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On a subclass of n-starlike functions associated with some hyperbola

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Dedicated to Professor Emil C. Popa on his 60th birthday

Abstract

In this paper we define a subclass of n-starlike functions associated with some hyperbola and we obtain some properties regarding this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : \ f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : \ f \text{ is univalent in } U\}$.

We recall here the definition of the well - known class of starlike functions:

$$S^* = \left\{ f \in A : Re\frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\},$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbb{C} \ , \quad Re \ a \geq 0.$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1, 2, 3, ...$ we obtain the Bernardi integral operator.

Let D^n be the Sălăgean differential operator (see [5]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

We observe that if $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$.

The purpose of this note is to define a subclass of n-starlike functions functions associated with some hyperbola and to obtain some estimations for the coefficients of the series expansion and some other properties regarding this class.

2 Preliminary results

Definition 2.1. *[6] A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2}-1\right) \right| < Re\left\{ \sqrt{2}\,\frac{zf'(z)}{f(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right),$$

for some α $(\alpha > 0)$ and for all $z \in U$.

Remark 2.1. *Geometric interpretation. Let*

$$\Omega(\alpha) = \left\{ \frac{zf'(z)}{f(z)} \,:\, z \in U\,,\, f \in SH(\alpha) \right\}.$$

Then $\Omega(\alpha) = \{w = u + i\cdot v \,:\, v^2 < 4\alpha u + u^2\,,\, u > 0\}$. Note that $\Omega(\alpha)$ is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.

Theorem 2.1. *[6] Let $f \in SH(\alpha)$ and $f(z) = z + b_2 z^2 + b_3 z^3 + \ldots$. Then*

$$|b_2| \le \frac{1+4\alpha}{1+2\alpha}\,,\ |b_3| \le \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3}\,.$$

The next theorem is result of the so called "admissible functions method" due to P.T. Mocanu and S.S. Miller (see [1], [2], [3]).

Theorem 2.2. *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } \ p(z) \prec h(z).$$

3 Main results

Definition 3.1. *Let $f \in S$ and $\alpha > 0$. We say that the function f is in the class $SH_n(\alpha)$, $n \in \mathbb{N}$, if*

$$\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < Re\left\{\sqrt{2}\,\frac{D^{n+1}f(z)}{D^n f(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right),\ z \in U .$$

Remark 3.1. *Geometric interpretation: If we denote with p_α the analytic and univalent functions with the properties $p_\alpha(0) = 1$, $p'_\alpha(0) > 0$ and $p_\alpha(U) = \Omega(\alpha)$ (see Remark 2.1), then $f \in SH_n(\alpha)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)} \prec p_\alpha(z)$, where the symbol "$\prec$" denotes the subordination in U. We have $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha$, $b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $Im\sqrt{w} \geq 0$. If we consider $p_\alpha(z) = 1 + C_1 z + \dots$, we have $C_1 = \frac{1+4\alpha}{1+2\alpha}$.*

Theorem 3.1. *Let $f \in SH_n(\alpha)$, $\alpha > 0$, and $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, then*

$$|a_2| \leq \frac{1}{2^n}\cdot\frac{1+4\alpha}{1+2\alpha}\ ,\ |a_3| \leq \frac{1}{3^n}\cdot\frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3} .$$

Proof. If we denote by $D^n f(z) = g(z)$, $g(z) = \sum_{j=2}^{\infty} b_j z^j$, we have:

$f \in SH_n(\alpha)$ if and only if $g \in SH(\alpha)$.

From the above series expansions we obtain $|a_j| \leq \frac{1}{j^n}\cdot|b_j|$, $j \geq 2$. Using the estimations from the Theorem 2.1 we obtain the needed results.

Theorem 3.2. *If $F(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$, and $f(z) = L_a F(z)$, where L_a is the integral operator defined by (1), then $f(z) \in SH_n(\alpha)$, $\alpha > 0$, $n \in \mathbb{N}$.*

Proof. By differentiating (1) we obtain $(1+a)F(z) = af(z) + zf'(z)\,.$

By means of the application of the linear operator D^{n+1} we obtain

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z)$$

Similarly, by means of the application of the linear operator D^n we obtain

$$(1+a)D^{n}F(z) = aD^{n}f(z) + D^{n+1}f(z)$$

Thus

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}f(z) + aD^nf(z)} =$$

$$= \frac{\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)} \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)} + a \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)}}{\dfrac{D^{n+1}f(z)}{D^nf(z)} + a} \tag{2}$$

With notation $\dfrac{D^{n+1}f(z)}{D^nf(z)} = p(z)$, where $p(z) = 1 + p_1z + ...$, we have

$$zp'(z) = z \cdot \left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)' =$$

$$= \frac{z\left(D^{n+1}f(z)\right)' \cdot D^nf(z) - D^{n+1}f(z) \cdot z\left(D^nf(z)\right)'}{\left(D^nf(z)\right)^2} =$$

$$= \frac{D^{n+2}f(z) \cdot D^nf(z) - \left(D^{n+1}f(z)\right)^2}{\left(D^nf(z)\right)^2}$$

and

$$\frac{1}{p(z)} \cdot zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^nf(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z)$$

From the above he have

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z)$$

Thus from (2) we obtain

$$\frac{D^{n+1}F(z)}{D^n F(z)} = \frac{p(z) \cdot \left(zp'(z) \cdot \frac{1}{p(z)} + p(z)\right) + a \cdot p(z)}{p(z) + a} = \tag{3}$$

$$= p(z) + \frac{1}{p(z) + a} \cdot zp'(z)$$

From Remark 3.1 we have $\dfrac{D^{n+1}F(z)}{D^n F(z)} \prec p_\alpha(z)$ and thus, using (3), we obtain

$$p(z) + \frac{1}{p(z) + a} zp'(z) \prec p_\alpha(z) \,.$$

We have from Remark 3.1 and from the hypothesis $Re\,(p_\alpha(z) + a) > 0$, $z \in U$. In this conditions from Theorem 2.2 we obtain $p(z) \prec p_\alpha(z)$ or $\dfrac{D^{n+1}f(z)}{D^n f(z)} \prec p_\alpha(z)$. This means that $f(z) = L_a F(z) \in SH(\alpha)$.

Theorem 3.3. *Let $a \in \mathbb{C}$, $Re\, a \geq 0$, $\alpha > 0$, and $n \in \mathbb{N}$. If $F(z) \in SH_n(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $f(z) = L_a F(z)$, $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L_a is the integral operator defined by (1), then*

$$|b_2| \leq \left|\frac{a+1}{a+2}\right| \cdot \frac{1}{2^n} \cdot \frac{1+4\alpha}{1+2\alpha}\ , \ |b_3| \leq \left|\frac{a+1}{a+3}\right| \cdot \frac{1}{3^n} \cdot \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3} \,.$$

Proof. From $f(z) = L_a F(z)$ we have $(1+a)F(z) = af(z) + zf'(z)$. Using the above series expansions we obtain

$$(1+a)z + \sum_{j=2}^{\infty}(1+a)a_j z^j = az + \sum_{j=2}^{\infty} ab_j z^j + z + \sum_{j=2}^{\infty} jb_j z^j$$

and thus $b_j(a+j) = (1+a)a_j\,,\ j \geq 2$. From the above we have $|b_j| \leq \left|\dfrac{a+1}{a+j}\right| \cdot |a_j|\,,\ j \geq 2$. Using the estimations from Theorem 3.1 we obtain the needed results.

For $a = 1$, when the integral operator L_a become the Libera integral operator, we obtain from the above theorem:

Corollary 3.1. *Let $\alpha > 0$ and $n \in \mathbb{N}$. If $F(z) \in SH_n(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $f(z) = L(F(z))$, $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L is Libera integral operator defined by $L(F(z)) = \dfrac{2}{z}\displaystyle\int_0^z F(t)dt$, then*

$$|b_2| \leq \frac{1}{2^{n-1}} \cdot \frac{1+4\alpha}{3+6\alpha}\ ,\ |b_3| \leq \frac{1}{3^n} \cdot \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{8(1+2\alpha)^3}\,.$$

Theorem 3.4. *Let $n \in \mathbb{N}$ and $\alpha > 0$. If $f \in SH_{n+1}(\alpha)$ then $f \in SH_n(\alpha)$.*

Proof. With notation $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$ we have (see the proof of the Theorem 3.2):

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z)\,.$$

From $f \in SH_{n+1}(\alpha)$ we obtain (see Remark 3.1) $p(z) + \dfrac{1}{p(z)} \cdot zp'(z) \prec p_\alpha(z)$. Using the definition of the function $p_\alpha(z)$ we have $Re\, p_\alpha(z) > 0$ and from Theorem 2.2 we obtain $p(z) \prec p_\alpha(z)$ or $f \in SH_n(\alpha)$.

Remark 3.2. *From the above theorem we obtain $SH_n(\alpha) \subset SH_0(\alpha) = SH(\alpha) \subset S^*$ for all $n \in \mathbb{N}$.*

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