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Continuity properties relative to the intermediate point in a mean value theorem

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Abstract

This article studies the existence and the value of the limit $\lim\limits_{x\uparrow a}\dfrac{x-a}{(c_x-a)^\alpha}$ where c_x is the intermediate point in a mean value theorem.

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1°. We shall consider the following mean value theorem:

Theorem 1. *Let us suppose that* $f:[a,b]\to\mathbb{R}$ *is continuous on* $[a,b]$ *and derivable on* (a,b). *If* $A(a,f(a))$, $B(b,f(b))$, *then for each* $M(x,y)\in AB$, $x\notin[a,b]$, *exists a point* $c\in(a,b)$, *such that:*

$$\frac{y-f(c)}{x-c}=f'(c).$$

The geometric significance of this theorem is the fact that from any point $M(x,y) \in AB$ where $x \notin [a,b]$ one can draw a tangent to the graphic representation of f (see [3]).

In [1], [3], [4], there are studies for the classical mean value theorems, under some hypothesis, the existence and the value of the limit $\lim\limits_{b\to a}\frac{c-a}{b-a}$, $c = c(b)$ being the intermediate point in the respective mean value theorem.

In the following we shall prove that:

$$\lim_{x\uparrow a}\frac{x-a}{(c_x-a)^\alpha} = 0, \text{ for } \alpha < 2, \text{ and}$$

$$(1)\qquad \lim_{x\uparrow a}\frac{x-a}{(c-a)^2} = \frac{f''(a)}{2[f'(a)-m_{AB}]},\ m_{AB} = \frac{f(b)-f(a)}{b-a},$$

where "c" is the intermediate point from Theorem 1 with the additional conditions that f should be twice derivable, $f'' < 0$ on $[a,b]$.

We can now prove the following:

Theorem 2. *If $f : [a,b] \to \mathbb{R}$ is two times differentiable on $[a,b]$ with $f'' < 0$ on $[a,b]$ and $A(a,f(a))$, $B(b,f(b))$ then for each point $M(x,y)$, $M \in AB$, $x \in (-\infty, a)$, there is an unique point $c \in (a,b)$ such that:*

$$\frac{y-f(c)}{x-c} = f'(c).$$

Proof. We consider that exists $a < c_1 < c_2 < b$ such that:

$$y - f(c_1) = (x-c_1)f'(c_1)$$

$$y - f(c_2) = (x-c_2)f'(c_2)$$

We have:

$$f'(c_2) < \frac{f(c_1)-f(c_2)}{c_1-c_2} = f'(\xi) < f'(c_1)$$

with $\xi \in (c_1, c_2)$.

Therefore

$$(c_1 - c_2)f'(c_2) > (x - c_2)f'(c_2) - (x - c_1)f'(c_1),$$

$$(c_1 - x)f'(c_2) > (c_1 - x)f'(c_1),$$

$$f'(c_2) > f'(c_1)$$

in contradiction with $f'(c_1) > f'(c_2)$.

Hence exists an unique point $c \in (a, b)$ with

$$\frac{y - f(c)}{x - c} = f'(c).$$

2°. Further we prove that the following theorem is valid.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ such that f'' exists and $f'' < 0$ on $[a, b]$. If $\widetilde{c} \in (a, b)$ is the unique point having the property*

$$f'(\widetilde{c}) = \frac{f(b) - f(a)}{b - a},$$

then:

i) whatever is $c \in (a, \widetilde{c})$, the tangent to the graphic representation of f in the point $(c, f(c))$ cuts AB in $M(x_c, y)$ with $x_c < a$.

ii) whatever is $c \in (\widetilde{c}, b)$, the tangent to the graphic representation of f in the point $(c, f(c))$ cuts AB in $N(x_c, y)$ with $x_c > b$.

Proof. i) The tangent to the graphic representation of f in the point $(c, f(c))$ cuts AB in a point having the abscissa:

$$x_c = \frac{cf'(c) - f(c) + f(a) - a \cdot m_{AB}}{f'(c) - m_{AB}}$$

where we have denoted

$$m_{AB} = \frac{f(b) - f(a)}{b - a}.$$

If

$$\Psi(t) = \frac{tf'(t) - f(t) + f(a) - a \cdot m_{AB}}{f'(t) - m_{AB}},\ t \in [a, \widetilde{c}),$$

we have:

$$\Psi'(t) = \frac{f(t) - [f(a) + (t - a)m_{AB}]}{[f'(t) - m_{AB}]^2} f''(t).$$

But $\Psi'(t) < 0$ on $(a, \widetilde{c})$, $\Psi(a) = a$ and from $\Psi(a) > \Psi(c)$ it results $a > x_c$.

In a similar way we prove the second part of the above theorem.

Remark 1. *Because f' is strictly decreasing on $[a, b]$ it follows the unicity of $\widetilde{c}$ and likewise that $f'(a) - m_{AB} \neq 0$.*

Theorem 4. *If $f : [a, b] \to \mathbb{R}$ with $f'' < 0$ on $[a, b]$ and $\widetilde{c} \in (a, b)$ is the unique point having the property*

$$f'(\widetilde{c}) = \frac{f(b) - f(a)}{b - a}$$

then whatever is $x \in (-\infty, a)$, there exists exactly one point $c \in (a, \widetilde{c})$ such that the tangent to the graphic representation of f in the point $(c, f(c))$ cuts AB in $M(x, y)$.

Proof. The existence and the uniqueness of $c \in (a, b)$ is given by Theorem 2. If $c \in (\widetilde{c}, b)$ then according to Theorem 3 ii), the tangent to the graphic representation of f in $(c, f(c))$ cuts AB in $N(x_c, y)$, $x_c > b$. Hence $c \in (a, \widetilde{c})$.

3°. Further let $f : [a, b] \to \mathbb{R}$ such that f'' exists and $f'' < 0$ on $[a, b]$ and $\widetilde{c} \in (a, b)$ the unique point which $f(\widetilde{c}) = \dfrac{f(b) - f(a)}{b - a}$. We define the function $\theta(c) = x_c$, $\theta : [a, \widetilde{c}) \to (-\infty, a]$ where c and x_c have the significance from Theorem 3. It is clear that $\theta(a) = a$ and according to the Theorem 3 and 4 the function θ is bijective.

If $c_0 \in (a, \widetilde{c})$ then

$$x_c = \frac{cf'(c) - f(c) + f(a) - a \cdot m_{AB}}{f'(c) - m_{AB}}$$

and when $c \to c_0$ we have

$$x_c \to \frac{c_0 f'(c_0) - f(c_0) + f(a) - a \cdot m_{AB}}{f'(c_0) - m_{AB}} = x_{c_0}.$$

So $\theta(c) \to \theta(c_0)$ and θ is continuous on $(a, \widetilde{c})$.

In addition, when $c \to a$ it is evidently that

$$x_c = \frac{af'(a) - a \cdot m_{AB}}{f'(a) - m_{AB}} = a.$$

Hence $\theta(c) \to \theta(a)$ and θ is also continuous in a, and on $[a, \widetilde{c})$.

Because θ is bijective and continuous it follows that θ^{-1} is also continuous on $(-\infty, a]$. We denote $\theta^{-1}(x) = c_x$ and when $x \to a$, $\theta^{-1}(x) \to \theta^{-1}(a)$ that is $c_x \to a$.

Finally we shall prove the following:

Theorem 5. *Under the hypothesis of the Theorem 2 we have:*

$$\lim_{x \uparrow a} \frac{x - a}{c_x - a} = 0,$$

and

$$\lim_{x \uparrow a} \frac{x - a}{(c_x - a)^2} = \frac{f''(a)}{2[f'(a) - m_{AB}]}.$$

Proof. From the relation

$$\frac{y - f(c_x)}{x - c_x} = f'(c_x)$$

where $y = f(a) + (x-a)m_{AB}$, we obtain:

$$f(a) - f(c_x) + (x-a)m_{AB} = (x-a)f'(c_x) + (a - c_x)f'(c_x)$$

which implies:

$$\frac{x-a}{c_x - a} = \frac{f'(c_x) - \dfrac{f(c_x) - f(a)}{c_x - a}}{f'(c_x) - m_{AB}} \tag{2}$$

and

$$\frac{x-a}{(c_x-a)^2} = \frac{(c_x-a)f'(c_x) - f(c_x) + f(a)}{(c_x-a)^2} \cdot \frac{1}{f'(c_x) - m_{AB}}.$$

For $x \to a$ we have $c_x \to a$ and

$$f'(c_x) \to f'(a),\ \ \frac{f(c_x) - f(a)}{c_x - a} \to f'(a).$$

According to (2) we find

$$\lim_{x \uparrow a} \frac{x-a}{c_x - a} = 0.$$

If we use l'Hôspital's rule, we have:

$$\lim_{c_x \to a} \frac{(c_x-a)f'(c_x) - f(c_x) + f(a)}{(c_x-a)^2} = \frac{1}{2}f''(a)$$

and so:

$$\lim_{x \uparrow a} \frac{x-a}{(c_x-a)^2} = \frac{f''(a)}{2[f'(a) - m_{AB}]}.$$

Remark 2. *It is clear that:*

$$\lim_{x \uparrow a} \frac{x-a}{(c_x-a)^\alpha} = \begin{cases} 0, & \alpha < 2 \\ \dfrac{f''(a)}{2[f'(a) - m_{AB}]}, & \alpha = 2 \\ -\infty, & \alpha > 2 \end{cases}.$$

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