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Data dependence for some integral equation via weakly Picard operators

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Abstract

In this paper we study data dependence for the following integral equation:

$$u(x) = h(x, u(a)) + \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} K(x, s, u(s))ds, x \in \prod_{i=1}^{m} [a_i, b_i]$$

by using c-WPOs .

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1 Introduction

Data dependence for integral-equations was study in [1], [2], [3].

Let (X, d) be a metric space and $A : X \to X$ an operator.We shall use the following notations:

$F_A := \{x \in X \mid Ax = x\}$ the fixed points set of A

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A

$$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N}$$

Definition 1.1. *(see [1]) An operator A is weakly Picard operator(WPO) if the sequence*

$$(A^n x)_{n \in N}$$

converges ,for all $x \in X$ and the limit(which depend on x) is a fixed point of A.

Definition 1.2. *(see [1]) If the operator A is WPO and $F_A = \{x^*\}$then by definition A is Picard operator.*

Definition 1.3. *(see [1]) If A is WPO ,then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n \to \infty} A^n x.$$

We remark that $A^\infty(X) = F_A$.

Definition 1.4. *(see [1]) Let be A an WPO and $c > 0$.The operator A is c-WPO if $d(x, A^\infty x) \leq d(x, Ax)$.*

We have the following characterization of the WPOs.

Theorem 1.1. *(see [1]) Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$;

(b) $A \mid: X_\lambda \to X_\lambda$ *is a Picard(c-Picard) operator, for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result.

Theorem 1.2. *(see [1]) Let* (X, d) *be a metric space and* $A_i : X \to X, i = 1, 2$ *an operator. We suppose that :*

(i) the operator A_i *is* $c_i - WPO$ *i=1,2.*

(ii) there exists $\eta > o$ *such that*

$$d(A_1x, A_2x) \leq \eta, \text{for all } x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

2 Main results

Next,we consider the integral equation

(1) $$u(x) = h(x, u(a)) + \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} K(x, s, u(s))ds, x \in \prod_{i=1}^{m}[a_i, b_i].$$

We denote $D = \prod_{i=1}^{m} [a_i, b_i]$. In [1] we have the following result:

Theorem 2.1. *We suppose that:*

(i) $h \in C(D \times \mathbb{R})$ *and* $K \in C(D \times D \times \mathbb{R})$.

(ii) $h(a, \alpha) = \alpha, (\forall)\alpha \in \mathbb{R}$.

(iii) $h(x, \cdot) and K(x, s, \cdot)$ *are monoton increasing for all* $x, s \in D$.

(iv) there exists $L_K > 0$ *such that*

$$\|K(x, s, u_1) - K(x, s, u_2)\|_{R^m} \leq L_K |u_1 - u_2|,$$

for all $x, s \in D$ *and* $u_1, u_2 \in \mathbb{R}$.

In these conditions the equation(1) has in C(D) an infinity of solutions. Moreover if u and v are solutions of the equations then

$$u(a) \leq v(a) \text{ implies that } u \leq v.$$

The result of this section is given by

Theorem 2.2. *We suppose that :*

(i) $h_i \in C(D \times \mathbb{R})$ *and* $K_i \in C(D \times D \times \mathbb{R})$*i=1,2.*

(ii) $h_i(a, \alpha) = \alpha, (\forall)\alpha \in \mathbb{R}$ *,i=1,2.*

(iii) there exists$L_{K_i} > 0$*such that*

$$\|K_i(x, s, u_1) - K_i(x, s, u_2)\|_{R^m} \leq L_{K_i} |u_1 - u_2|,$$

for all $x, s \in D$ *and* $u_1, u_2 \in \mathbb{R}$.

(iv) exists $\eta_1, \eta_2 > 0$ *such that*

$$|h_1(x, u) - h_2(x, u)| \leq \eta_1$$

$$\|K_1(x,s,u) - K_2(x,s,u)\|_{R^m} \le \eta_2,$$

for all $x, s \in D, u \in \mathbb{R}$. *Then*

$$H(F_{A_1}, F_{A_2}) \le (\eta_1 + \eta_2 \prod_{i=1}^{m}(b_i - a_i)) max\{L_{K_1} + 1, L_{K_2} + 1\}.$$

Proof. We consider the following operators

$$A_i : (C(D), \|\cdot\|_B) \to (C(D), \|\cdot\|_B)),$$

$$A_i u(x) = h_i(x, u(a)) + \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} K_i(x, s, u(s)) ds, i = 1, 2$$

Here

$$|f|_B = \max_{x \in D} |f(x)| e^{-\tau \sum_{i=1}^{m}(x_i - a_i)}$$

We have

$$|A_1 u(x) - A_2 u(x)| \le$$

$$\le |h_1(x, u(a)) - h_2(x, u(a))| + \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} \|K_1(x, s, u(s)) - K_2(x, s, u(s))\|_{R^m} ds \le$$

$$\le \eta_1 + \eta_2 \prod_{i=1}^{m}(b_i - a_i).$$

We consider

$$X_\lambda = \{u \in C(D) \mid u(a) = \lambda\}, \lambda \in R.$$

We have

$$X = \bigcup_{\lambda \in R} X_\lambda.$$

For $u, v \in X_\lambda, x \in D$ we have

$$|A_i u(x) - A_i v(x)| \leq \frac{L_{K_i}}{\tau^m}|u - v|_B e^{\tau \sum_{i=1}^{m}(x_i - a_i)} \text{ which implies}$$

$$|A_i u - A_i v|_B \leq \frac{L_{K_i}}{\tau^m}|u - v|_B$$

We take $\tau = \sqrt[m]{L_{K_i+1}}$,it follow that $A \mid X_\lambda$ is $L_{K_i} + 1$ PO and A_i is $L_{K_i} + 1$ WPO.

From this we have conclusion.

References

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