General Mathematics Vol. 13, No. 1 (2005), 73–82

Multidimensional Integral Inequalities with Homogeneous Weights

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Dedicated to Professor Emil C. Popa on the occasion of his 60th birthday

Abstract

We give upper and lower bounds estimates for the best constant which appears in a multidimensional integral inequality, in the particular case of homogeneous weights.

2000 Mathematics Subject Classification: 26D15, 26B99

Keywords and phrases: Multidimensional embedding, decreasing functions, integral inequalities, best constant.

1 Introduction

Let $\mathbb{R}_+^n := \{(x_1, ..., x_n) : x_i \geq 0, i = 1, 2, ..., n\}$. Assume that $f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is decreasing which means that it is decreasing with respect to each variable. We denote $f \downarrow$ when f is decreasing (=nonicreasing). Throughout this paper u and v are positive, locally integrable functions defined on $\mathbb{R}_+^n, n \geq 1$. χ_D denotes the characteristic function of a set D. In order to motivate the results of this paper and put them into a frame we use Section 2 to remind the characterization of the inequality

$$(1)\qquad \left(\int_{\mathbb{R}_+^n} f^q u\right)^{1/q} \leq C \left(\int_{\mathbb{R}_+^n} f^p v\right)^{1/p}, 0 < p \leq q < \infty,$$

for all $f \downarrow$ and some of the cases when the best constant in the above inequality can be easily computed or estimated. In Section 3 we present some new facts about the best constant in the inequality (1) in the case of homogeneous weights. More precisely we give upper and lower estimates for the best constant. Section 4 is reserved for particular cases and open questions.

2 Weighted inequalities for decreasing functions of several variables

In the one-dimensional case the inequality (1) was characterized in [7] as follows:
If $n = 1$, $0 < p \leq q < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$C_1 = \sup_{t>0} \left(\int_0^t u \right)^{1/q} \left(\int_0^t v \right)^{-1/p} < \infty$$

and the constant $C = C_1$ is sharp.
The multidimensional case was treated in [2] as follows:
If $0 < p \leq q < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$C_n = \sup_{D \in \mathcal{D}} \frac{\left(\int_D u \right)^{1/q}}{\left(\int_D v \right)^{1/p}} < \infty \tag{2}$$

and the constant $C = C_n$ is sharp. Here $\mathcal{D}$ is the set of all "decreasing domains" , i.e. for which the characteristic function is a decreasing function in each variable. It is not always easy to calculate the constant defined in (2). It was shown that in the case of weights of product type the supremum in (2) can be reduced to supremum over a much simpler class of decreasing sets (the rectangles), which is something much simpler to compute, as the following theorem shows. For it's complete proof see [2].

Theorem 1. *If $0 < p \le q < \infty$, $u(x_1, \dots, x_n) = u_1(x_1)\dots u_n(x_n)$ and $v(x_1, \dots, x_n) = v_1(x_1)\dots v_n(x_n)$ then*

$$(3)\qquad \sup_{D\in\mathcal{D}} \frac{\left(\int_D u\right)^{1/q}}{\left(\int_D v\right)^{1/p}} = \sup_{a_i>0} \frac{\left(\int_0^{a_1}\cdots\int_0^{a_n} u(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/q}}{\left(\int_0^{a_1}\cdots\int_0^{a_n} v(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/p}}.$$

In [3] it was given an example which shows that the equality (3) does not hold for arbitrary weights. The natural and important question is now whether the constants C_n and

$$A_n = \sup_{a_i>0} \frac{\left(\int_0^{a_1}\cdots\int_0^{a_n} u(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/q}}{\left(\int_0^{a_1}\cdots\int_0^{a_n} v(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/p}}$$

are comparable in the general case. Clearly $A_n \le C_n$ and we may ask if the converse inequality $C_n \le cA_n$ holds with a constant c independent on the weights. An answer to this question was given in [3] with the help of the following result of independent interest where explicit lower and upper estimates are given for the constant C_n when the weights are of the form: $u(xy), v(xy)$.

Theorem 2. *Let $0 < p \le q < \infty$ and $u(s) \ge 0$ and $v(s) \ge 0$ be two measurable functions on $\mathbb{R}_+$ such that $U(t) = \int_0^t u < \infty, V(t) = \int_0^t v < \infty$ for all $t > 0$. Then the inequality*

$$\left(\int_{\mathbb{R}^2_+} f^q(x,y)u(xy)dxdy\right)^{1/q} \le C\left(\int_{\mathbb{R}^2_+} f^p(x,y)v(xy)dxdy\right)^{1/p}.$$

holds for all $f \ge 0$ decreasing in x and y with a finite constant $C > 0$ independent on f if and only if

$$A = A_{p,q} := \sup_{t>0}\left(\frac{U(t)}{V(t)}\right)^{1/q}\left(\int_0^t V(x)\frac{dx}{x}\right)^{1/q-1/p} < \infty$$

Moreover, $C = A_{p,p}$ and

$$2^{-1/p}A \le C \le \left(\frac{p}{q}\right)^{1/q} A,\ \ \text{if } p < q.$$

Using the above theorem it was proved that A_n and C_n are comparable in the sense that either both are finite or both are infinite, but the estimate $C_n \leq cA_n$ is no longer uniform. For more details see [3].

3 Upper and lower estimates of the best constant in the case of homogeneous weights

In this section we study the same type of questions as above but for the case when the weights are homogeneous functions. The next Lemma gives important information about the best constant in the inequality (1) when the weights are homogeneous functions.

Lemma 1. *Let $0 < p \leq q < \infty$, $\alpha > -n$, $\beta > -n$, $u(\varepsilon x) = \varepsilon^\alpha u(x)$, $v(\varepsilon x) = \varepsilon^\beta v(x)$, $x \in \mathbb{R}^n_+$ and $\varepsilon > 0$. If $C_n < \infty$ then*

$$\frac{\alpha + n}{q} = \frac{\beta + n}{p}.$$

Proof. Let $S = \{\sigma \in \mathbb{R}^n_+ : |\sigma| = 1\}$ and suppose that

$$\frac{\alpha + n}{q} \neq \frac{\beta + n}{p}.$$

Then

$$\begin{aligned} C_n &= \sup_{D \in \mathcal{D}} \frac{\left(\int_D u\right)^{1/q}}{\left(\int_D v\right)^{1/p}} \\ &\geq \sup_{r>0} \frac{\left(\int_0^r \int_S u(\rho\sigma)\rho^{n-1} d\rho d\sigma\right)^{1/q}}{\left(\int_0^r \int_S v(\rho\sigma)\rho^{n-1} d\rho d\sigma\right)^{1/p}} \\ &= \sup_{r>0} \frac{\left(\int_0^r \int_S \rho^{\alpha+n-1} u(\sigma) d\rho d\sigma\right)^{1/q}}{\left(\int_0^r \int_S \rho^{\beta+n-1} v(\sigma) d\rho d\sigma\right)^{1/p}} \\ &= \sup_{r>0} r^{(\alpha+n)/q - (\beta+n)/p} C(u, v, p, q) = \infty. \end{aligned}$$

where $C(u, v, p, q) < \infty$ and depends only on u, v, p and q. Hence in the case of homogeneous weights the only interesting case is when

$$\frac{\alpha + n}{q} = \frac{\beta + n}{p}.$$

In fact, for the case $n = 2$ we have the following result:

Theorem 3. *Let* $0 < p \leq q$, $\alpha > -2$, $\beta > -2$, $u(\varepsilon x, \varepsilon y) = \varepsilon^{\alpha} u(x, y)$, $v(\varepsilon x, \varepsilon y) = \varepsilon^{\beta} v(x, y)$, $(x, y) \in \mathbb{R}^2_+$, $\varepsilon > 0$.

Denote by $U(s) = \int_0^s u(1, y)dy < \infty$ *and* $V(s) = \int_0^s v(1, y)dy < \infty$ *for all* $s > 0$. *Then the inequality (1) holds for all* $f \downarrow$ *if and only if*

$$A := \sup_{t>0} \left(t^{(\beta-\alpha)} \frac{U(t)}{V(t)} \right)^{1/q} \left(\int_t^{\infty} V(s) \frac{ds}{s^{\beta+3}} \right)^{1/q-1/p} < \infty.$$

Moreover,

$$A \leq C_2 \leq (p/q)^{1/q} A, if p < q.$$

and

$$A = C_2 \text{ if } p = q.$$

Proof. By Lemma 1 we may suppose that $\frac{\alpha+2}{q} = \frac{\beta+2}{p}$. We will first prove the upper bound. We know from (2) that

$$C_2 = \sup_{t>0, h\downarrow} \mathcal{I}_h(t) = \sup_{t>0, h\downarrow} \frac{\left(\int_0^t dx \int_0^{h(x)} u(x, y)dy \right)^{1/q}}{\left(\int_0^t dx \int_0^{h(x)} v(x, y)dy \right)^{1/p}} \tag{4}$$

and by using the homogeneity and changing variables, we find that

$$\mathcal{I}_h(t) = \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{h(x)}{x}\right) dx \right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{h(x)}{x}\right) dx \right)^{1/p}}.$$

We observe that

$$U(t) \leq A^q \left(\int_t^{\infty} V(s) \frac{ds}{s^{\beta+3}} \right)^{q/p-1} V(t) t^{(\beta+2)(q/p-1)}, \forall t > 0.$$

By substituting now t by $\frac{h(x)}{x}$ and multiplying by $x^{\alpha+1}$ we get

$$x^{\alpha+1}U\left(\frac{h(x)}{x}\right)$$

$$\leq A^q\left(\int_{\frac{h(x)}{x}}^{\infty}V(s)\frac{ds}{s^{\beta+3}}\right)^{q/p-1}V\left(\frac{h(x)}{x}\right)x^{\beta+1}h^{(\beta+2)(q/p-1)}(x)$$

i.e.

$$x^{\alpha+1}U\left(\frac{h(x)}{x}\right)\leq A^q\left(\int_{\frac{h(x)}{x}}^{\infty}V(s)\left(\frac{h(x)}{s}\right)^{(\beta+2)}\frac{ds}{s}\right)^{q/p-1}V\left(\frac{h(x)}{x}\right)x^{\beta+1}.$$

By integrating now with respect to x, from 0 to t, changing the variables $\xi=\dfrac{h(x)}{s}$ and using the facts that h and V are decreasing, respectively increasing we obtain

$$\begin{aligned}\int_0^t x^{\alpha+1}U\left(\frac{h(x)}{x}\right)dx &\leq A^q\int_0^t\left(\int_0^x V\left(\frac{h(x)}{\xi}\right)\xi^{\beta+1}d\xi\right)^{q/p-1}V\left(\frac{h(x)}{x}\right)x^{\beta+1}dx\\ &\leq \int_0^t\left(\int_0^x V\left(\frac{h(\xi)}{\xi}\right)\xi^{\beta+1}d\xi\right)^{q/p-1}V\left(\frac{h(x)}{x}\right)x^{\beta+1}dx\\ &= \frac{p}{q}A^q\left(\int_0^t V\left(\frac{h(x)}{x}\right)x^{\beta+1}dx\right)^{q/p}. \qquad (5)\end{aligned}$$

This implies that

$$\mathcal{I}_h\leq\left(\frac{p}{q}\right)^{1/q}A$$

for all $t>0$ and $h\downarrow$ and we got the desired upper bound.
For the lower bound suppose first that $p=q$, thus also that $\alpha=\beta$. Consider a strictly increasing sequence $(a_n)_{n\geq 1}$, $a_1=0$ which converges to 1 and take $h(x)\equiv 1$. For a fix $t>0$, by using the fact that U is increasing and $y^{-(\alpha+3)}$ is decreasing we get

$$\int_0^{a_nt}x^{\alpha+1}U\left(\frac{1}{x}\right)dx = \sum_{k=2}^{n}\int_{a_{k-1}t}^{a_kt}x^{\alpha+1}U\left(\frac{1}{x}\right)dx$$

$$
\begin{aligned}
& = \sum_{k=2}^{n} \int_{\frac{1}{a_k t}}^{\frac{1}{a_{k-1} t}} y^{-(\alpha+3)} U(y) dy \\
& \geq \sum_{k=2}^{n} U\left(\frac{1}{a_k t}\right) (a_{k-1} t)^{\alpha+3} \\
& \geq t^{\alpha+3} U\left(\frac{1}{t}\right) \sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}. \qquad (6)
\end{aligned}
$$

In the same way since V is also increasing we get

$$
\begin{aligned}
\int_0^{a_n t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx & = \int_0^{a_2 t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx + \sum_{k=3}^{n} \int_{\frac{1}{a_k t}}^{\frac{1}{a_{k-1} t}} y^{-(\alpha+3)} V(y) dy \\
& \leq \int_0^{a_2 t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx + t^{\alpha+3} \sum_{k=3}^{n} V\left(\frac{1}{a_{k-1} t}\right) {a_k}^{\alpha+3}. \qquad (7)
\end{aligned}
$$

By (6) and (7) we have

$$
\frac{\int_0^{a_n t} x^{\alpha+1} U\left(\frac{1}{x}\right) dx}{\int_0^{a_n t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx} \geq \frac{t^{\alpha+3} U\left(\frac{1}{t}\right) \sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}}{\int_0^{a_2 t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx + t^{\alpha+3} \sum_{k=3}^{n} V\left(\frac{1}{a_{k-1} t}\right) {a_k}^{\alpha+3}}.
$$

Dividing now by $t^{\alpha+3} \sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}$ we get

$$
\frac{\int_0^{a_n t} x^{\alpha+1} U\left(\frac{1}{x}\right) dx}{\int_0^{a_n t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx} \geq \frac{U\left(\frac{1}{t}\right)}{\dfrac{\int_0^{a_2 t} x^{\alpha+1} V\left(\frac{1}{x}\right) dx}{t^{\alpha+3} \sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}} + \dfrac{\sum_{k=3}^{n} V\left(\frac{1}{a_{k-1} t}\right) {a_k}^{\alpha+3}}{\sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}}}. \qquad (8)
$$

Since V is continuous, we get by a theorem of Stolz-Cesaro (see e.g Theorem 2.10.2 in [5]) that

$$
\lim_{n \to \infty} \frac{\sum_{k=3}^{n} V\left(\frac{1}{a_{k-1} t}\right) {a_k}^{\alpha+3}}{\sum_{k=2}^{n} {a_{k-1}}^{\alpha+3}} = V\left(\frac{1}{t}\right).
$$

Since t was arbitrarily chosen, letting now $n \to \infty$ in (8) we finally get

$$C_2 \geq \sup_{t>0} \left(\frac{U(t)}{V(t)} \right)^{1/q}.$$

If now $p < q$, i.e. $\alpha \neq \beta$ one can prove exactly as above that for any $t > 0$,

$$\frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx \right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx \right)^{1/q}} \geq \left(t^{\alpha-\beta} \frac{U(\frac{1}{t})}{V(\frac{1}{t})} \right)^{1/q}.$$

Hence

$$\begin{aligned} C_2 &\geq \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx \right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx \right)^{1/p}} \\ &= \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx \right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx \right)^{1/q}} \left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx \right)^{1/p-1/q} \\ &\geq \left(t^{\alpha-\beta} \frac{U(\frac{1}{t})}{V(\frac{1}{t})} \right)^{1/q} \left(\int_{\frac{1}{t}}^{\infty} V(s) \frac{ds}{s^{\beta+3}} \right)^{1/p-1/q}. \end{aligned}$$

Since $t > 0$ was arbitrarily chosen, the above inequality implies the upper bound for the constant and this completes the proof.

Remark. Let

$$A_2 := \sup_{a,b>0} \frac{\left(\int_0^a \int_0^b u(x,y) dx dy \right)^{1/q}}{\left(\int_0^a \int_0^b v(x,y) dx dy \right)^{1/p}}.$$

it is clear from (4), with $h(x) \equiv b$ and changing variables that we obtain

$$A_2 = \sup_{t,b>0} \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{b}{x}\right) dx \right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{b}{x}\right) dx \right)^{1/p}}$$

and $A_2 \geq \widetilde{A}$ where

$$\widetilde{A} = \sup_{t>0} \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx\right)^{1/q}}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx\right)^{1/p}}.$$

Obviously, Theorem 3 yields

$$\widetilde{A} \leq A_2 \leq C_2 \leq A.$$

Applying the Lospital test we note that

$$\lim_{t\to 0} \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx\right)}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx\right)^{q/p}} = \frac{p}{q} \lim_{t\to\infty} t^{\beta-\alpha} \frac{U(t)}{V(t)} \left(\int_t^\infty V(s) \frac{ds}{s^{\beta+3}}\right)^{1-q/p}$$

and

$$\lim_{t\to\infty} \frac{\left(\int_0^t x^{\alpha+1} U\left(\frac{1}{x}\right) dx\right)}{\left(\int_0^t x^{\beta+1} V\left(\frac{1}{x}\right) dx\right)^{q/p}} = \frac{p}{q} \lim_{t\to 0} t^{\beta-\alpha} \frac{U(t)}{V(t)} \left(\int_t^\infty V(s) \frac{ds}{s^{\beta+3}}\right)^{1-q/p}.$$

Since the functions involved are continuous we conclude that A and $\widetilde{A}$ are comparable in the sense that if $A < \infty$ then $\widetilde{A} < \infty$ and the other way round. The same is true for A and A_2. This does not give an answer to the question if the two quantities are comparable or not in the general case.

4 Conclusions

By (8) and Theorem 3 one could prove results for power weights which generalize some of the results proved in [1]. For further details see [2]. Also for the case $n = 2$ the same results can be derived from Theorem 3 since the power weights are also homogeneous functions. Observe that if $p = q$ the constant does not depend on the degree of homogeneity. Also we get the same constant if the weights are $u(x, y) = \frac{xy}{x^2+y^2}$ and $v(x, y) = \frac{x^2}{y^2+xy}$ as if they are $u(x, y) = \frac{x^3y}{x^2+y^2}$ and $v(x, y) = \frac{x^4}{y^2+xy}$ For other related results as well as for the results for increasing functions see also [6].

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