let the function $f(z)$ defined by (1.2) be in the class $UT(\Phi, \Psi, \gamma, k)$. If $\{\sigma(\gamma, k, n)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence. Then we have*

$$|D_z \xi f(z)| \geq \frac{|z|^{1-\xi}}{\Gamma(2-\xi)} \left(1 - \frac{4(1-\gamma)}{(2-\xi)\sigma(\gamma, k, 2)} |z|\right) \tag{3.24}$$

and

$$|D_z \xi f(z)| \leq \frac{|z|^{1-\xi}}{\Gamma(2-\xi)} \left(1 + \frac{4(1-\gamma)}{(2-\xi)\sigma(\gamma, k, 2)} |z|\right) \tag{3.25}$$

for $0 \leq \xi < 1$, $z \in \mathcal{U}$. The result is sharp for the function given by (3.23).

Letting $\xi = 0$ in Corollary 3.10, we have

Corollary 3.12 ([3]). *Let the function* $f(z)$ *defined by (1.2) be in the class* $UT(\Phi, \Psi, \gamma, k)$. *If* $\{\sigma(\gamma, k, n)\}_{n=2}^{\infty}$ *is a non-decreasing sequence. Then we have*

$$1 - \frac{1-\gamma}{\sigma(\gamma, k, 2)}\,|z| \leq |f(z)| \leq 1 + \frac{1-\gamma}{\sigma(\gamma, k, 2)}\,|z| \tag{3.26}$$

for $\xi > 0$, $z \in \mathcal{U}$. *The result is sharp for the function*

$$f(z) = z - \frac{1-\gamma}{\sigma(\gamma, k, 2)} z^2. \tag{3.27}$$

Letting $\xi \to 1$ in Corollary 3.11,we have

Corollary 2.7 ([3]).*Let the function* $f(z)$ *defined by (1.2) be in the class* $UT(\Phi, \Psi, \gamma, k)$. *If* $\{\sigma(\gamma, k, n)/n\}_{n=2}^{\infty}$ *is a non-decreasing sequence. Then we have*

$$1 - \frac{2(1-\gamma)}{\sigma(\gamma, k, 2)}\,|z| \leq |f'(z)| \leq 1 + \frac{2(1-\gamma)}{\sigma(\gamma, k, 2)}\,|z| \tag{3.28}$$

for $0 \leq \xi < 1$, $z \in \mathcal{U}$. *The result is sharp for the function given by (3.27).*

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